PROBLEMS AND SOLUTIONS

Edited by Gerald A. Edgar, Doug Hensley, Douglas B. West

Proposed problems and solutions should be sent in duplicate to the MONTHLY problems address on the back of the title page. Proposed problems should never be under submission concurrently to more than one journal. Submitted solutions should arrive before December 31, 2011. Additional information, such as generalizations and references, is welcome. The problem number and the solver's name and address should appear on each solution. An asterisk (*) after the number of a problem or a part of a problem indicates that no solution is currently available.

PROBLEMS

11586. Proposed by Takis Konstantopoulos, Uppsala University, Uppsala, Sweden. Let \( A_0, B_0, \) and \( C_0 \) be noncollinear points in the plane. Let \( p \) be a line that meets lines \( B_0C_0, C_0A_0, \) and \( A_0B_0 \) at \( A^*, B^*, \) and \( C^* \) respectively. For \( n \geq 1, \) let \( A_n \) be the intersection of \( B^*B_{n-1} \) with \( C^*C_{n-1} \), and define \( B_n, C_n \) similarly. Show that all three sequences converge, and describe their respective limits.

11587. Proposed by Andrei Ciupan, Harvard University, Cambridge, MA, and Bozgan Francisc, UCLA, Los Angeles, CA. For which pairs \((a, b)\) of positive integers do there exist infinitely many positive integers \( n \) such that \( n^2 \) divides \( a^n + b^n \)?

11588. Proposed by Taras Banakh, Ivan Franko National University of Lviv, Lviv, Ukraine, and Igor Protasov, Taras Shevchenko National University of Kyiv, Kyiv, Ukraine. Show that \( \mathbb{R} - \{0\} \) can be partitioned into countably many subsets, each of which is linearly independent over \( \mathbb{Q} \), if and only if the continuum hypothesis holds.

11589. Proposed by Catalin Barboianu, Infarom Publishing, Craiova, Romania. Let \( P \) be a polynomial over \( \mathbb{R} \) given by \( P(x) = x^3 + a_2x^2 + a_1x + a_0 \), with \( a_i > 0 \). Show that \( P \) has at least one zero between \(-a_0/a_1\) and \(-a_2\).

11590. Proposed by Khodakhast Bibak, University of Waterloo, Waterloo, Ontario, Canada. Let \( m \) balls numbered 1 to \( m \) each be painted with one of \( n \) colors, with \( n \geq 2 \) and at least two balls of each color. For each positive integer \( k \), let \( P(k) \) be the number of ways to put these balls into urns numbered 1 through \( k \) so that no urn is empty and no urn gets two or more balls of the same color. Prove that

\[
\sum_{k=1}^{m} \frac{(-1)^k}{k} P(k) = 0.
\]

http://dx.doi.org/10.4169/amer.math.monthly.118.07.653
11596. Proposed by Mehmet Sahin (student) Ankara University, Ankara, Turkey. Let \(a, b, c\) be the side lengths of a triangle, and let \(r_a, r_b, r_c\) be the corresponding exradii. Prove that
\[
\frac{a^2}{r_a^2} + \frac{b^2}{r_b^2} + \frac{c^2}{r_c^2} = 8 \left( \frac{r_a + r_b + r_c}{a + b + c} \right)^2 - 2.
\]

11597. Proposed by Michel Bataille, Rouen, France. Let \(f(x) = x/\log(1 - x)\). Prove that for \(0 < x < 1\),
\[
\sum_{n=1}^{\infty} \frac{x^n(1-x)^n}{n!} f^{(n)}(x) = \frac{1}{2} x f(x).
\]

11598. Proposed by Mowaffaq Hajja, Yarmouk University, Irbid, Jordan. Let \(S\) be an additive semigroup of positive integers. Show that there is a finite subset \(T\) of \(S\) that generates \(S\) and that is contained in every generating set of \(S\).

11599. Proposed by Fred Galvin, University of Kansas, Lawrence, KS, and Péter Komjáth, Eötvös Loránd University, Budapest, Hungary. Prove that the following statement is equivalent to the axiom of choice: for any finite family \(A_1, \ldots, A_n\) of sets, there is a finite set \(F\) such that \(|A_i \cap F| < |A_j \cap F|\) whenever \(|A_i| < |A_j|\).

Here, equivalence is to be judged in the context of Zermelo-Fraenkel set theory, not assuming the axiom of choice, and to say that \(|C| < |D|\) is to say that there is an injection from \(C\) to \(D\), but none from \(D\) to \(C\).

SOLUTIONS

A Generic Lower Bound for \(a^2 + b^2 + c^2\) in a Triangle

11460 [2009, 844]. Proposed by Cosmin Pohoata, Tudor Vianu National College of Informatics, Bucharest, Romania. Given a triangle of area \(S\) with sides of lengths \(a, b, c\), and positive numbers \(x, y, z\), show that
\[
a^2 + b^2 + c^2 \geq 4\sqrt{3}S + \frac{2}{x + y + z} \left( a^2 \frac{x^2 - yz}{x} + b^2 \frac{y^2 - zx}{y} + c^2 \frac{z^2 - xy}{z} \right).
\]

Solution by Marian Dinca, Romania. Since the proposed inequality is homogeneous in \(x, y, z\), we may assume without loss of generality that \(x + y + z = 1\). The inequality may be written as
\[
ma^2 + nb^2 + pc^2 \geq 4\sqrt{3}S,
\]
where \(m = 1 - 2(x^2 - yz)/x, n = 1 - 2(y^2 - zx)/y,\) and \(p = 1 - 2(z^2 - xy)/z\). A corollary of the Neuberg-Pedoe inequality (see comment below) tells us that
\[
ma^2 + nb^2 + pc^2 \geq 4\sqrt{3}mn + np + pm.
\]
It now suffices to show that \(mn + np + pm = 3\), which may be done as follows: Let \(t = 2(xy + yz + zx)\), so that \(m = 1 - 2(1 - y - z) + 2yz/x = (2xy + 2yz + 2zx - x)/x = (t - x)/x, n = (t - y)/y,\) and \(p = (t - z)/z\). Then \(mn = (t - y)(t - y)/(xy) = (zt^2 - (xx + yz)t + xy)/(xyz),\) etc., so \(mn + np + pm = (x + y + z)t^2 - t^3 + 3xyz)/(xyz) = 3\).

Editorial comment. Several solvers made note of the connection between this inequality and others already in the literature: