

# TRANSITION SEMIGROUPS OF BANACH SPACE VALUED ORNSTEIN-UHLENBECK PROCESSES<sup>1</sup>

B. GOLDYS AND J.M.A.M. VAN NEERVEN

ABSTRACT. We investigate the transition semigroup of the solution to a stochastic evolution equation

$$dX(t) = AX(t) dt + dW_H(t), \quad t \geq 0,$$

where  $A$  is the generator of a  $C_0$ -semigroup  $\mathbf{S}$  on a separable real Banach space  $E$  and  $\{W_H(t)\}_{t \geq 0}$  is cylindrical white noise with values in a real Hilbert space  $H$  which is continuously embedded in  $E$ . Various properties of these semigroups, such as the strong Feller property, the spectral gap property, and analyticity, are characterized in terms of the behaviour of  $\mathbf{S}$  in  $H$ . In particular we investigate the interplay between analyticity of the transition semigroup,  $\mathbf{S}$ -invariance of  $H$ , and analyticity of the restricted semigroup  $\mathbf{S}_H$ .

## CONTENTS

1. Introduction	2
2. Preliminaries	3
2.1. Reproducing kernel Hilbert spaces	3
2.2. The general setting	6
3. Invariance of the reproducing kernel Hilbert space $H$	9
4. The Liapunov equation $AX + XA^* = -Q$ and $Q$ -symmetry	16
5. Spectral gap conditions	19
6. The Ornstein-Uhlenbeck semigroup in $C_b(E)$	22
6.1. The strong Feller property	28
7. The Ornstein-Uhlenbeck semigroup in $L^2(E, \mu_\infty)$	29
8. Analyticity of the Ornstein-Uhlenbeck semigroup	32
9. Analyticity and invariance of $H$	35
References	41

---

2000 *Mathematics Subject Classification*. Primary: 35R15, 60H15; Secondary: 47D03.

*Key words and phrases*. Ornstein-Uhlenbeck process, transition semigroup, reproducing kernel Hilbert space, strong Feller property, spectral gap, analytic semigroup, stochastic PDE of parabolic type.

<sup>1</sup>This is an update of the published version in *Acta Appl. Math.* **76** (2003), 283–330. Changes are marked with footnoted

## 1. INTRODUCTION

In this paper we study transition semigroups associated with stochastic linear Cauchy problems

$$(1.1) \quad \begin{aligned} dX(t) &= AX(t) + dW_H(t), & t \geq 0, \\ X(0) &= x. \end{aligned}$$

We assume that  $A$  is the generator of a  $C_0$ -semigroup  $\mathbf{S} = \{S(t)\}_{t \geq 0}$  of bounded linear operators on a separable real Banach space  $E$  and  $\mathbf{W}_H = \{W_H(t)\}_{t \geq 0}$  is a cylindrical Wiener process with the Cameron-Martin space  $H$  which is continuously imbedded into  $E$ .

If  $E$  is a Hilbert space, an explicit condition is known (see for example [15]) which ensures the existence of a unique solution to (1.1) of the form

$$(1.2) \quad X(t, x) = S(t)x + \int_0^t S(t-s) dW_H(s).$$

The solution  $\{X(t, x)\}_{t \geq 0}$  is called the *Ornstein-Uhlenbeck* process associated with  $\mathbf{S}$  and  $\mathbf{W}_H$ . It is a Markov process on  $E$  whose transition semigroup is given by

$$(1.3) \quad P(t)\phi(x) = \mathbb{E}\phi(X(t, x)) = \int_E \phi(S(t)x + y) d\mu_t(y),$$

where  $\{\mu_t : t \geq 0\}$  is a family of centred Gaussian measures on  $E$  associated with  $\mathbf{S}$  and  $H$ ; see Section 6 for details. This semigroup is also called the *Ornstein-Uhlenbeck semigroup* associated with  $\mathbf{S}$  and  $H$ .

If  $E$  is a Banach space, there seems to be no general satisfactory theory of stochastic integration to give a rigorous meaning to the integral appearing in (1.2). However, in many important cases it can be shown that formula (1.2) is meaningful (at least in a weak sense) and defines again a Markov process on  $E$  with transition semigroup  $\mathbf{P} = \{P(t)\}_{t \geq 0}$  given by (1.3); see for example [4], [5], [6]. The aim of this paper is to study the transition semigroup  $\mathbf{P}$  and its generator under the sole assumption that the process  $\{X(t, x)\}_{t \geq 0}$  is well defined and admits an invariant measure  $\mu_\infty$ .

Apart from the case where  $E$  is itself a Hilbert space and  $H = E$ , many aspects of Ornstein-Uhlenbeck semigroups are not well understood. For example, the existing criteria for the strong Feller property are difficult to check in general. Similarly, it is very difficult to check whether  $\mathbf{P}$  is analytic in  $L^2(E, \mu_\infty)$  or whether its generator has the spectral gap property.

The main idea of this paper, already exploited in [9, 10], is to study the transition semigroup under the assumption that  $\mathbf{S}$  restricts to a  $C_0$ -semigroup  $\mathbf{S}_H$  on  $H$ . In this setting we obtain explicit conditions for some properties of  $\mathbf{P}$  in terms of the behaviour of the semigroup  $\mathbf{S}_H$ . In particular we provide necessary and sufficient conditions for the strong Feller property of  $\mathbf{P}$  and for the existence of a spectral gap. We also obtain conditions for analyticity of  $\mathbf{P}$  in terms of analyticity of the restricted semigroup  $\mathbf{S}_H$  which seem to be close to optimal.

Our results extend and complement various results from [9, 10, 12, 13, 15, 20, 22, 23, 29].

Let us now describe the contents of the paper in more detail. Since many properties of  $\mathbf{P}$  are determined by the behaviour of the semigroup  $\mathbf{S}$  on the spaces  $H$  and the reproducing kernel Hilbert spaces  $H_t$  associated with the measures  $\mu_t$ , Sections 2 and 3 are devoted to a study of interactions between the semigroup  $\mathbf{S}$ , the space  $H$  and the spaces  $H_t$ . We also investigate in detail the situation when  $H$  is invariant under the semigroup  $\mathbf{S}$ . In Section 4 the Liapunov equation is considered and conditions are given for the symmetry of  $\mathbf{S}$  acting in  $H$ .

In Section 5 we give several characterizations of the spectral gap property of the generator  $A$  of  $\mathbf{S}$  when considered in  $H$  and reproducing kernel Hilbert space  $H_\infty$  associated with the invariant measure  $\mu_\infty$ . In the case when  $E$  is a Hilbert space, it was shown in [9] that this property is equivalent to the logarithmic Sobolev inequality for the generator of the associated Ornstein-Uhlenbeck semigroup. We also show that more accurate information can be obtained if  $H$  is  $\mathbf{S}$ -invariant.

In Section 6 we introduce the Ornstein-Uhlenbeck semigroup  $\mathbf{P}$ . It is studied in the space  $C_b(E)$  endowed with the mixed topology  $\tau_{\text{mixed}}$  under the minimal assumption that (1.3) is meaningful. We extend the results from [24] by showing that  $\mathbf{P}$  is  $C_0$ -semigroup in  $(C_b(E), \tau_{\text{mixed}})$  and by giving an explicit formula for its generator  $L$  on a suitable core. Let us note that  $\mathbf{P}$  is not strongly continuous, in fact not even strongly measurable, in  $C_b(E)$  endowed with the supremum norm. We also provide a new explicit condition for the strong Feller property of  $\mathbf{P}$  in the case when  $H$  is  $\mathbf{S}$ -invariant.

Under the assumption of the existence of an invariant measure  $\mu_\infty$ , in Section 7 we study the semigroup  $\mathbf{P}$  in  $L^2(E, \mu_\infty)$ . In particular we extend the existing criteria for the symmetry of  $\mathbf{P}$  and the existence of spectral gap for  $L$ .

In Sections 8 and 9 we are concerned with analyticity of the Ornstein-Uhlenbeck semigroup in  $L^2(E, \mu_\infty)$ . We obtain necessary and sufficient conditions for analyticity in terms of  $H$ . We establish connections between the analyticity of  $\mathbf{P}$ , the invariance of  $H$  under  $\mathbf{S}$ , and the analyticity of the restricted semigroup  $\mathbf{S}_H$ . We apply our criteria to prove analyticity of Ornstein-Uhlenbeck semigroups associated with some stochastic partial differential equations of parabolic type.

## 2. PRELIMINARIES

**2.1. Reproducing kernel Hilbert spaces.** Stochastic evolution equations in Banach spaces are studied conveniently by using the language of reproducing kernel Hilbert spaces. We start by recalling some elementary properties of these spaces.

Throughout this paper  $E$  denotes a real Banach space. The dual of  $E$  is denoted by  $E^*$ . A bounded linear operator  $Q \in \mathcal{L}(E^*, E^*)$  is called *positive* if

$$\langle Qx^*, x^* \rangle \geq 0, \quad x^* \in E^*,$$

and *symmetric* if

$$\langle Qx^*, y^* \rangle = \langle Qy^*, x^* \rangle, \quad x^*, y^* \in E^*.$$

More generally these definitions make sense for operators  $Q \in \mathcal{L}(E^*, E^{**})$ .

If  $Q \in \mathcal{L}(E^*, E)$  is positive and symmetric, then the bilinear map on the range of  $Q$  defined by

$$(Qx^*, Qy^*) \mapsto \langle Qx^*, y^* \rangle, \quad x^*, y^* \in E^*,$$

is easily checked to be a well defined inner product on the range of  $Q$ . The Hilbert space completion of range  $Q$  with respect to this inner product is called the *reproducing kernel Hilbert space* (RKHS) associated with  $Q$  and is denoted by  $(H_Q, [\cdot, \cdot]_{H_Q})$ . It is well known that the inclusion mapping from range  $Q$  into  $E$  extends to a continuous injection from  $H_Q$  into  $E$ . Denoting this extension by  $i_Q$ , we have

$$Q = i_Q \circ i_Q^*.$$

This factorization immediately implies that  $Q$  is weak\*-to-weakly continuous, and that  $H_Q$  is separable whenever  $E$  is separable.

Conversely, if  $i : H \hookrightarrow E$  is a continuous embedding of a real Hilbert space  $H$  into  $E$ , then  $Q := i \circ i^*$  is positive and symmetric. As subsets of  $E$  we have  $H = H_Q$  and the map  $i^*x^* \mapsto i_Q^*x^*$  defines an isometrical isomorphism of  $H$  onto  $H_Q$ .

*Example 2.1.*

- (1) If  $B$  is a bounded operator from a real Hilbert space  $\mathcal{H}$  into  $E$ , then  $Q := B \circ B^* \in \mathcal{L}(E^*, E)$  is positive and symmetric. As subsets of  $E$  we have  $H_Q = \text{range } B$ , and the inner product of  $H_Q$  is given by

$$[Bg, Bh]_{H_Q} = [Pg, Ph]_{\mathcal{H}}, \quad g, h \in \mathcal{H}.$$

Here  $P$  denotes the orthogonal projection in  $\mathcal{H}$  onto the orthogonal complement of  $\ker B$ .

- (2) As a special case of (1) let  $E$  be a real Hilbert space and let  $Q \in \mathcal{L}(E)$  be a positive and selfadjoint operator. Identifying the dual space  $E^*$  with  $E$  in the natural way, we have  $H_Q = \text{range } Q^{\frac{1}{2}}$ , with inner product

$$[Q^{\frac{1}{2}}x, Q^{\frac{1}{2}}y]_{H_Q} = [Px, Py]_E, \quad x, y \in E.$$

Here  $P$  denotes the orthogonal projection in  $E$  onto the orthogonal complement of  $\ker Q^{\frac{1}{2}}$ .

It will be useful to compare the RKHS's associated with different positive symmetric operators in  $\mathcal{L}(E^*, E)$ . In this direction we have the following easy fact; cf. [15, Appendix B]. If  $Q$  and  $R$  are positive and symmetric operators in  $\mathcal{L}(E^*, E)$ , the following assertions are equivalent:

- (1)  $i_Q(H_Q) \subseteq i_R(H_R)$ ;  
(2) There exists a constant  $M \geq 0$  such that

$$(2.1) \quad \langle Qx^*, x^* \rangle \leq M \langle Rx^*, x^* \rangle, \quad x^* \in E^*.$$

Whenever it is convenient, we shall identify an embedded Hilbert space with its image in  $E$ . Thus, instead of  $i_Q(H_Q) \subseteq i_R(H_R)$  we shall simply write  $H_Q \subseteq H_R$ .

Another simple observation about RKHS's will be useful. Suppose  $E$  and  $F$  are real Banach spaces,  $j : E \hookrightarrow F$  a continuous inclusion, and  $Q_E \in \mathcal{L}(E^*, E)$  and  $Q_F \in \mathcal{L}(F^*, F)$  are positive symmetric operators such that the following diagram commutes:

$$\begin{array}{ccc} E & \xrightarrow{\quad} & F \\ & \uparrow Q_E & \uparrow Q_F \\ E^* & \xleftarrow{\quad} & F^* \end{array}$$

Thus,  $Q_F = j \circ Q_E \circ j^*$ . Let  $i_E : H_E \hookrightarrow E$  and  $i_F : H_F \hookrightarrow F$  denote the RKHS's associated with  $Q_E$  and  $Q_F$ , respectively. Then the mapping

$$I_{E,F} : i_E^* j^* y^* \mapsto i_F^* y^*, \quad y^* \in F^*,$$

extends uniquely to an isometry from  $H_E$  onto  $H_F$ . Moreover, as subsets of  $F$ , the spaces  $H_E$  and  $H_F$  are identical.

Indeed, we compute:

$$\|i_E^* j^* y^*\|_{H_E}^2 = \langle Q_E j^* y^*, j^* y^* \rangle = \langle Q_F y^*, y^* \rangle = \|i_F^* y^*\|_{H_F}^2, \quad y^* \in F^*.$$

Since  $i_E$  and  $j$  are injective,  $i_E^* \circ j^*$  has dense range in  $E^*$ , and since  $i_F$  is injective,  $i_F^*$  has dense range in  $F^*$ . This shows that  $I_{E,F}$  uniquely extends to an isometry of  $H_E$  onto  $H_F$ . From  $I_{E,F} \circ i_E^* \circ j^* = i_F^*$  it follows, moreover, that  $j \circ i_E \circ I_{E,F}^* = i_F$  and therefore,

$$j(i_E(H_E)) = (j \circ i_E \circ I_{E,F}^* \circ I_{E,F}^*)(H_E) = (i_F \circ I_{E,F})(H_E) = i_F(H_F).$$

This shows that  $H_E$  and  $H_F$  are identical as subsets of  $F$  and we obtain the commuting diagram

$$\begin{array}{ccc} E & \xrightarrow{j} & F \\ \uparrow i_E & & \uparrow i_F \\ H_E & \xrightarrow[\simeq]{} & H_F \end{array}$$

The following observation will be useful:

**Proposition 2.2.** *Let  $R \in \mathcal{L}(E^*, E^{**})$  be a positive symmetric operator. Suppose there exists a positive symmetric operator  $Q \in \mathcal{L}(E^*, E)$  and a constant  $C \geq 0$  such that*

$$\langle x^*, Rx^* \rangle \leq C \langle Qx^*, x^* \rangle, \quad x^* \in E^*.$$

Then  $R \in \mathcal{L}(E^*, E)$ .

*Proof.* Fix  $x^* \in E^*$ . On the range of  $i_Q^*$  we define a linear form  $\phi_{x^*}$  by

$$\phi_{x^*}(i_Q^* y^*) := \langle y^*, Rx^* \rangle, \quad y^* \in E^*.$$

By the Cauchy-Schwarz inequality applied to the symmetric bilinear form  $(x^*, y^*) \mapsto \langle y^*, Rx^* \rangle$ ,

$$\begin{aligned} |\phi_{x^*}(i_Q^* y^*)| &= |\langle y^*, Rx^* \rangle| \leq \langle x^*, Rx^* \rangle^{\frac{1}{2}} \langle y^*, Ry^* \rangle^{\frac{1}{2}} \\ &\leq C \langle Qx^*, x^* \rangle^{\frac{1}{2}} \langle Qy^*, y^* \rangle^{\frac{1}{2}} = \|i_Q x^*\|_{H_Q} \|i_Q^* y^*\|_{H_Q}. \end{aligned}$$

It follows that  $\phi_{x^*}$  is well defined and extends to a bounded linear form on  $H_Q$  of norm  $\leq C \|i_Q^* x^*\|_{H_Q}$ . By the Riesz representation theorem, we may identify  $\phi_{x^*}$  with an element of  $H_Q$ . For all  $y^* \in E^*$  we then have

$$\langle i_Q \phi_{x^*}, y^* \rangle = [\phi_{x^*}, i_Q^* y^*]_{H_Q} = \phi_{x^*}(i_Q^* y^*) = \langle y^*, Rx^* \rangle.$$

This shows that  $Rx^* = i_Q \phi_{x^*} \in E$ . ■

**2.2. The general setting.** We consider a  $C_0$ -semigroup  $\mathbf{S} = \{S(t)\}_{t \geq 0}$  of bounded linear operators on  $E$  and a real Hilbert space  $H$  which is continuously embedded into  $E$ . The embedding will be denoted by  $i : H \hookrightarrow E$ . The inner product of  $H$  will be denoted by  $[\cdot, \cdot]_H$ . The operator  $Q := i \circ i^* \in \mathcal{L}(E^*, E)$  is positive and symmetric, and  $H$  is its RKHS.

By [29, Proposition 1.2], the  $E$ -valued function  $s \mapsto S(s)QS^*(s)x^*$  is strongly measurable and we may define, for each  $t > 0$ , the positive symmetric operator  $Q_t \in \mathcal{L}(E^*, E)$  by

$$(2.2) \quad Q_t x^* := \int_0^t S(s)QS^*(s)x^* ds, \quad x^* \in E^*.$$

The RKHS associated with  $Q_t$  will be denoted by  $H_t$  and the embedding  $H_t \hookrightarrow E$  by  $i_t$ . From (2.1) it is immediate that  $H_s \subseteq H_t$  whenever  $s \leq t$  and the inclusion mapping is contractive [29, Corollary 1.5]. Whenever it is convenient we further put  $Q_0 := 0$  and  $H_0 = \{0\}$ .

We will frequently consider the following hypothesis:

- **( $\mathbf{H}Q_\infty$ )**: For all  $x^* \in E^*$ ,  $\text{weak-}\lim_{t \rightarrow \infty} Q_t x^*$  exists in  $E$ .

Here, ‘ $\text{weak-}\lim$ ’ denotes the limit in the weak topology of  $E$ . This hypothesis is slightly more general than the one in [29, Section 6] where strong limits are taken, but the results proved there remain true under **( $\mathbf{H}Q_\infty$ )** without change in the proofs.

Assuming **( $\mathbf{H}Q_\infty$ )**, we may define a bounded operator  $Q_\infty : E^* \rightarrow E$  by

$$Q_\infty x^* := \text{weak-}\lim_{t \rightarrow \infty} Q_t x^*, \quad x^* \in E^*.$$

Clearly,  $Q_\infty$  is positive and symmetric. The RKHS associated with  $Q_\infty$  will be denoted by  $H_\infty$  and the embedding  $H_\infty \hookrightarrow E$  by  $i_\infty$ . From (2.1) it is immediate that  $H_t \subseteq H_\infty$  for all  $t > 0$ ; by an obvious modification of [29, Corollary 1.5] the inclusion mapping is contractive.

Necessary and sufficient conditions for **( $\mathbf{H}Q_\infty$ )** to be satisfied will be given in Section 4. Hypothesis **( $\mathbf{H}Q_\infty$ )** is trivially satisfied if  $\mathbf{S}$  is *uniformly exponentially stable*, i.e. if there exist constants  $M \geq 0$  and  $\omega > 0$  such that  $\|S(t)\| \leq Me^{-\omega t}$  for all  $t \geq 0$ . In this case we have

$$(2.3) \quad Q_\infty x^* = \int_0^\infty S(s)QS^*(s)x^* ds, \quad x^* \in E^*,$$

the integral being convergent as a Bochner integral in  $E$ .

Even in the case when  $E$  is separable, we do not know whether the integral in (2.3) always exists as a Bochner integral. We shall prove next that the integral always does exist as a Pettis integral. For more information on Pettis integrals we refer the reader to [16].

**Proposition 2.3** (**( $\mathbf{H}Q_\infty$ )**). *For all  $x^* \in E^*$  we have*

$$Q_\infty x^* = \int_0^\infty S(s)QS^*(s)x^* ds,$$

*the integral being convergent as a Pettis integral in  $E$ .*

*Proof.* Let  $x^* \in E^*$  be fixed. First we prove the following claim: for all  $y^* \in E^*$  the real-valued function  $s \mapsto \langle S(s)QS^*(s)x^*, y^* \rangle$  is Lebesgue integrable on  $[0, \infty)$  and

$$(2.4) \quad \langle Q_\infty x^*, y^* \rangle = \int_0^\infty \langle S(s)QS^*(s)x^*, y^* \rangle ds.$$

First we take  $y^* = x^*$ . We have

$$\langle S(s)QS^*(s)x^*, x^* \rangle = \langle QS^*(s)x^*, S^*(s)x^* \rangle \geq 0, \quad s \geq 0.$$

Hence by monotone convergence,

$$\begin{aligned} \int_0^\infty |\langle S(s)QS^*(s)x^*, x^* \rangle| ds &= \int_0^\infty \langle S(s)QS^*(s)x^*, x^* \rangle ds \\ &= \lim_{t \rightarrow \infty} \int_0^t \langle S(s)QS^*(s)x^*, x^* \rangle ds = \lim_{t \rightarrow \infty} \langle Q_t x^*, x^* \rangle = \langle Q_\infty x^*, x^* \rangle. \end{aligned}$$

This proves the claim for  $y^* = x^*$ .

Next let  $y^* \in E^*$  be arbitrary. For all  $t > 0$  we have

$$\begin{aligned} \int_0^t |\langle S(s)QS^*(s)x^*, y^* \rangle| ds &= \int_0^t |[i^* S^*(s)x^*, i^* S^*(s)y^*]_H| ds \\ &\leq \left( \int_0^t \|i^* S^*(s)x^*\|_H^2 ds \right)^{\frac{1}{2}} \cdot \left( \int_0^t \|i^* S^*(s)y^*\|_H^2 ds \right)^{\frac{1}{2}} \\ &= \left( \int_0^t \langle QS^*(s)x^*, S^*(s)x^* \rangle ds \right)^{\frac{1}{2}} \cdot \left( \int_0^t \langle QS^*(s)y^*, S^*(s)y^* \rangle ds \right)^{\frac{1}{2}} \\ &\leq \left( \int_0^\infty \langle QS^*(s)x^*, S^*(s)x^* \rangle ds \right)^{\frac{1}{2}} \cdot \left( \int_0^\infty \langle QS^*(s)y^*, S^*(s)y^* \rangle ds \right)^{\frac{1}{2}} \\ &= \langle Q_\infty x^*, x^* \rangle^{\frac{1}{2}} \cdot \langle Q_\infty y^*, y^* \rangle^{\frac{1}{2}}. \end{aligned}$$

Passing to the limit  $t \rightarrow \infty$ , we obtain

$$\int_0^\infty |\langle S(s)QS^*(s)x^*, y^* \rangle| ds \leq \langle Q_\infty x^*, x^* \rangle^{\frac{1}{2}} \cdot \langle Q_\infty y^*, y^* \rangle^{\frac{1}{2}}.$$

It follows that  $s \mapsto \langle S(s)QS^*(s)x^*, y^* \rangle$  is Lebesgue integrable in  $[0, \infty)$ . The identity (2.4) now follows from the dominated convergence theorem. This concludes the proof of the claim.

In order to prove that  $t \mapsto S(t)QS^*(t)x^*$  is Pettis integrable, we have to show next that for all measurable subsets  $B \subseteq [0, \infty)$  there exists an element  $x_{B,x^*} \in E$  such that

$$\langle x_{B,x^*}, y^* \rangle = \int_B \langle S(t)QS^*(t)x^*, y^* \rangle dt, \quad y^* \in E^*.$$

To this end, define the positive symmetric operator  $Q_B \in \mathcal{L}(E^*, E^{**})$  by

$$\langle y^*, Q_B x^* \rangle := \int_B \langle S(t)QS^*(t)x^*, y^* \rangle dt. \quad x^*, y^* \in E^*.$$

Clearly, for all  $x^* \in E^*$  we have  $\langle Q_B x^*, x^* \rangle \leq \langle Q_\infty x^*, x^* \rangle$ , and therefore  $Q_B \in \mathcal{L}(E^*, E)$  by Proposition 2.2. Then  $x_{B,x^*} := Q_B x^*$  does the job.  $\blacksquare$

The space  $H_\infty$  displays some remarkable properties, some of which we shall discuss next.

**Proposition 2.4** ( $\mathbf{H}Q_\infty$ ). *The space  $H_\infty$  is invariant under the action of  $\mathbf{S}$ , and the restriction  $\mathbf{S}_\infty$  of  $\mathbf{S}$  to  $H_\infty$  defines a strongly continuous contraction semigroup on  $H_\infty$ . Its adjoint  $\mathbf{S}_\infty^*$  is strongly stable, i.e. for all  $h_\infty \in H_\infty$  we have*

$$\lim_{t \rightarrow \infty} \|S_\infty^*(t)h_\infty\|_{H_\infty} = 0.$$

*Proof.* The first assertion is proved in [8] (for Hilbert spaces  $E$ ) and [29].

Noting that  $S(t) \circ i_\infty = i_\infty \circ S_\infty(t)$ , for all  $x^* \in E^*$  we have

$$\begin{aligned} \lim_{t \rightarrow \infty} \|S_\infty^*(t)i_\infty^*x^*\|_{H_\infty}^2 &= \lim_{t \rightarrow \infty} \|i_\infty^*S^*(t)x^*\|_{H_\infty}^2 = \lim_{t \rightarrow \infty} \langle Q_\infty S^*(t)x^*, S^*(t)x^* \rangle \\ &= \lim_{t \rightarrow \infty} \int_0^\infty \langle S(s)QS^*(s)S^*(t)x^*, S^*(t)x^* \rangle ds \\ &= \lim_{t \rightarrow \infty} \int_t^\infty \langle S(\sigma)QS^*(\sigma)x^*, x^* \rangle d\sigma = 0. \end{aligned}$$

Since the range of  $i_\infty^*$  is dense in  $H_\infty$  and  $\mathbf{S}_\infty$  is a contraction semigroup on  $H_\infty$ , the strong stability of  $\mathbf{S}_\infty^*$  follows from this.  $\blacksquare$

For later reference we recall from [8] and [29]:

**Proposition 2.5** ( $\mathbf{H}Q_\infty$ ). *For  $t > 0$  fixed, the following assertions are equivalent:*

- (1)  $H_t = H_\infty$  with equivalent norms;
- (2)  $\|S_\infty(t)\|_{H_\infty} < 1$ .

The following result gives a relation between  $H_\infty$  and  $H$ :

**Proposition 2.6** ( $\mathbf{H}Q_\infty$ ). *We have  $\overline{H} \subseteq \overline{H_\infty}$ , the closures being taken in  $E$ .*

*Proof.* Suppose  $y^* \in E^*$  is such that  $\langle h_\infty, y^* \rangle = 0$  for all  $h_\infty \in H_\infty$ ; we have to prove that  $\langle h, y^* \rangle = 0$  for all  $h \in H$ .

First note that from  $H_t \subseteq H_\infty$  it follows that  $\langle Q_t x^*, y^* \rangle = 0$  for all  $t > 0$  and  $x^* \in E^*$ . Now fix  $x^\circ \in E^\circ$ , where  $E^\circ$  denotes the closed linear subspace of  $E^*$  of all elements whose orbit under the adjoint semigroup  $\mathbf{S}^*$  is strongly continuous. Then for all  $t > 0$  we have

$$\int_0^t \langle S(s)QS^*(s)x^\circ, y^* \rangle ds = \langle Q_t x^\circ, y^* \rangle = 0,$$

and since the integrand is a continuous function, this implies that

$$\langle S(s)QS^*(s)x^\circ, y^* \rangle = 0, \quad s \geq 0.$$

In particular,  $\langle Qx^\circ, y^* \rangle = 0$ .

Since  $Q$  is symmetric, it follows that  $\langle Qy^*, x^\circ \rangle = 0$  for all  $x^\circ \in E^\circ$ , and  $E^\circ$  being weak\*-dense in  $E^*$  this implies that  $Qy^* = 0$ . Then  $\langle Qx^*, y^* \rangle = \langle Qy^*, x^* \rangle = 0$  for all  $x^* \in E^*$ , and since the range of  $Q$  is dense in  $H$  it follows that  $\langle h, y^* \rangle = 0$  for all  $h \in H$ .  $\blacksquare$

It need not be the case that  $H \subseteq H_\infty$ . In fact, as we will show in Section 5 it often happens that  $H_\infty \subseteq H$  (in which case of course  $\overline{H} = \overline{H_\infty}$ ).

3. INVARIANCE OF THE REPRODUCING KERNEL HILBERT SPACE  $H$ 

In many important examples,  $H$  is invariant under the action of  $\mathbf{S}$  and  $\mathbf{S}$  restricts to a  $C_0$ -semigroup on  $H$ . For example, we will show that this happens if the Ornstein-Uhlenbeck semigroup  $\mathbf{P}$  in  $L^2(E, \mu_\infty)$  is selfadjoint (Section 4) or analytic with a spectral gap (Section 9). A further example is when  $E$  is a Hilbert space and  $S(t)Q = QS(t)$  holds for all  $t \geq 0$ ; see [9].

In this section we will investigate the situation where  $\mathbf{S}$  restricts to a  $C_0$ -semigroup on  $H$  in some detail. It will turn out that the restricted semigroup enjoys some interesting regularizing properties. These will be used to study the strong Feller property of Ornstein-Uhlenbeck semigroups.

We begin with a simple criterion for invariance. If  $T \in \mathcal{L}(E)$  is a bounded operator satisfying  $T(H) \subseteq H$ , then we denote the restriction of  $T$  to  $H$  by  $T_H$ ; by the closed graph theorem,  $T_H$  is a bounded operator on  $H$ . Note that  $T \circ i = i \circ T_H$ .

**Proposition 3.1.** *For a bounded operator  $T \in \mathcal{L}(E)$  the following assertions are equivalent:*

- (1)  $T(H) \subseteq H$ ;
- (2) There exists a constant  $M \geq 0$  such that for all  $x^* \in E^*$  we have

$$\|i^* T^* x^*\|_H \leq M \|i^* x^*\|_H.$$

- (3) There exists a constant  $M \geq 0$  such that for all  $x^*, y^* \in E^*$  we have

$$|\langle TQx^*, y^* \rangle| \leq M \|i^* x^*\|_H \|i^* y^*\|_H.$$

In this situation the restriction  $T_H$  is bounded on  $H$  and satisfies  $\|T_H\|_H \leq M$ , where  $M$  is either one of the constants in (2) or (3).

*Proof.* (1)  $\Rightarrow$  (2): From  $T \circ i = i \circ T_H$  we have, for all  $x^* \in E^*$ ,

$$\|i^* T^* x^*\|_H = \|T_H^*(i^* x^*)\|_H \leq \|T_H\|_H \|i^* x^*\|_H.$$

This gives (2), with  $M = \|T_H\|_H$ .

(2)  $\Rightarrow$  (3): From  $Q = i \circ i^*$  we then have, for all  $x^*, y^* \in E^*$ ,

$$|\langle TQx^*, y^* \rangle| = |\langle i(i^* x^*), T^* y^* \rangle| = |[i^* x^*, i^* T^* y^*]_H| \leq \|T_H\|_H \|i^* x^*\|_H \|i^* y^*\|_H.$$

This gives (3), with the same constant  $M$ .

(3)  $\Rightarrow$  (1): By assumption, the mapping  $\phi : i^* y^* \mapsto \langle TQx^*, y^* \rangle$  is well defined and uniquely extends to a bounded linear functional  $\phi$  on  $H$  of norm  $\leq M \|i^* x^*\|_H$ . By the Riesz representation theorem we identify  $\phi$  with an element  $h \in H$  of norm  $\leq M \|i^* x^*\|_H$ . Then for all  $y^* \in E^*$  we have

$$\langle ih, y^* \rangle = [i^* y^*, h]_H = \phi(i^* y^*) = \langle TQx^*, y^* \rangle,$$

and therefore  $TQx^* = ih$ . Defining  $T_H(i^* x^*) := h$ , we have  $\|T_H(i^* x^*)\| = \|h\|_H \leq M \|i^* x^*\|_H$ . Hence we obtain a well defined bounded operator  $T_H$  on  $H$  of norm  $\leq M$ . Finally, for all  $x^*, y^* \in E^*$  we have

$$\langle (i \circ T_H)(i^* x^*), y^* \rangle = \langle ih, y^* \rangle = \langle TQx^*, y^* \rangle = \langle (T \circ i)(i^* x^*), y^* \rangle,$$

which shows that  $i \circ T_H = T \circ i$ . ■

The implication (2)  $\Rightarrow$  (1) admits the following, even shorter, direct proof. By assumption of (2), the mapping  $S_H : i^*x^* \mapsto i^*T^*x^*$  is well defined and extends uniquely to a bounded operator  $S_H$  on  $H$  of norm  $\leq M$ . From  $S_H \circ i^* = i^* \circ T^*$  we obtain, by dualizing,  $i \circ S_H^* = T \circ i$ . Hence  $T$  maps  $H$  into itself and the restriction of  $T$  to  $H$  equals  $S_H^*$ . Next we address the question of strong continuity of  $\mathbf{S}_H$ .

**Proposition 3.2.** *Assume that  $S(t)H \subseteq H$  for all  $t \geq 0$ . Then the semigroup  $\mathbf{S}_H$  is strongly continuous on  $[0, \infty)$  if and only if*

$$\limsup_{t \downarrow 0} \|S_H(t)\|_H < \infty.$$

*Proof.* For all  $x^*, y^* \in E^*$  we have

$$\begin{aligned} \lim_{t \downarrow 0} [S_H(t)i^*x^* - i^*x^*, i^*y^*]_H &= \lim_{t \downarrow 0} [i^*x^*, i^*S^*(t)y^* - i^*y^*]_H = 0 \\ &= \lim_{t \downarrow 0} \langle Qx^*, S^*(t)y^* - y^* \rangle = 0. \end{aligned}$$

By assumption,  $\mathbf{S}_H$  is locally bounded near 0, and it follows that  $\mathbf{S}_H$  is weakly continuous. By a standard result from semigroup theory, this implies that  $\mathbf{S}_H$  is strongly continuous.  $\blacksquare$

It may happen that  $\mathbf{S}_H$  fails to be strongly continuous at 0, even if  $E$  is a Hilbert space:

*Example 3.3.* For  $n = 1, 2, \dots$  we define the Hilbert space  $H_n$  to be  $L^2[0, 1]$  with the norm

$$\|f\|_{H_n}^2 := \int_0^{1-\frac{1}{n}} |f(t)|^2 dt + n^2 \int_{1-\frac{1}{n}}^{1-\frac{1}{2n}} |f(t)|^2 dt + \int_{1-\frac{1}{2n}}^1 |f(t)|^2 dt.$$

The nilpotent left shift semigroup  $\mathbf{S}_{H_n}$  on  $H_n$ ,

$$S_{H_n}(t)f(s) = \begin{cases} f(s+t), & 0 \leq s+t \leq 1 \\ 0, & \text{else} \end{cases}$$

is strongly continuous and we have

$$\|S_{H_n}(t)f(s)\|_{H_n} = \begin{cases} n, & t \in [0, \frac{1}{n}) \\ 1, & t \in [\frac{1}{n}, 1) \\ 0, & \text{else.} \end{cases}$$

Let  $E_n = L^2[0, 1]$  with the usual norm and let  $\mathbf{S}_{E_n}$  denote the nilpotent left shift semigroup on  $E_n$ . Now consider the Hilbert space direct sums

$$H := \bigoplus_{n=1}^{\infty} H_n, \quad E := \bigoplus_{n=1}^{\infty} E_n.$$

Note that  $H \subseteq E$  with a continuous inclusion map, which we denote by  $i$ .

The semigroups  $\mathbf{S}_H := \bigoplus_{n=1}^{\infty} \mathbf{S}_{H_n}$  and  $\mathbf{S}_E := \bigoplus_{n=1}^{\infty} \mathbf{S}_{E_n}$  act in  $H$  and  $E$ , respectively, and  $\mathbf{S}_H$  is the restriction to  $H$  of  $\mathbf{S}_E$ . We have

$$\|S_H(t)f(s)\|_H = \begin{cases} 1, & t = 0 \\ n, & t \in [\frac{1}{n+1}, \frac{1}{n}), \quad n = 1, 2, \dots \\ 0, & \text{else.} \end{cases}$$

Thus,  $\limsup_{t \downarrow 0} \|S_H(t)\|_H = \infty$  and  $\mathbf{S}_H$  fails to be strongly continuous in  $H$  at 0. On the other hand  $\mathbf{S}_E$  is strongly continuous at 0.

The infinitesimal generator of  $\mathbf{S}$  will be denoted by  $A$ . The next result gives necessary and sufficient conditions for  $\mathbf{S}_H$  to be contractive:

**Theorem 3.4.** *The following assertions are equivalent:*

- (1) For all  $t \geq 0$  we have  $S(t)H \subseteq H$ , and  $\mathbf{S}_H$  is a  $C_0$ -semigroup of contractions on  $H$ ;
- (2) For all  $x^* \in E^*$ , the function  $t \mapsto \|i^* S^*(t)x^*\|_H$  is nonincreasing on  $[0, \infty)$ ;
- (3) For all  $x^* \in \mathcal{D}(A^*)$ , the domain of  $A^*$ , we have  $-\langle Qx^*, A^*x^* \rangle \geq 0$ .

*Proof.* (1)  $\Rightarrow$  (3): Fix  $x^* \in \mathcal{D}(A^*)$ . Then for all  $h \in H$ ,

$$\lim_{t \downarrow 0} \frac{1}{t} [S_H(t)i^*x^* - i^*x^*, h]_H = \lim_{t \downarrow 0} \frac{1}{t} \langle ih, S^*(t)x^* - x^* \rangle = \langle ih, A^*x^* \rangle = [i^*A^*x^*, h]_H.$$

Let  $A_H$  denote the infinitesimal generator of  $\mathbf{S}_H$ . By a standard result from semigroup theory [32, Theorem 2.1.3], the above identities imply that  $i^*x^* \in \mathcal{D}(A_H^*)$  and  $A_H^*(i^*x^*) = i^*A^*x^*$ . The fact that  $A_H$  generates a contraction semigroup on  $H$  then gives, using [32, Theorem 1.4.3],

$$-\langle Qx^*, A^*x^* \rangle = -[i^*x^*, i^*A^*x^*]_H = -[i^*x^*, A_H^*(i^*x^*)]_H \geq 0.$$

(3)  $\Rightarrow$  (2): First consider a fixed  $x^* \in \mathcal{D}(A^*)$ . It is easy to see that the function  $u \mapsto \|i^* S^*(u)x^*\|_H^2 = \langle QS^*(u)x^*, S^*(u)x^* \rangle$  is differentiable and

$$\begin{aligned} \frac{d}{du} \|i^* S^*(u)x^*\|_H^2 &= \langle QS^*(u)A^*x^*, S^*(u)x^* \rangle + \langle QS^*(u)x^*, S^*(u)A^*x^* \rangle \\ &= 2\langle QS^*(u)x^*, A^*S^*(u)x^* \rangle, \end{aligned}$$

where we used the symmetry of  $Q$ . Hence for all  $t \geq s \geq 0$  we have

$$\begin{aligned} \|i^* S^*(s)x^*\|_H^2 - \|i^* S^*(t)x^*\|_H^2 &= - \int_s^t \frac{d}{du} \|i^* S^*(u)x^*\|_H^2 du \\ &= -2 \int_s^t \langle QS^*(u)x^*, A^*S^*(u)x^* \rangle du \geq 0. \end{aligned}$$

For general  $x^* \in E^*$ , the result follows by approximation: noting that

$$\|i^* S^*(\tau)y^*\|_H = \lim_{\lambda \rightarrow \infty} \lambda \|(\lambda - A_H^*)^{-1} i^* S^*(\tau)y^*\|_H = \lim_{\lambda \rightarrow \infty} \lambda \|i^* S^*(\tau)(\lambda - A^*)^{-1} y^*\|_H,$$

we can apply the above to  $x^* = (\lambda - A^*)^{-1} y^* \in \mathcal{D}(A^*)$ .

(2)  $\Rightarrow$  (1): Since  $t \mapsto \|i^* S^*(t)x^*\|_H$  is nonincreasing on  $[0, \infty)$ , for all  $t \in [0, \infty)$  and all  $x^* \in E^*$  we have  $\|i^* S^*(t)x^*\|_H \leq \|i^* x^*\|_H$ . Then from Proposition 3.1 it follows that the operators  $S(t)$  restrict to contractions on  $H$ . The strong continuity of  $\mathbf{S}_H$  follows from Proposition 3.2.  $\blacksquare$

Recalling that  $i : H \hookrightarrow E$  denotes the inclusion map, we define the positive symmetric operator  $Q \in \mathcal{L}(E^*, E)$  by

$$Q := i \circ i^*.$$

Using this operator, we define the positive symmetric operators  $Q_t \in \mathcal{L}(E, E^*)$  by (2.2). As before we let  $H_t$  be the RKHS associated with  $Q_t$ , and  $i_t : H_t \hookrightarrow E$  is the natural inclusion mapping.

**Theorem 3.5.** *Assume that  $\mathbf{S}$  restricts to a  $C_0$ -semigroup  $\mathbf{S}_H$  on  $H$ .*

- (1) *For all  $t, s > 0$  we have  $H_t = H_s$  with equivalent norms;*
- (2) *For all  $t > 0$  we have  $H_t \subseteq H$  with dense inclusion;*
- (3) *For all  $t > 0$  we have  $S(t)H \subseteq H_t$  and*

$$\limsup_{t \downarrow 0} \sqrt{t} \|S(t)\|_{\mathcal{L}(H, H_t)} \leq \limsup_{t \downarrow 0} \|S_H(\cdot)\|_H.$$

*Proof.* We start with some general observations. For  $t > 0$  define the positive selfadjoint operator  $R_t \in \mathcal{L}(H)$  by

$$R_t h := \int_0^t S_H(s) S_H^*(s) h \, ds, \quad h \in H.$$

Let  $G_t$  denote the RKHS associated with  $R_t$  and let  $j_t : G_t \hookrightarrow H$  denote the inclusion mapping. By [29, Theorem 1.11], as subsets of  $E$  we have  $H_t = G_t$ . It follows that  $H_t \subseteq H$ .

Denoting by  $i_t : H_t \hookrightarrow E$  the inclusion mapping, the map

$$i_t^* x^* \mapsto j_t^*(i_t^* x^*), \quad x^* \in E^*,$$

establishes an isometrical isomorphism of  $H_t$  and  $G_t$ . Thanks to this observation, in the rest of the proof we may identify  $H_t$  with  $G_t$ .

For each  $h \in H$ , the function  $f_h(s) := S_H(s)h$  belongs to  $L^2([0, t]; H)$ , and hence

$$(3.1) \quad t \cdot S(t)h = \int_0^t S(t)h \, ds = \int_0^t S(t-s)f_h(s) \, ds.$$

Hence by [15, Appendix B],  $S(t)h \in G_t$ . This shows that  $S(t)H \subseteq G_t = H_t$ .

Let

$$Y_t = \{S_H(s)h : 0 < s \leq t, h \in H\}.$$

By the strong continuity of  $\mathbf{S}_H$ , the set  $Y$  is dense in  $H$ . On the other hand, for all  $0 < s \leq t$  we have  $H_s \subseteq H_t$  and therefore

$$S_H(s)h \in H_s \subseteq H_t.$$

It follows that  $Y \subseteq H_t \subseteq H$ , and  $H_t$  is dense in  $H$ . This proves (2).

Fix  $t_0 > 0$ . From (2) we have

$$S(t_0)H_{t_0} \subseteq S(t_0)H \subseteq H_{t_0}.$$

Therefore by [29, Theorem 1.9], for all  $t \geq t_0$  we have  $H_t = H_{t_0}$  with equivalent norms. Since  $t_0 > 0$  is arbitrary, this gives (1).

Fix  $h \in H$  and  $t_0 > 0$ . Using the language of control theory of [15, Appendix B], (3.1) shows that the function  $u(s) := t_0^{-1} S_H(s)h$  is a control for reaching  $S(t_0)h$  at time  $t_0$ . The the minimum energy for a control to reach  $S(t_0)h$  being equal  $\|S(t_0)h\|_{H_{t_0}}$ , it follows that

$$\|S(t_0)h\|_{H_{t_0}}^2 \leq \|u\|_{L^2([0, t_0]; H)}^2 = \frac{1}{t_0^2} \int_0^{t_0} \|S_H(s)h\|_H^2 \, ds.$$

Therefore,

$$\limsup_{t \downarrow 0} \sqrt{t} \|S(t)\|_{\mathcal{L}(H, H_t)} \leq \limsup_{t \downarrow 0} \|S_H(\cdot)\|_H.$$

This gives (3). ■

In case  $E$  is a Hilbert space and  $Q = I$ , these estimates are well known; cf. [15].

Assertion (3) admits a control theoretic interpretation. In order to explain this, we need to introduce some terminology.

Let  $\mathcal{H}$  be a real Hilbert space, let  $B : \mathcal{H} \rightarrow E$  a bounded linear operator, and let  $t_0 > 0$  be given. We say that the pair  $(\mathbf{S}, B)$  is *null controllable in time  $t_0$*  if for every  $x \in E$  there exists a function  $f \in L^2((0, t_0); \mathcal{H})$  such that the unique mild solution  $u$  of the equation

$$\begin{aligned} u'(t) &= Au(t) + Bf(t), & t \geq 0, \\ u(0) &= x \end{aligned}$$

satisfies

$$u(t_0) = 0.$$

It is well known that the pair  $(\mathbf{S}, B)$  is null controllable in time  $t_0$  if and only if  $S(t_0)$  maps  $E$  into  $H_{R_{t_0}}$ , the RKHS associated with the positive symmetric operator  $R_{t_0} \in \mathcal{L}(E^*, E)$  defined by

$$(3.2) \quad R_{t_0}x^* := \int_0^{t_0} S(s)BB^*S^*(s)x^* ds, \quad x^* \in E^*,$$

cf. [15, 30].

**Theorem 3.6.** *In the above situation, let there exist  $\delta > 0$  and a constant  $M \geq 0$  such that for all  $t \in [0, \delta]$  and all  $x^* \in E^*$  we have*

$$(3.3) \quad \|B^*S^*(t)x^*\|_{\mathcal{H}} \leq M\|B^*x^*\|_{\mathcal{H}}.$$

*Then the following assertions are equivalent:*

- (1) *The pair  $(\mathbf{S}, B)$  is null controllable for all  $t > 0$ ;*
- (2)  *$S(t)E \subseteq \text{range } B$  for all  $t > 0$ .*

*Proof.* Let  $R := B \circ B^*$  and let  $i_R : H_R \hookrightarrow E$  denote the RKHS associated with  $R$ . As outlined in Example 2.1 (1), we may identify  $H_R$  with the range of  $B$ . By Proposition 3.1, the estimate (3.3) implies that  $H_R$  is  $S(t)$ -invariant for all  $t \in [0, \delta]$ , and that the restricted operators are uniformly bounded on  $H_R$ . Then Proposition 3.2 implies that  $\mathbf{S}$  restricts to a  $C_0$ -semigroup on  $H_R$ , and we are in a position to apply Theorem 3.5.

(1)  $\Rightarrow$  (2): If  $(\mathbf{S}, B)$  is null controllable at time  $t$ , then  $S(t)(E) \subseteq H_{R_t} \subseteq H_R$ , where the second inclusion follows from Theorem 3.5 (2).

(2)  $\Rightarrow$  (1): For all  $t > 0$  we have  $S(t)x = S(\frac{t}{2})(S(\frac{t}{2})x) \in S(\frac{t}{2})H_R \subseteq H_{R_{\frac{t}{2}}} = H_{R_t}$ , where we used Theorem 3.5 (1), (3). ■

In the previous section we showed that Hypothesis  $(\mathbf{H}Q_\infty)$  implies the inclusion  $\overline{H} \subseteq \overline{H_\infty}$ . If  $H$  is  $\mathbf{S}$ -invariant we can prove more:

**Proposition 3.7**  $(\mathbf{H}Q_\infty)$ . *If  $\mathbf{S}$  restricts to a  $C_0$ -semigroup on  $H$ , then  $H \cap H_\infty$  is dense in both  $H$  and  $H_\infty$ . In particular we have  $\overline{H} = \overline{H_\infty}$ .*

*Proof.* By Theorem 3.5, for all  $h \in H$  we have  $S_H(t)h \in H_t \subseteq H \cap H_\infty$ . Hence from  $\lim_{t \downarrow 0} S_H(t)h = h$  strongly in  $H$  it follows that  $H \cap H_\infty$  is dense in  $H$ .

For all  $t > 0$  and  $x^* \in E^*$  we have  $i_t^* x^* \in H_t \subseteq H \cap H_\infty$ . We claim that  $\lim_{t \rightarrow \infty} i_t^* x^* = i_\infty^* x^*$  weakly in  $H_\infty$ . Since the range of  $i_\infty^*$  is dense in  $H_\infty$ , this will show that  $H \cap H_\infty$  is weakly dense in  $H_\infty$ , and therefore dense in  $H_\infty$ .

To prove the claim we first recall from Section 2 that the inclusion mapping  $H_t \hookrightarrow H_\infty$  is contractive. Therefore,

$$(3.4) \quad \|i_t^* x^*\|_{H_\infty}^2 \leq \|i_t^* x^*\|_{H_t}^2 = \langle Q_t x^*, x^* \rangle \leq \langle Q_\infty x^*, x^* \rangle = \|i_\infty^* x^*\|_{H_\infty}^2.$$

Moreover, for all  $y^* \in E^*$  we have

$$\begin{aligned} \lim_{t \rightarrow \infty} [i_t^* x^*, i_\infty^* y^*]_{H_\infty} &= \lim_{t \rightarrow \infty} \langle i_\infty i_t^* x^*, y^* \rangle = \lim_{t \rightarrow \infty} \langle i_t i_t^* x^*, y^* \rangle \\ &= \lim_{t \rightarrow \infty} \langle Q_t x^*, y^* \rangle = \langle Q_\infty x^*, y^* \rangle = [i_\infty^* x^*, i_\infty^* y^*]_{H_\infty}. \end{aligned}$$

Using once more the density of the range of  $i_\infty^*$ , together with the uniform bound (3.4) this proves the claim.  $\blacksquare$

If  $\mathbf{S}_H$  is a  $C_0$ -semigroup of normal operators, then for individual orbits we have the following version of the estimate in Theorem 3.5 (3):

**Theorem 3.8.** *If  $\mathbf{S}_H$  is a  $C_0$ -semigroup of normal operators on  $H$ , then for all  $t > 0$  and  $h \in H$  we have*

$$\int_0^t \|S_H(s)h\|_{H_t}^2 ds = \|h\|_H^2.$$

Note that the right hand side is independent of the semigroup  $\mathbf{S}$ .

*Proof.* For  $t > 0$  define the positive selfadjoint operators  $R_t, R_{*t} \in \mathcal{L}(H)$  by

$$R_t h := \int_0^t S_H(s) S_H^*(s) h ds, \quad R_{*t} h := \int_0^t S_H^*(s) S_H(s) h ds, \quad h \in H.$$

Let  $G_t$  and  $G_{*t}$  denote the RKHS's associated with  $R_t$  and  $R_{*t}$ , respectively, and let  $j_t : G_t \hookrightarrow H$  and  $j_{*t} : G_{*t} \hookrightarrow H$  denote the inclusion mappings. From

$$\|j_t^* h\|_{G_t}^2 = \int_0^t [S_H(s) S_H^*(s) h, h]_H ds = \int_0^t [S_H^*(s) S_H(s) h, h]_H ds = \|(j_{*t})^* h\|_{G_{*t}}^2$$

it follows that  $G_t$  and  $G_{*t}$  are canonically isometrically isomorphic as Hilbert spaces, and identical as subsets of  $H$ . Moreover, as we saw in the proof of Theorem 3.5,  $H_t$  and  $G_t$  are canonically isometrically isomorphic as Hilbert spaces and we have  $H_t = G_t$  as subsets of  $E$ . It follows that it suffices to prove that

$$\int_0^t \|S_H(s)h\|_{G_{*t}}^2 ds = \|h\|_H^2.$$

From the normality of each operator  $S_H(t)$  it is not difficult to see that

$$(3.5) \quad S_H(t) S_H^*(s) = S_H^*(s) S_H(t), \quad t, s \geq 0.$$

Indeed, by the semigroup property this is true whenever both  $t$  and  $s$  are integer multiples of a common fixed real number. The set of all pairs  $(t, s)$  with this property being dense in  $[0, \infty) \times [0, \infty)$ , the general case follows by strong continuity of  $\mathbf{S}_H$  and its adjoint  $\mathbf{S}_H^*$ .

Using (3.5) we see that

$$R_{*t}S_H(\tau) = S_H(\tau)R_{*t}, \quad R_{*t}S_H^*(\tau) = S_H^*(\tau)R_{*t}, \quad \tau \geq 0,$$

and hence,

$$(3.6) \quad R_{*t}^{\frac{1}{2}}S_H(\tau) = S_H(\tau)R_{*t}^{\frac{1}{2}}, \quad R_{*t}^{\frac{1}{2}}S_H^*(\tau) = S_H^*(\tau)R_{*t}^{\frac{1}{2}}, \quad \tau \geq 0.$$

Define the convolution operator  $A_t$  from  $L^2([0, t]; H)$  into  $H$  by

$$A_t\psi := \int_0^t S_H^*(t-s)\psi(s) ds.$$

By (3.6),

$$R_{*t}^{\frac{1}{2}}A_t = A_tR_{*t}^{\frac{1}{2}}.$$

It is trivially checked that

$$A_t^*h = S_H(t - \cdot)h, \quad h \in H.$$

Hence, the kernel of  $A_t$  is equal to the orthonormal complement in  $L^2([0, t]; H)$  of the closed linear subspace  $V_t$  spanned by all functions of the form  $s \mapsto S_H(t-s)h$ ,  $h \in H$ . Let  $\pi_t$  denote the orthonormal projection in  $L^2([0, t]; H)$  onto  $V_t$ .

Now fix  $h \in H$ . Then by Example 2.1(2),  $R_{*t}^{\frac{1}{2}}h \in G_{*t}$ , the RKHS associated with  $R_{*t}$ . Therefore by [15, Appendix B] there exists a function  $\phi \in L^2([0, t]; H)$  such that

$$R_{*t}^{\frac{1}{2}}h = A_t\phi.$$

Noting that  $A_t\phi = A_t(\pi_t\phi)$ , it follows that

$$(3.7) \quad R_{*t}h = R_{*t}^{\frac{1}{2}}A_t\phi = R_{*t}^{\frac{1}{2}}A_t(\pi_t\phi) = A_t(R_{*t}^{\frac{1}{2}}(\pi_t\phi)).$$

On the other hand,

$$(3.8) \quad R_{*t}h = \int_0^t S_H^*(t-s)S_H(t-s)h ds = A_t(S_H(t - \cdot)h).$$

Observing that  $S_H(t - \cdot)h \in V_t$ , and noting that from  $\pi_t\phi \in V_t$  it follows that also  $R_{*t}^{\frac{1}{2}}(\pi_t\phi) \in V_t$ , from (3.7) and (3.8) we now deduce that

$$S_H(t - \cdot)h = R_{*t}^{\frac{1}{2}}(\pi_t\phi).$$

But clearly,  $R_{*t}^{\frac{1}{2}}(\pi_t\phi) \in L^2([0, t]; G_{*t})$ . It follows that  $S_H(\cdot)h \in L^2([0, t]; G_{*t})$ .

Finally, because  $R_{*t}^{\frac{1}{2}}$  is an isometry from  $H$  onto  $G_{*t}$  and  $A_t$  is an isometry from  $V_t$  onto  $G_{*t}$ ,

$$\begin{aligned} \int_0^t \|S_H(s)h\|_{G_{*t}}^2 ds &= \int_0^t \|S(t-s)h\|_{G_{*t}}^2 ds \\ &= \int_0^t \|R_{*t}^{\frac{1}{2}}(\pi_t\phi(s))\|_{G_{*t}}^2 ds = \int_0^t \|\pi_t\phi(s)\|_H^2 ds \\ &= \|\pi_t\phi\|_{L^2([0, t]; H)}^2 = \|A_t(\pi_t\phi)\|_{G_{*t}}^2 = \|R_{*t}^{\frac{1}{2}}h\|_{G_{*t}}^2 = \|h\|_H^2. \end{aligned}$$

■

*Remark 3.9.* By [26, Theorem 22.4.1], for normal semigroups we always have an estimate

$$(3.9) \quad \|S_H(t)\|_H \leq e^{at}, \quad t \geq 0,$$

for some  $a \in \mathbb{R}$ .

Let us now assume that  $\mathbf{S}_H$  is an analytic semigroup which satisfies (3.9) for some  $a < 0$ . These assumptions imply that  $D_{A_H}(\frac{1}{2}, 2) = \mathcal{D}((-A_H)^{\frac{1}{2}})$ ; we use standard notations as can be found, e.g., in [15]. From this, in turn, it follows that there exists a constant  $C \geq 0$  such that

$$(3.10) \quad \int_0^t \|S_H(s)h\|_{D_{A_H}(\frac{1}{2}, 2)}^2 ds \leq C\|h\|_H, \quad h \in H, \quad t > 0;$$

see [15, Appendix A]. On the other hand, for all  $t > 0$  we have  $H_t = D_{A_H}(\frac{1}{2}, 2)$  with equivalent norms see [15, Appendix B]. Therefore (3.10) implies

$$(3.11) \quad \int_0^t \|S_H(s)h\|_{H_t}^2 ds \leq C_t\|h\|_H, \quad h \in H, \quad t > 0,$$

with a constant  $C_t$  depending on  $t$ . Theorem 3.8 shows that in the normal case one has equality in (3.11) with  $C_t = 1$ .

#### 4. THE LIAPUNOV EQUATION $AX + XA^* = -Q$ AND $Q$ -SYMMETRY

In this section we study the *Liapunov equation*

$$(4.1) \quad AX + XA^* = -Q$$

and apply the results to the case where we have  $S(t) \circ Q = Q \circ S^*(t)$  for all  $t \geq 0$ .

The following result shows that the operator  $Q_\infty$ , if exists, ‘solves’ this equation:

**Proposition 4.1 (H $Q_\infty$ ).** *For all  $x^* \in \mathcal{D}(A^*)$  we have  $Q_\infty x^* \in \mathcal{D}(A)$  and*

$$AQ_\infty x^* + Q_\infty A^* x^* = -Qx^*.$$

*Proof.* Take  $x^*, y^* \in \mathcal{D}(A^*)$ . Differentiating the identity

$$\langle Q_\infty S^*(t)x^*, S^*(t)y^* \rangle = \langle Q_\infty x^*, y^* \rangle - \langle Q_t x^*, y^* \rangle$$

on both sides with respect to  $t$ . Evaluating at  $t = 0$  gives

$$(4.2) \quad \langle Q_\infty x^*, A^* y^* \rangle + \langle Q_\infty A^* x^*, y^* \rangle = -\langle Qx^*, y^* \rangle.$$

It follows that  $Q_\infty x^* \in \mathcal{D}(A)$  and that  $AQ_\infty x^* + QA^* x^* = -Qx^*$ . ■

This result motivates the following definition.

**Definition 4.2.** *A solution of equation (4.1) is a bounded operator  $X \in \mathcal{L}(E^*, E)$  such that for all  $x^* \in \mathcal{D}(A^*)$  we have  $Xx^* \in \mathcal{D}(A)$  and  $AXx^* + XA^* x^* = -Qx^*$ .*

We recall the following observation from [37]; since our setting is slightly different we include a proof.

**Proposition 4.3.** *If  $X$  is a positive symmetric solution of the equation (4.1), then for all  $t > 0$  we have*

$$(4.3) \quad X - S(t)XS^*(t) = Q_t.$$

*Proof.* From (4.1) we have, for  $x^*, y^* \in \mathcal{D}(A^*)$ ,

$$\begin{aligned} \langle Q_t x^*, y^* \rangle &= - \int_0^t \langle S(s)(AX + XA^*)S^*(s)x^*, y^* \rangle ds \\ &= - \int_0^t \frac{d}{ds} \langle S(s)XS^*(s)x^*, y^* \rangle ds = \langle Xx^*, y^* \rangle - \langle S(t)XS^*(t)x^*, y^* \rangle. \end{aligned}$$

Since  $\mathcal{D}(A^*)$  is weak\*-dense in  $E^*$ , it follows that  $Q_t x^* = Xx^* - S(t)XS^*(t)x^*$  for all  $x^* \in \mathcal{D}(A^*)$ . Finally, since both  $Q_t$  and  $X$  are positive and symmetric, and therefore weak\*-to-weakly continuous, it follows from this that  $Q_t x^* = Xx^* - S(t)XS^*(t)x^*$  for all  $x^* \in E^*$ .  $\blacksquare$

After these preparations we can state and prove our main result about the Liapunov equation. Under somewhat more restrictive conditions, this result was proved in [41] for the case when  $E$  is a Hilbert space; see also [15, Theorem 11.7].

**Theorem 4.4.** *The following assertions are equivalent:*

- (1) Equation (4.1) has a positive symmetric solution;
- (2) Hypothesis  $(\mathbf{H}Q_\infty)$  holds.

*If these equivalent conditions are satisfied, the operator  $Q_\infty$  is a positive symmetric solution of (4.1), which is minimal in the sense that if  $R$  is another positive symmetric solution of (4.1), then for all  $x^* \in E^*$  we have*

$$\langle Q_\infty x^*, x^* \rangle \leq \langle R x^*, x^* \rangle.$$

*Proof.* (1)  $\Rightarrow$  (2): Suppose  $R$  is a positive symmetric solution of (4.1). By (4.3), for all  $x^* \in E^*$  we have

$$\langle Q_t x^*, x^* \rangle = \langle R x^*, x^* \rangle - \langle R S^*(t)x^*, S^*(t)x^* \rangle.$$

Since by assumption  $R$  is positive, this implies

$$\langle Q_t x^*, x^* \rangle \leq \langle R x^*, x^* \rangle$$

for all  $x^* \in E^*$ . Hence,  $t \mapsto \langle Q_t x^*, x^* \rangle$  is a bounded function. Since this function is also non-decreasing, it follows that the limit  $\lim_{t \rightarrow \infty} \langle Q_t x^*, x^* \rangle$  exists for all  $x^* \in E^*$  and we have

$$\lim_{t \rightarrow \infty} \langle Q_t x^*, x^* \rangle \leq \langle R x^*, x^* \rangle.$$

By polarization, the limit  $\lim_{t \rightarrow \infty} \langle Q_t x^*, y^* \rangle$  exists for all  $x^*, y^* \in E^*$ . We now define a linear operator  $Q_\infty \in \mathcal{L}(E^*, E^{**})$  by

$$\langle y^*, Q_\infty x^* \rangle := \lim_{t \rightarrow \infty} \langle Q_t x^*, y^* \rangle.$$

By the uniform boundedness theorem,  $Q_\infty$  is bounded. We claim that  $Q_\infty$  actually takes values in  $E$ . Indeed, for all  $x^* \in E^*$  we have

$$\langle x^*, Q_\infty x^* \rangle = \lim_{t \rightarrow \infty} \langle Q_t x^*, x^* \rangle \leq \langle R x^*, x^* \rangle, \quad x^* \in E^*,$$

and the claim follows from Proposition 2.2.

(2)  $\Rightarrow$  (1): This is the content of Proposition 4.1.  $\blacksquare$

The semigroup  $\mathbf{S}$  is said to be *Q-symmetric* if for all  $t \geq 0$  we have

$$S(t)Q = QS^*(t).$$

It is easy to check that the following assertions are equivalent:

- (1)  $\mathbf{S}$  is  $Q$ -symmetric;
- (2) For all  $x^* \in \mathcal{D}(A^*)$  we have  $Qx^* \in \mathcal{D}(A)$  and  $AQx^* = QA^*x^*$ .

If Hypothesis  $(\mathbf{H}Q_\infty)$  holds, then these assertions are equivalent to:

- (3) For all  $t \geq 0$  we have  $S(t)Q_\infty = Q_\infty S^*(t)$ ;
- (4) For all  $x^* \in \mathcal{D}(A^*)$  we have  $Q_\infty x^* \in \mathcal{D}(A)$  and  $AQ_\infty x^* = Q_\infty A^* x^*$ .

It follows from (4) and Liapunov equation that

$$AQ_\infty = Q_\infty A^* = -\frac{1}{2}Q.$$

**Theorem 4.5** ( $\mathbf{H}Q_\infty$ ). *If  $\mathbf{S}$  is  $Q$ -symmetric, then  $\mathbf{S}$  restrict to a selfadjoint and strongly stable  $C_0$ -semigroup of contractions on  $H$ .*

*Proof.* For all  $x^* \in \mathcal{D}(A^*)$  we have

$$-\langle Qx^*, A^*x^* \rangle = 2\langle AQ_\infty x^*, A^*x^* \rangle = 2\langle Q_\infty A^*x^*, A^*x^* \rangle \geq 0$$

since  $Q_\infty$  is a positive operator. Therefore by Theorem 3.4,  $\mathbf{S}$  maps  $H$  into itself and the restricted semigroup  $\mathbf{S}_H$  is a  $C_0$ -semigroup of contractions on  $H$ .

Selfadjointness of  $\mathbf{S}_H$  follows from

$$\begin{aligned} [i^*x^*, S_H^*(t)i^*y^*]_H &= [S_H(t)i^*x^*, i^*y^*]_H = [i^*S^*(t)x^*, i^*y^*]_H \\ &= \langle Qy^*, S^*(t)x^* \rangle = \langle S(t)Qy^*, x^* \rangle \\ &= \langle iS_H(t)i^*y^*, x^* \rangle = [i^*x^*, S_H(t)i^*y^*]_H. \end{aligned}$$

It remains to prove strong stability of  $\mathbf{S}_H$ . Fix  $x^* \in E^*$ . From

$$\int_0^\infty \|S_H^*(t)i^*x^*\|_H^2 dt = \langle Q_\infty x^*, x^* \rangle < \infty$$

and a standard argument it follows that

$$\lim_{t \rightarrow \infty} \|S_H(t)i^*x^*\|_H = \lim_{t \rightarrow \infty} \|S_H^*(t)i^*x^*\|_H = 0.$$

By a density argument, this gives the strong stability of  $\mathbf{S}_H$ . ■

In the following result, which extends a result from [10], we do not assume Hypothesis  $(\mathbf{H}Q_\infty)$ :

**Proposition 4.6.** *The following assertions are equivalent:*

- (1)  $\mathbf{S}$  is  $Q$ -symmetric;
- (2)  $H$  is  $\mathbf{S}$ -invariant and  $\mathbf{S}_H$  is a selfadjoint semigroup on  $H$ ;
- (3)  $H$  is  $\mathbf{S}$ -invariant and  $\mathbf{S}_H$  is a selfadjoint  $C_0$ -semigroup on  $H$ .

*Proof.* (1)  $\Rightarrow$  (3): By rescaling  $\mathbf{S}$  we may assume that  $\mathbf{S}$  is uniformly exponentially stable. Then Hypothesis  $(\mathbf{H}Q_\infty)$  holds, and the assertion follows from Theorem 4.5.

(3)  $\Rightarrow$  (2): Trivial.

(2)  $\Rightarrow$  (1): For all  $x^*, y^* \in E^*$ ,

$$\begin{aligned} \langle S(t)Qx^*, y^* \rangle &= \langle iS_H(t)i^*x^*, y^* \rangle = \langle iS_H^*(t)i^*x^*, y^* \rangle \\ &= \langle i i^* S^*(t)x^*, y^* \rangle = \langle QS^*(t)x^*, y^* \rangle. \end{aligned}$$

■

## 5. SPECTRAL GAP CONDITIONS

In this section we shall prove some results for the semigroup  $\mathbf{S}_\infty$ , which will be applied in Section 7 to obtain a necessary and sufficient condition for the existence of a spectral gap for the generator of the Ornstein-Uhlenbeck semigroup associated with  $\mathbf{S}$  and  $Q$ .

We start with a simple but useful lemma.

**Lemma 5.1** ( $\mathbf{H}Q_\infty$ ). *The set  $i_\infty^*(\mathcal{D}(A^*))$  is a core for  $A_\infty^*$ .*

*Proof.* Since  $\mathcal{D}(A^*)$  is weak\*-dense in  $E^*$  and  $i_\infty^*$  is weak\*-to-weakly continuous, the set  $i_\infty^*(\mathcal{D}(A^*))$  is weakly dense, and hence dense, in  $H_\infty$ .

From  $S_\infty^*(t)i_\infty^* = i_\infty^* \circ S^*(t)$  it follows that  $i_\infty^*(\mathcal{D}(A^*))$  is  $\mathbf{S}_\infty^*$ -invariant, and by another standard result from the theory of  $C_0$ -semigroups [18, Proposition II.1.7] this implies that  $i_\infty^*(\mathcal{D}(A^*))$  is a core for  $A_\infty^*$ . ■

The next result extends a result from [9] to the Banach space setting.

**Lemma 5.2** ( $\mathbf{H}Q_\infty$ ). *Let  $M > 0$  be given. The following statements are equivalent:*

- (1)  $\langle Q_\infty x^*, x^* \rangle \leq M \langle Qx^*, x^* \rangle$  for all  $x^* \in E^*$ ;
- (2)  $\|S_\infty(t)\|_{H_\infty} \leq \exp(-\frac{t}{2M})$  for all  $t \geq 0$ ;

*Proof.* Before we start the proof we note that for all  $x^* \in \mathcal{D}(A^*)$  we have the identity

$$(5.1) \quad \begin{aligned} \langle Qx^*, x^* \rangle &= -2\langle x^*, AQ_\infty x^* \rangle. \\ &= -2[i_\infty^* A^* x^*, i_\infty^* x^*]_{H_\infty} = -2[A_\infty^* i_\infty^* x^*, i_\infty^* x^*]_{H_\infty}. \end{aligned}$$

(1)  $\Rightarrow$  (2): By (5.1),

$$(5.2) \quad \|i_\infty^* x^*\|_{H_\infty}^2 = \langle Q_\infty x^*, x^* \rangle \leq M \langle Qx^*, x^* \rangle = -2M[A_\infty^* i_\infty^* x^*, i_\infty^* x^*]_{H_\infty}.$$

Hence by Lemma 5.1, for all  $h \in \mathcal{D}(A_\infty^*)$  we obtain

$$(5.3) \quad [A_\infty^* h, h]_{H_\infty} \leq -\frac{1}{2M} \|h\|_{H_\infty}^2.$$

By standard results on contraction semigroups in Hilbert spaces, this is equivalent to (2).

(2)  $\Rightarrow$  (1): If (2) holds, then (5.3) holds for all  $h \in \mathcal{D}(A_\infty^*)$ . Taking  $h = i_\infty^* x^*$  with  $x^* \in \mathcal{D}(A^*)$ , it follows that (5.2) holds for all  $x^* \in \mathcal{D}(A^*)$ . It remains to prove that (5.2) holds for all  $x^* \in E^*$ .

Let  $x^* \in E^*$  be arbitrary and fixed, and let  $(x_n^*)$  be a sequence in  $\mathcal{D}(A^*)$  converging to  $x^*$  weak\* in  $E^*$ . Then  $i^* x_n^* \rightarrow i^* x^*$  weakly in  $H$  and  $i_\infty^* x_n^* \rightarrow i_\infty^* x^*$  weakly in  $H_\infty$ . Choose a sequence  $(y_n^*)$  consisting of convex combinations of elements from  $(x_n^*)$  for which  $i^* y_n^* \rightarrow i^* x^*$  strongly in  $H$  and  $i_\infty^* y_n^* \rightarrow i_\infty^* x^*$  strongly in  $H_\infty$ .

Applying (5.2) to  $y_n^*$  and passing to the limit for  $n \rightarrow \infty$  gives

$$\langle Q_\infty x^*, x^* \rangle = \|i_\infty^* x^*\|_{H_\infty}^2 \leq M \|i^* x^*\|_H^2 = M \langle Qx^*, x^* \rangle$$

for all  $x^* \in E^*$ , and we obtain (1). ■

Combined with Proposition 2.5 this implies:

**Corollary 5.3** ( $\mathbf{H}Q_\infty$ ). *If  $H_\infty \subseteq H$ , then for all  $t > 0$  we have  $H_t = H_\infty$  with equivalent norms.*

In case  $H$  is invariant, we can say more. For the  $Q$ -symmetric case in Hilbert spaces, the following result was obtained in [10]. Let  $A_H$  denote the generator of the semigroup  $\mathbf{S}_H$ .

**Theorem 5.4** ( $\mathbf{H}Q_\infty$ ). *If  $\mathbf{S}$  restricts to a  $C_0$ -semigroup on  $H$ , the following assertions are equivalent:*

- (1)  $\mathbf{S}_H$  is uniformly exponentially stable;
- (2)  $\mathbf{S}_\infty$  is uniformly exponentially stable;
- (3) For some  $t > 0$  we have  $H_t = H_\infty$  with equivalent norms;
- (4) For all  $t > 0$  we have  $H_t = H_\infty$  with equivalent norms;
- (5)  $H_\infty \subseteq H$ .

In this situation, the inclusion  $H_\infty \subseteq H$  is dense.

*Proof.* (2)  $\Leftrightarrow$  (3): Recall that  $\mathbf{S}_\infty$  is uniformly exponentially stable if and only if there exists  $t_0 > 0$  such that  $\|S_\infty(t_0)\|_{H_\infty} < 1$ . Therefore the equivalence follows from Proposition 2.5.

(3)  $\Rightarrow$  (4): This follows from Theorem 3.5.

(4)  $\Rightarrow$  (5): Fix  $t_0 > 0$ . Then by Proposition 3.2,  $H_\infty = H_{t_0} \subseteq H$ .

(5)  $\Rightarrow$  (2): This follows from Lemma 5.2.

(5)  $\Rightarrow$  (1) : By assumptions there is a constant  $K$  such that for all  $x^* \in E^*$  we have

$$\int_0^\infty \|S_H(t)i^*x^*\|_H^2 dt = \langle Q_\infty x^*, x^* \rangle \leq K \langle Qx^*, x^* \rangle = K \|i^*x^*\|_H^2.$$

Hence the map  $i^*x^* \mapsto S_H(\cdot)i^*x^*$  has a bounded extension, of norm  $\leq \sqrt{K}$ , to a bounded operator from  $H$  into  $L^2([0, \infty); H)$ . But then the Datko-Pazy theorem [32] implies (1).

(1)  $\Rightarrow$  (5): For all  $x^* \in E^*$  we may estimate

$$\begin{aligned} \langle Q_\infty x^*, x^* \rangle &= \int_0^\infty \|S_H(t)i^*x^*\|_H^2 dt \\ &\leq \int_0^\infty Me^{-at} dt \cdot \|i^*x^*\|_H^2 = Ma^{-1} \langle Qx^*, x^* \rangle. \end{aligned}$$

This gives (5).

The final assertion follows from Proposition 3.7. Alternatively, one could observe that by Proposition 3.2, the inclusions  $H_t \subseteq H$  are dense. Therefore the result follows from the fact that  $H_t = H_\infty$  with equivalent norms.  $\blacksquare$

In view of Lemma 5.2, it seems natural to ask whether assertions (2) and (5) in Theorem 5.4 are always equivalent (i.e., even when  $H$  fails to be invariant). The following example shows that this is not the case.

*Example 5.5.* Let  $E = \mathbb{R}^2$ , let

$$Q = \begin{pmatrix} 0 & 0 \\ 0 & 1 \end{pmatrix}$$

and let the semigroup  $\mathbf{S}$  on  $E$  be given by

$$S(t) = e^{-t} \begin{pmatrix} 1 & t \\ 0 & 1 \end{pmatrix}.$$

An easy computation gives

$$Q_\infty = \frac{1}{4} \begin{pmatrix} 1 & 1 \\ 1 & 2 \end{pmatrix},$$

cf. [20, Example 4.3]. This matrix is invertible, so  $H_\infty = E = \mathbb{R}^2$  (with equivalent norms). On the other hand,  $H$  is the one-dimensional subspace of  $E$  spanned by the vector  $\begin{pmatrix} 0 \\ 1 \end{pmatrix}$ . It follows that

$$H_\infty \not\subseteq H.$$

On the other hand it is clear that  $\mathbf{S}_\infty$  is strongly stable, hence uniformly exponentially stable since  $H_\infty$  is finite-dimensional.

There is no contradiction with Theorem 5.2: the point is that there exists no  $\omega > 0$  such that  $\|S_\infty(t)\|_{H_\infty} \leq e^{-\omega t}$  for all  $t \geq 0$ . This can be checked by the following direct computation that will be useful in the next section as well.

We use the simple fact that  $\|S_\infty(t)\|^2$  is the largest eigenvalue of  $S_\infty(t)S_\infty^*(t)$ . Noting that  $i_\infty^*$  is a surjection from  $E$  onto  $H_\infty$ , the number  $\lambda(t)$  is an eigenvalue of  $S_\infty(t)S_\infty^*(t)$  if and only if there exists a vector  $x^* \in E$  such that for all  $y \in E^*$  we have

$$[S_\infty(t)S_\infty^*(t)i_\infty^*x^*, i_\infty^*y^*]_{H_\infty} = \lambda(t)[i_\infty^*x^*, i_\infty^*y^*]_{H_\infty},$$

or equivalently,

$$\langle S(t)Q_\infty S^*(t)x^*, y^* \rangle = \lambda(t)\langle Q_\infty x^*, y^* \rangle.$$

Thus we have to solve the equation

$$\det [S(t)Q_\infty S^*(t) - \lambda(t)Q_\infty] = 0.$$

An elementary computation gives

$$S(t)Q_\infty S^*(t) - \lambda(t)Q_\infty = \frac{1}{4} \begin{pmatrix} e^{-2t}(2t^2 + 2t + 1) - \lambda(t) & e^{-2t}(2t + 1) - \lambda(t) \\ e^{-2t}(2t + 1) - \lambda(t) & 2e^{-2t} - 2\lambda(t) \end{pmatrix}$$

from which we deduce that

$$\lambda_\pm(t) = e^{-2t} \left( t \pm \sqrt{t^2 + 1} \right)^2.$$

We finally obtain

$$\|S_\infty(t)\| = e^{-t} (t + \sqrt{t^2 + 1}).$$

Denoting the right hand side by  $f(t)$ , we have  $f(0) = 1$ ,  $\lim_{t \rightarrow \infty} f(t) = 0$  monotonously, and  $f'(0) = 0$ . Clearly, a function  $f(t)$  with these properties cannot be dominated by a negative exponential  $e^{-\omega t}$ .

6. THE ORNSTEIN-UHLENBECK SEMIGROUP IN  $C_b(E)$ 

Positive symmetric operators from  $E^*$  into  $E$  arise naturally as the covariance operators of Gaussian Borel measures on  $E$ . However, not every positive symmetric operator is a Gaussian covariance operator, and for this reason we will frequently consider the following hypothesis:

- $(\mathbf{H}\mu_t)$ :  $E$  is separable, and for all  $t > 0$  the operator  $Q_t$  is the covariance of a centred Gaussian Borel measure  $\mu_t$  on  $E$ .

Whenever it is convenient we further put  $\mu_0 := \delta_0$ , the Dirac measure concentrated at the origin.

The separability in Hypothesis  $(\mathbf{H}\mu_t)$  is added in order to avoid certain measure theoretic complications.

If  $E$  is a separable real Hilbert space, then  $Q_t$  is a Gaussian covariance if and only if  $Q_t$  is a trace class operator, and this happens if and only if the inclusion  $i_t : H_t \hookrightarrow E$  is a Hilbert-Schmidt operator.

The relevance of Hypothesis  $(\mathbf{H}\mu_t)$  is explained by the following result from [6]:

**Proposition 6.1**  $(\mathbf{H}\mu_t)$ . *Let  $A$  denote the generator of the semigroup  $\mathbf{S}$ . The stochastic evolution equation*

$$(6.1) \quad \begin{aligned} dX(t) &= AX(t) dt + dW_H(t), & t \geq 0, \\ X(0) &= x, \end{aligned}$$

*has a unique weak solution  $\{X(t, x)\}_{t \geq 0}$  if and only if Hypothesis  $(\mathbf{H}\mu_t)$  holds. In this situation the process  $\{X(t, x)\}_{t \geq 0}$  is Gaussian. For all  $t > 0$  we have  $X(t, x) = S(t)x + X(t, 0)$  almost surely, and the distribution of  $X(t, 0)$  equals  $\mu_t$ .*

Assuming Hypothesis  $(\mathbf{H}\mu_t)$ , we define the transition semigroup  $\mathbf{P} = \{P(t)\}_{t \geq 0}$  of  $\{X(t, \cdot)\}_{t \geq 0}$  on the space  $B_b(E)$  of all bounded Borel functions on  $E$  by

$$P(t)f(x) = \mathbb{E}(f(X(t, x))) = \int_E f(S(t)x + y) d\mu_t(y), \quad t \geq 0, x \in E.$$

The semigroup  $\mathbf{P}$  is contractive on  $B_b(E)$  and it maps  $C_b(E)$ , the space of all bounded continuous functions on  $E$ , into itself.

In general, the semigroup  $\mathbf{P}$  is not strongly continuous on  $C_b(E)$ , and not even on its closed subspace  $BUC(E)$  of all bounded and uniformly continuous functions on  $E$ .

We will show next that  $\mathbf{P}$  is strongly continuous on  $C_b(E)$  endowed with the *mixed topology* which is defined as the finest locally convex topology on  $C_b(E)$  that agrees on every norm-bounded set with the topology of uniform convergence on compact sets. For Hilbert spaces  $E$ , this fact was proved in [24]. By the results in [35], this definition agrees with the one in [24]. Clearly,

$$\tau_{\text{uniform-on-compact}} \subset \tau_{\text{mixed}} \subset \tau_{\text{uniform}}.$$

We have the following characterization of sequential convergence in the mixed topology: *a sequence  $(f_n)$  in  $C_b(E)$  converges to  $f \in C_b(E)$  if and only if*

- (1)  $\sup_n \|f_n\|_\infty < \infty$ ;
- (2)  $\lim_{n \rightarrow \infty} f_n = f$  uniformly on compact subsets of  $E$ .

We will also need the fact that the dual space  $(C_b(E), \tau_{\text{mixed}})^*$  can be identified in the natural way with the space of finite Borel measures on  $E$  [19].

For more information about the mixed topology we refer the interested reader to the papers [35, 38, 39] and the references therein.

**Theorem 6.2** ( $\mathbf{H}\mu_t$ ). *The semigroup  $\mathbf{P}$  is strongly continuous on  $C_b(E)$  in its mixed topology.*

*Proof.* Following the arguments of [24], we see that it suffices to prove that for all  $f \in C_b(E)$  and all compact subsets  $K \subseteq E$  we have

$$\lim_{t \downarrow 0} \left( \sup_{x \in K} |P(t)f(x) - f(x)| \right) = 0.$$

For Hilbert spaces  $E$ , this can be proved easily by probabilistic arguments. Here we give a direct, analytical proof.

Fix  $f \in C_b(E)$  and  $K \subseteq E$  compact. We may assume that  $K$  is convex. As was observed in [29], we have weak convergence  $\mu_t \rightarrow \mu_0 = \delta_0$ , the Dirac measure concentrated at 0. Fixing an arbitrary  $\varepsilon > 0$ , by tightness we may choose a compact set  $L$  in  $E$  such that  $\mu_t(L) \geq 1 - \varepsilon$  for all  $t \in [0, 1]$ . We may assume that  $L$  is convex. Keeping in mind that  $\mu_0 = \delta_0$ , we necessarily have  $0 \in L$ .

For all  $t \geq 0$  and  $x \in E$  we have

$$(6.2) \quad \begin{aligned} & |P(t)f(x) - f(x)| \\ & \leq \int_E |f(S(t)x + y) - f(x + y)| d\mu_t(y) + \int_E |f(x + y) - f(x)| d\mu_t(y). \end{aligned}$$

We will estimate the two integrals on the right hand side separately. For all  $x \in K$  and  $t \in [0, 1]$  we have

$$\int_E |f(S(t)x + y) - f(x + y)| d\mu_t(y) \leq 2\varepsilon \|f\|_\infty + \int_L |f(S(t)x + y) - f(x + y)| d\mu_t(y).$$

By the strong continuity of  $\mathbf{S}$ , which is uniform on compact sets, and the uniform continuity of  $f$  on the compact set  $\{S(t)x + y : (t, x, y) \in [0, 1] \times K \times L\}$  we may choose  $0 < t_0 \leq 1$  so small that

$$\sup_{x \in K, y \in L} |f(S(t)x + y) - f(x + y)| < \varepsilon, \quad t \in [0, t_0].$$

Thus, for all  $t \in [0, t_0]$  we obtain

$$(6.3) \quad \sup_{x \in K} \int_E |f(S(t)x + y) - f(x + y)| d\mu_t(y) \leq 2\varepsilon \|f\|_\infty + \varepsilon.$$

Next we estimate the second integral on the right hand side of (6.2). As above, for all  $x \in K$  and  $t \in [0, 1]$  we have

$$(6.4) \quad \int_E |f(x + y) - f(x)| d\mu_t(y) \leq 2\varepsilon \|f\|_\infty + \int_L |f(x + y) - f(x)| d\mu_t(y).$$

Hence it remains to show that

$$(6.5) \quad \lim_{t \downarrow 0} \left( \sup_{x \in K} \int_L |f(x + y) - f(x)| d\mu_t(y) \right) = 0.$$

The restriction of  $f$  to  $K + L$  being uniformly continuous, we introduce its modulus of continuity,

$$\eta(\delta) := \sup\{|f(z) - f(z')| : z, z' \in K + L, \|z - z'\| \leq \delta\}.$$

Then, recalling that  $0 \in L$ ,

$$\sup_{x \in K} \int_L |f(x + y) - f(x)| d\mu_t(y) \leq \int_L \eta(\|y\|) d\mu_t(y).$$

The function  $\zeta(y) := \eta(\|y\|)$  is bounded, nonnegative, and continuous on  $L$ . By the Tietze-Urysohn extension theorem [17, Theorem 2.1.8], it can be extended to a bounded, nonnegative, and continuous function  $\zeta$  on all of  $E$ . The weak convergence  $\mu_t \rightarrow \delta_0$  then implies

$$\limsup_{t \downarrow 0} \int_L \eta(\|y\|) d\mu_t(y) \leq \lim_{t \downarrow 0} \int_E \zeta(y) d\mu_t(y) = \zeta(0) = 0.$$

This proves (6.5). ■

In the remainder of this section we will always consider  $\mathbf{P}$  as a strongly continuous semigroup on  $C_b(E)$  in its mixed topology. The *infinitesimal generator*  $(L, \mathcal{D}(L))$  of  $\mathbf{P}$  is defined by

$$\begin{aligned} \mathcal{D}(L) &= \left\{ f \in C_b(E) : \lim_{t \downarrow 0} \frac{P(t)f - f}{t} \text{ exists} \right\}, \\ Lf &= \lim_{t \downarrow 0} \frac{P(t)f - f}{t} \quad (f \in \mathcal{D}(L)), \end{aligned}$$

where the limits are taken with respect to the mixed topology. In a similar way we define the weak generator: we say that  $\phi \in \mathcal{D}(L_w)$  if there exists a (necessarily unique) function  $f \in C_b(E)$  such that

$$(6.6) \quad \lim_{t \downarrow 0} \int_E \frac{P(t)\phi(x) - \phi(x)}{t} d\nu(x) = \int_E f(x) d\nu(x)$$

for each finite Borel measure  $\nu$  on  $E$ ; then we define  $L_w\phi := f$ .

**Proposition 6.3** ( $\mathbf{H}\mu_t$ ). *We have  $L = L_w$ .*

*Proof.* The proof is a simple modification of the proof of [32, Corollary 1.2]. Obviously  $L \subseteq L_w$ . Let, conversely,  $\phi \in \mathcal{D}(L_w)$ . Then, as in [32], for each finite Borel measure  $\nu$  on  $E$  we find

$$\int_E P(t)\phi(x) - \phi(x) d\nu(x) = \int_E \left( \int_0^t P(s)L_w\phi(x) ds \right) d\nu(x).$$

Since this holds for all finite Borel measures  $\nu$  and the functions  $x \mapsto \phi(x)$ ,  $x \mapsto P(t)\phi(x)$  and  $x \mapsto \int_0^t P(s)L_w\phi(x) ds$  are continuous we obtain, for all  $x \in E$ ,

$$P(t)\phi(x) - \phi(x) = \int_0^t P(s)L_w\phi(x) ds.$$

Hence if  $K \subseteq E$  is compact, then

$$\begin{aligned} & \lim_{t \downarrow 0} \left( \sup_{x \in K} \left[ \frac{P(t)\phi(x) - \phi(x)}{t} - L_w \phi(x) \right] \right) \\ &= \lim_{t \downarrow 0} \left( \sup_{x \in K} \left[ \frac{1}{t} \int_0^t P(s) L_w \phi(x) - L_w \phi(x) ds \right] \right) = 0 \end{aligned}$$

where in the last step we used that  $\lim_{t \downarrow 0} P(t) L_w \phi(x) = L_w \phi(x)$  uniformly on  $K$ . By the definition of  $L$  and the afore-mentioned criterium for sequential convergence in the mixed topolgy, this shows that  $\phi \in \mathcal{D}(L)$ . This concludes the proof.  $\blacksquare$

As an immediate corollary we have the following result, which shows that our definition of  $(L, \mathcal{D}(L))$  agrees with the one in [33].

**Corollary 6.4.** *We have  $\phi \in \mathcal{D}(L)$  if and only if the following two conditions hold:*

- (1)  $\limsup_{t \downarrow 0} \frac{\|P(t)\phi - \phi\|}{t} < \infty$ ;
- (2) *There exists a function  $f \in C_b(E)$  such that for all  $x \in E$ ,*

$$\lim_{t \downarrow 0} \frac{P(t)\phi(x) - \phi(x)}{t} = f(x).$$

*In this situation we have  $L\phi = f$ .*

Our next aim is to obtain an explicit representation of  $L$  on a suitable core.

**Lemma 6.5** ( $\mathbf{H}\mu_t$ ). *Let  $\{X(t, x_0)\}_{t \geq 0}$  be the unique weak solution of (6.1). Let  $\phi \in C^2(\mathbb{R}^d)$ ,  $x_1^*, \dots, x_d^* \in \mathcal{D}(A^*)$ , and  $x_0 \in E$ . Then for all  $t \geq 0$  the following identity holds almost surely:*

$$\begin{aligned} & \phi(\langle X(t, x_0), x_1^* \rangle, \dots, \langle X(t, x_0), x_d^* \rangle) \\ &= \phi(\langle x_0, x_1^* \rangle, \dots, \langle x_0, x_d^* \rangle) \\ &+ \sum_{j=1}^d \int_0^t \frac{\partial \phi}{\partial x_j}(\langle X(s, x_0), x_1^* \rangle, \dots, \langle X(s, x_0), x_d^* \rangle) dW_H(s) i^* x_j^* \\ &+ \sum_{j=1}^d \int_0^t \frac{\partial \phi}{\partial x_j}(\langle X(s, x_0), x_1^* \rangle, \dots, \langle X(s, x_0), x_d^* \rangle) \langle X(s, x_0), A^* x_j^* \rangle ds \\ &+ \frac{1}{2} \sum_{j=1}^d \sum_{k=1}^d [i^* x_j^*, i^* x_k^*]_H \int_0^t \frac{\partial^2 \phi}{\partial x_j \partial x_k}(\langle X(s, x_0), x_1^* \rangle, \dots, \langle X(s, x_0), x_d^* \rangle) ds. \end{aligned}$$

*Proof.* For  $j = 1, \dots, d$  and  $t \geq 0$  we define

$$\xi_j(t) := \langle x_0, x_j^* \rangle + \int_0^t \langle X(s, x_0), A^* x_j^* \rangle ds + W_H(t) i^* x_j^*.$$

By the definition of a weak solution, for all  $t \geq 0$  we have  $\xi_j(t) = \langle X(t, x_0), x_j^* \rangle$  almost surely, so  $\{\xi_j(t)\}_{t \geq 0}$  is a modification of the process  $\{\langle X(t, x_0), x_j^* \rangle\}_{t \geq 0}$ . Since almost surely, the trajectories of  $\{\langle X(t, x_0), A^* x_j^* \rangle\}_{t \geq 0}$  are locally integrable, we see that almost surely the trajectories of the process  $\{V_j(t)\}_{t \geq 0}$  defined by

$$(6.7) \quad V_j(t) := \int_0^t \langle X(s, x_0), A^* x_j^* \rangle ds.$$

are continuous and locally of bounded variation. By redefining the random variables  $V_j(t)$  to be 0 on a common null set we obtain a modification of  $\{\langle X(t, x_0), x_j^* \rangle\}_{t \geq 0}$ , still denoted by  $\{\xi_j(t)\}_{t \geq 0}$ . From the representation

$$\xi_j(t) = \langle x_0, x_j^* \rangle + V_j(t) + M_j(t)$$

where  $M_j(t) := W_H(t) i^* x_j^*$ , we see that  $\{\xi_j(t)\}_{t \geq 0}$  is a continuous semimartingale. Define  $F : \mathbb{R}^d \times \mathbb{R}^d \rightarrow \mathbb{R}$  by

$$F(u, v) := \phi(\langle x_0, x_1^* \rangle, \dots, \langle x_0, x_d^* \rangle) + u + v.$$

Put

$$\xi(t) := (\xi_1(t), \dots, \xi_d(t)), \quad M(t) := (M_1(t), \dots, M_d(t)), \quad V(t) := (V_1(t), \dots, V_d(t)).$$

By the Itô formula [11, Theorem 5.10] almost surely we have, for all  $t \geq 0$ ,

$$\begin{aligned} \phi(\xi(t)) - \phi(\xi(0)) &= F(M(t), V(t)) - F(M(0), V(0)) \\ &= \sum_{j=1}^d \int_0^t \frac{\partial F}{\partial u_j}(M(s), V(s)) dM_j(s) \\ &\quad + \sum_{j=1}^d \int_0^t \frac{\partial F}{\partial v_j}(M(s), V(s)) dV_j(s) \\ &\quad + \frac{1}{2} \sum_{j=1}^d \sum_{k=1}^d [i^* x_j^*, i^* x_k^*]_H \int_0^t \frac{\partial^2 F}{\partial u_j \partial v_k}(M(s), V(s)) ds \\ &= \sum_{j=1}^d \int_0^t \frac{\partial \phi}{\partial x_j}(\xi(s)) dW_H(s) i^* x_j^* \\ &\quad + \sum_{j=1}^d \int_0^t \frac{\partial \phi}{\partial x_j}(\xi(s)) \langle X(s, x_0), A^* x_j^* \rangle ds \\ &\quad + \frac{1}{2} \sum_{j=1}^d \sum_{k=1}^d [i^* x_j^*, i^* x_k^*]_H \int_0^t \frac{\partial^2 \phi}{\partial x_j \partial x_k}(\xi(s)) ds, \end{aligned} \tag{6.8}$$

where we used (6.7) and the fact that the mutual quadratic variation of  $M_j(s)$  and  $M_k(s)$  equals  $s \cdot [i^* x_j^*, i^* x_k^*]_H$ .

We claim that almost surely,

$$\xi_j(s) = \langle X(s, x_0), x_j^* \rangle \quad \text{for almost all } s \geq 0. \tag{6.9}$$

To see this, note that  $\{\xi(t)\}_{t \geq 0}$  is progressively measurable, being a process with continuous trajectories. Also,  $\{\langle X(t, x_0), x^* \rangle\}_{t \geq 0}$  is progressively measurable, being predictable. The claim follows from Fubini's theorem.

The proposition now follows by combining (6.8) and (6.9).  $\blacksquare$

We will now identify a suitable core for  $L$  consisting of cylindrical functions satisfying the assumptions of Lemma 6.5. To this end let us define  $\mathcal{F} = \mathcal{F}C_b^2(\mathcal{D}(A^*))$  as the space of all functions  $f : E \rightarrow \mathbb{R}$  of the form

$$f(x) = \phi(\langle x, x_1^* \rangle, \dots, \langle x, x_d^* \rangle) \tag{6.10}$$

for some  $d \geq 1$ , with  $x_j^* \in \mathcal{D}(A^*)$  for all  $j = 1, \dots, d$  and  $\phi \in C_b^2(\mathbb{R}^d)$ . Let

$$\mathcal{F}_0 = \{f \in \mathcal{F} : \langle \cdot, A^* Df(\cdot) \rangle \in C_b(E)\}.$$

For  $f \in \mathcal{F}_0$  we define  $L_0 f \in C_b(E)$  by

$$(6.11) \quad L_0 f(x) := \frac{1}{2} \text{trace } D_H^2 f(x) + \langle x, A^* Df(x) \rangle, \quad x \in E.$$

Here  $Df : E \rightarrow E^*$  is the Fréchet derivative of  $f$ ,

$$Df(x) = \sum_{j=1}^d \frac{\partial \phi}{\partial x_j}(\langle x, x_1^* \rangle, \dots, \langle x, x_d^* \rangle) \otimes x_j^*$$

and  $D_H f : E \rightarrow H$  is defined by

$$(6.12) \quad D_H f(x) = \sum_{j=1}^d \frac{\partial \phi}{\partial x_j}(\langle x, x_1^* \rangle, \dots, \langle x, x_d^* \rangle) \otimes i^* x_j^*.$$

In a slightly different setting, the space  $\mathcal{F}_0$  was introduced first by Cerrai and Gozzi [7]; see also [24]. Extending the results from these papers to the Banach space setting, we will show in a moment that  $\mathcal{F}_0$  is a core for the generator  $L$  and that  $Lf = L_0 f$  for  $f \in \mathcal{F}_0$ .

**Theorem 6.6** ( $\mathbf{H}\mu_t$ ).  *$\mathcal{F}_0$  is a core for  $L$ , and for all  $f \in \mathcal{F}_0$  we have  $Lf = L_0 f$ .*

*Proof.* We will show first that  $L_0 \subseteq L$ . Clearly,  $L_0 f \in C_b(E)$  for any  $f \in \mathcal{F}_0$ . Let  $f(x) = \phi(\langle x, x_1^* \rangle, \dots, \langle x, x_d^* \rangle)$  with  $x_j^* \in \mathcal{D}(A^*)$  for all  $j = 1, \dots, d$  and  $\phi \in C_b^2(\mathbb{R}^d)$ . First we note that

$$D_H^2 \phi(\langle x, x_1^* \rangle, \dots, \langle x, x_d^* \rangle) = \sum_{j=1}^d \sum_{k=1}^d \frac{\partial^2 \phi}{\partial x_j \partial x_k}(\langle x, x_1^* \rangle, \dots, \langle x, x_d^* \rangle) \otimes (i^* x_j^* \otimes B^* x_k^*).$$

Now we apply Lemma 6.5 and take on both sides the expectation. This gives

$$\begin{aligned} & \frac{1}{t} (P(t)f(x) - f(x)) \\ &= \mathbb{E} \frac{1}{t} (f(X(t, x)) - f(x)) \\ &= \mathbb{E} \frac{1}{t} \int_0^t \sum_{j=1}^d \frac{\partial \phi}{\partial x_j}(\langle X(s, x), x_1^* \rangle, \dots, \langle X(s, x), x_d^* \rangle) \langle X(s, x), A^* x_j^* \rangle ds \\ & \quad + \mathbb{E} \frac{1}{2t} \sum_{j=1}^d \sum_{k=1}^d [i^* x_j^*, i^* x_k^*]_H \int_0^t \frac{\partial^2 \phi}{\partial x_j \partial x_k}(\langle X(s, x), x_1^* \rangle, \dots, \langle X(s, x), x_d^* \rangle) ds \\ &= \frac{1}{t} \int_0^t P(s) (\langle \cdot, A^* Df(\cdot) \rangle)(x) ds + \frac{1}{2t} \int_0^t P(s) (\text{trace } D_H^2 f)(x) ds. \end{aligned}$$

The assumption  $f \in \mathcal{F}_0$  implies that the functions

$$x \mapsto \langle x, A^* Df(x) \rangle \quad \text{and} \quad x \mapsto \frac{1}{2} \text{trace } D_H^2 f(x)$$

belong to  $C_b(E)$ . Therefore by Theorem 6.2 we can pass to the limit  $t \downarrow 0$  and by using Corollary 6.4 we obtain  $L_0 \subseteq L$ . By [24, Lemma 4.6]  $\mathcal{F}_0$  is dense in  $C_b(E)$ , and since  $\mathcal{F}_0$  is also  $\mathbf{P}$ -invariant,  $\mathcal{F}_0$  is a core for  $L$  by [24, Lemma 4.7]; *mutatis mutandis*, the proofs of these results extend to the Banach space case.  $\blacksquare$

**6.1. The strong Feller property.** We say that  $\mathbf{P}$  has the *strong Feller property* if for every  $t > 0$ ,  $P(t)$  maps  $B_b(E)$  into  $C_b(E)$ . We start with a characterization of the strong Feller property in terms of the mixed topology on  $C_b(E)$ . Although a more general version of the following result below appears to be known to specialists, it is not easily available and for the convenience of reader we include a straightforward proof for the case of Ornstein-Uhlenbeck semigroups.

**Proposition 6.7.** *The following conditions are equivalent:*

- (1) *The semigroup  $\mathbf{P}$  has the strong Feller property.*
- (2) *For each  $t > 0$ , the mapping*

$$x \mapsto \mu_t(x, \cdot)$$

*is continuous from  $E$  into  $(C_b(E), \tau_{\text{mixed}})^*$  with the variation norm.*

- (3) *For each  $t > 0$ , the operator*

$$P(t) : (C_b(E), \tau_{\text{uniform}}) \rightarrow (C_b(E), \tau_{\text{mixed}})$$

*is compact.*

*Proof.* (1)  $\Rightarrow$  (2): Assume that (1) holds. As we have already mentioned, the dual space  $(C_b(E), \tau_{\text{mixed}})^*$  may be identified as the space of finite Borel measures on  $E$ . By [15, Theorem 9.19], for each  $t > 0$  there exists  $c_t \geq 0$  such that

$$\|\mu_t(x, \cdot) - \mu_t(y, \cdot)\|_{\text{var}} \leq c_t \|x - y\|, \quad x, y \in E,$$

and (2) follows.

(2)  $\Rightarrow$  (1): If (2) holds, then by definition of the variation norm,

$$|P(t)\phi(x) - P(t)\phi(y)| \leq \|\mu_t(x, \cdot) - \mu_t(y, \cdot)\|_{\text{var}} \|\phi\|,$$

which implies (1).

(2)  $\Leftrightarrow$  (3): Assume that (2) holds and let

$$B = \{\phi \in C_b(E) : \|\phi\| \leq 1\}.$$

The set  $P(t)B$  is bounded and by (6.1) uniformly equicontinuous on compacts. Therefore  $P(t)B$  is relatively compact by [24, Theorem 2.4] and (3) follows.

(3)  $\Leftrightarrow$  (2): Assume now that (3) holds. Then there exists a sequence  $(\phi_n)$  in  $B$  such that

$$\|\mu_t(x, \cdot) - \mu_t(y, \cdot)\|_{\text{var}} = \lim_{n \rightarrow \infty} |P(t)\phi_n(x) - P(t)\phi_n(y)|.$$

and since the set  $P(t)B$  is relatively compact we may assume that  $\lim_{n \rightarrow \infty} P(t)\phi_n = \psi$  in the mixed topology. Hence,  $\psi$  is continuous and (b) follows.  $\blacksquare$

It is well known that  $\mathbf{P}$  has the strong Feller property if and only if the pair  $(\mathbf{S}, i)$  is null controllable [15, 29]. Under the assumption that  $H$  is  $\mathbf{S}$ -invariant, from Theorem 3.6 we thus obtain the following explicit necessary and sufficient condition for the strong Feller property. It extends a previous result from [10].

**Theorem 6.8 ( $H\mu_t$ ).** *If  $\mathbf{S}$  restricts to a  $C_0$ -semigroup on  $H$ , then the following assertions are equivalent:*

- (1)  *$\mathbf{P}$  has the strong Feller property;*

(2)  $S(t)E \subseteq H$  for all  $t > 0$ .

### 7. THE ORNSTEIN-UHLENBECK SEMIGROUP IN $L^2(E, \mu_\infty)$

In order to be able to study the semigroup  $\mathbf{P}$  in an  $L^2$ -context, we consider the following hypothesis:

- $(\mathbf{H}\mu_\infty)$ :  $E$  is separable, Hypothesis  $(\mathbf{H}Q_\infty)$  holds, and the operator  $Q_\infty$  is the covariance of a centred Gaussian Borel measure  $\mu_\infty$  on  $E$ .

By (2.1) and a standard tightness argument,  $(\mathbf{H}\mu_\infty)$  implies  $(\mathbf{H}\mu_t)$ . If  $E$  is a Hilbert space, then  $(\mathbf{H}\mu_\infty)$  holds if and only if  $(\mathbf{H}\mu_t)$  holds and

$$\sup_{t>0} (\text{trace } Q_t) < \infty,$$

in particular, if  $\mathbf{S}$  is uniformly exponentially stable [15, Theorem 11.11]. Extensions of these results to Banach spaces not containing a closed subspace isomorphic to  $c_0$  have been obtained in [31].

In general it is not true that  $(\mathbf{H}\mu_t)$  and  $(\mathbf{H}Q_\infty)$  imply  $(\mathbf{H}\mu_\infty)$ , as is shown by the following example.

*Example 7.1.* Let  $E = \ell^2$ ; we identify  $E$  and its dual in a natural way. The standard unit basis of  $E$  will be denoted by  $(x_n)_{n=1}^\infty$ .

Let  $(q_n)_{n=1}^\infty$  be a bounded sequence of strictly positive real numbers and define  $Q \in \mathcal{L}(E)$  by

$$Qx_n := q_n x_n.$$

Let  $(a_n)_{n=1}^\infty$  be a sequence of strictly positive real numbers and define the operator  $(A, \mathcal{D}(A))$  by

$$Ax_n := -a_n x_n$$

with maximal domain. Then  $A$  generates a strongly stable  $C_0$ -semigroup  $\mathbf{S}$  on  $E$ , given by  $S(t)x_n = e^{-a_n t} x_n$ .

It is easy to check that

$$H = \left\{ (b_n)_{n=1}^\infty \in E : \sum_{n=1}^\infty \frac{1}{q_n} b_n^2 < \infty \right\}.$$

For all  $t > 0$  we have  $Q_t x_n = \frac{q_n}{2a_n} (1 - e^{-2a_n t}) x_n$ , hence

$$H_t = \left\{ (b_n)_{n=1}^\infty \in E : \sum_{n=1}^\infty \frac{2a_n}{q_n} (1 - e^{-2a_n t})^{-1} b_n^2 < \infty \right\}.$$

Hypothesis  $(\mathbf{H}Q_\infty)$  holds if and only if  $\sup_{n \geq 1} \frac{q_n}{a_n} < \infty$ . In this case we have  $Q_\infty x_n =$

$\frac{q_n}{2a_n} x_n$  and

$$H_\infty = \left\{ (b_n)_{n=1}^\infty \in E : \sum_{n=1}^\infty \frac{a_n}{q_n} b_n^2 < \infty \right\}.$$

Therefore, Hypothesis  $(\mathbf{H}\mu_\infty)$  is satisfied if and only if  $\sum_{n=1}^\infty \frac{q_n}{a_n} < \infty$ .

Let us now assume that the sequence  $(a_n)_{n=1}^\infty$  is *bounded*. Then there exists, for every  $t > 0$ , a constant  $M_t \geq 1$  such that

$$1 \leq \frac{1 - e^{-2a_n t}}{2a_n} \leq M_t$$

for all  $n \geq 1$ . From this it follows that

$$H_t = \left\{ (b_n)_{n=1}^\infty \in E : \sum_{n=1}^\infty \frac{1}{q_n} b_n^2 < \infty \right\}.$$

Thus, for all  $t > 0$  we have  $H_t = H$  up to an equivalent norm. By computing the trace of  $Q_t$  we see that Hypothesis  $(\mathbf{H}\mu_t)$  is satisfied if and only if  $\sum_{n=1}^\infty q_n < \infty$ . For  $q_n = 1/n^2$  and  $a_n = 1/n$  we obtain an example where  $(\mathbf{H}Q_\infty)$  and  $(\mathbf{H}\mu_t)$  hold, but not  $(\mathbf{H}\mu_\infty)$ .

This example is interesting for another reason. It is shown in [30] that  $(\mathbf{H}Q_\infty)$  implies that  $A^*$  has no point spectrum in the closed right half plane  $\{z \in \mathbb{C} : \operatorname{Re} z \geq 0\}$ , and that  $A$  has no point spectrum on the imaginary axis if in addition we assume that  $\mathbf{S}$  is uniformly bounded. It may happen that  $\sigma(A) \cap i\mathbb{R}$  is non-empty, however, even in the presence of Hypothesis  $(\mathbf{H}\mu_\infty)$ . For example, take  $q_n = 1/n^3$  and  $a_n = 1/n$ ; then  $0 \in \sigma(A)$  and Hypothesis  $(\mathbf{H}\mu_\infty)$  holds.

Let us note that Example 7.1 can easily be extended to  $E = \ell^p$  ( $1 \leq p < \infty$ ) by using the fact [36, Theorem V.5.6] that a positive diagonal operator  $(x_n)_{n=1}^\infty \mapsto (c_n x_n)_{n=1}^\infty$  from  $E^*$  to  $E$  is a Gaussian covariance operator if and only if  $\sum_{n=1}^\infty c_n^{p/2} < \infty$ .

If Hypothesis  $(\mathbf{H}\mu_\infty)$  holds, the measure  $\mu_\infty$  is *invariant* under the semigroup  $\mathbf{P}$ , that is, for all  $f \in B_b(E)$  we have

$$\int_E (P(t)f)(x) d\mu_\infty(x) = \int_E f(x) d\mu_\infty(x), \quad t \geq 0.$$

By standard arguments, cf. [40, Theorem XIII.1], it follows that  $\mathbf{P}$  extends to a  $C_0$ -contraction semigroup, also denoted by  $\mathbf{P}$ , on  $L^p(E, \mu_\infty)$  for all  $p \in [1, \infty)$ . The space  $\mathcal{F}_0$ , being norm-dense and  $\mathbf{P}$ -invariant, is a core for the generator  $(L, \mathcal{D}(L))$ . We have the following integration by parts formula:

**Lemma 7.2**  $(\mathbf{H}\mu_\infty)$ . *For all  $f, g \in \mathcal{F}_0$  we have*

$$(7.1) \quad \int_E f Lg + g Lf d\mu_\infty = - \int_E [D_H f, D_H g]_H d\mu_\infty.$$

*Proof.* Observe that  $\mathcal{F}_0$  is closed under multiplication. Hence if  $f, g \in \mathcal{F}_0$ , then  $fg \in \mathcal{F}_0 \subseteq \mathcal{D}(L_0) \subseteq \mathcal{D}(L)$  and a simple calculation based on (6.11) gives

$$(7.2) \quad \begin{aligned} L(fg) &= L_0(fg) = f L_0 g + g L_0 f + [D_H f, D_H g]_H \\ &= f Lg + g Lf + [D_H f, D_H g]_H. \end{aligned}$$

Since  $\mu_\infty$  is an invariant measure, we have

$$\int_E P(t)(fg) d\mu_\infty = \int_E fg d\mu_\infty$$

from which it is immediate that

$$\int_E L(fg) d\mu_\infty = 0.$$

Therefore, for  $f, g \in \mathcal{F}_0$  the desired result follows by integrating (7.2) over  $E$ .  $\blacksquare$

*Remark 7.3.* The identity (7.1) extends to arbitrary elements  $f, g \in \mathcal{D}(L)$  if  $D_H$  is closable. Necessary and sufficient conditions for closability of  $D_H$ , as well as simple examples where  $D_H$  fails to be closable, were obtained in [23]. In Proposition 8.7 below we show that  $D_H$  is closable if  $\mathbf{P}$  is analytic on  $L^2(E, \mu_\infty)$ .

On  $L^2(E, \mu_\infty)$  we have the representation

$$P(t) = \Gamma(S_\infty^*(t)), \quad t \geq 0,$$

where  $\Gamma$  denotes the second quantization functor; cf. [8], [29]. This result permits one to study the semigroup  $\mathbf{P}$  through the semigroup  $\mathbf{S}_\infty^*$ . We give two simple illustrations. The first is a characterization of selfadjointness.

**Theorem 7.4** ( $\mathbf{H}\mu_\infty$ ). *The following assertions are equivalent:*

- (1) *The semigroup  $\mathbf{P}$  is selfadjoint on  $L^2(E, \mu_\infty)$ ;*
- (2) *The semigroup  $\mathbf{S}$  is  $Q$ -symmetric.*

*Proof.* We will show that  $\mathbf{S}$  is  $Q$ -symmetric if and only if  $\mathbf{S}_\infty$  is selfadjoint. The proposition is then a consequence of the identities  $P(t) = \Gamma(S_\infty^*)$  and  $P^*(t) = \Gamma(S_\infty)$ , where  $\Gamma$  denotes the second quantization functor.

If  $\mathbf{S}$  is  $Q$ -symmetric, then for all  $t \geq 0$  and  $x^*, y^* \in E^*$  we have

$$\begin{aligned} [S_\infty(t)i_\infty^*x^*, i_\infty^*y^*]_{H_\infty} &= [i_\infty^*x^*, S_\infty^*(t)i_\infty^*y^*]_{H_\infty} = [i_\infty^*x^*, i_\infty^*S^*(t)y^*]_{H_\infty} \\ &= \langle Q_\infty x^*, S^*(t)y^* \rangle = \langle Q_\infty S^*(t)y^*, x^* \rangle \\ &= \langle S(t)Q_\infty y^*, x^* \rangle = [i_\infty^*y^*, i_\infty^*S^*(t)x^*]_{H_\infty} \\ &= [i_\infty^*y^*, S_\infty^*(t)i_\infty^*x^*]_{H_\infty} = [S_\infty^*(t)i_\infty^*x^*, i_\infty^*y^*]_{H_\infty}. \end{aligned}$$

It follows that  $\mathbf{S}_\infty$  is selfadjoint. Conversely if  $\mathbf{S}_\infty$  is selfadjoint, then a similar argument shows that  $\mathbf{S}$  is  $Q$ -symmetric.  $\blacksquare$

The second illustration concerns the spectral gap of the generator of  $\mathbf{P}$ . If  $(B, \mathcal{D}(B))$  is a negative operator in a Hilbert space  $K$ , i.e., if  $[Bk, k]_K \leq 0$  for all  $k \in \mathcal{D}(B)$ , then its spectrum  $\sigma(B)$  is contained in the interval  $(-\infty, 0]$ . We say that  $B$  has a *spectral gap* if  $0 \in \sigma(B)$  and there exists  $\omega > 0$  such that  $\sigma(B) \setminus \{0\} \subseteq (-\infty, -\omega]$ . The largest such  $\omega > 0$  is called the *spectral gap* of  $B$ .

As an application of the results of Section 5 we shall give a necessary and sufficient condition for the existence of a spectral gap for the generator  $L$  of the Ornstein-Uhlenbeck semigroup  $\mathbf{P}$  in  $L^2(E, \mu_\infty)$ .

Let  $\mathcal{H}_1 = L^2(E, \mu_\infty) \ominus \mathbb{R}$  denote the orthogonal complement in  $L^2(E, \mu_\infty)$  of the constant functions. By second quantization and the properties of the Wiener-Itô decomposition, we obtain immediately that  $\mathbf{P}$  restricts to a  $C_0$ -semigroup of contractions on  $\mathcal{H}_1$  satisfying  $\|P(t)\|_{\mathcal{H}_1} = \|S_\infty(t)\|$  for all  $t \geq 0$ . Let us denote the generator of  $\mathbf{P}_1$  by  $L_1$ . The following result may now be deduced from Lemma 5.2, Theorem 5.4, and the spectral theory of  $C_0$ -semigroups.

**Theorem 7.5** ( $\mathbf{H}\mu_\infty$ ). *The following assertions are equivalent:*

- (1)  $L_1$  has a spectral gap;
- (2)  $H_\infty \subseteq H$ .

If  $H_\infty \subseteq H$ , the spectral gap of  $L_1$  equals the exponential growth bound of the semigroup generated by  $A_\infty$ .

If  $\mathbf{S}$  restricts to a  $C_0$ -semigroup on  $H$ , then (1) and (2) are equivalent to:

- (3)  $\mathbf{S}_H$  is uniformly exponentially stable.

## 8. ANALYTICITY OF THE ORNSTEIN-UHLENBECK SEMIGROUP

In this section we investigate conditions under which the complexified semigroup  $\mathbf{P}^\mathbb{C} = \{P^\mathbb{C}(t)\}_{t \geq 0}$  is analytic on  $L_2^\mathbb{C}(E, \mu_\infty)$ , the complexification of  $L^2(E, \mu_\infty)$ . Here  $\mathbf{P}$  denote the Ornstein-Uhlenbeck semigroup on  $L^2(E, \mu_\infty)$  associated with  $\mathbf{S}$  and  $H$ ; cf. Section 2.

Recall that a semigroup  $\mathbf{T} = \{T(t)\}_{t \geq 0}$  on a complex Banach space is called an *analytic contraction semigroup* if  $\mathbf{T}$  is analytic and  $\|T(z)\| \leq 1$  for all  $z \in \mathbb{C}$  belonging to some sector containing the positive real axis.

Our first result generalizes to Banach spaces a result from [22]; cf. also [20, Theorem 3.6].

**Theorem 8.1** ( $\mathbf{H}\mu_\infty$ ). *The following assertions are equivalent:*

- (1)  $\mathbf{P}^\mathbb{C}$  extends to an analytic semigroup on  $L_2^\mathbb{C}(E, \mu_\infty)$ ;
- (2)  $\mathbf{P}^\mathbb{C}$  extends to an analytic contraction semigroup on  $L_2^\mathbb{C}(E, \mu_\infty)$ ;
- (3)  $\mathbf{S}_\infty^\mathbb{C}$  extends to an analytic contraction semigroup on  $H_\infty^\mathbb{C}$ ;
- (4) There exists a constant  $M \geq 0$  such that

$$|[A_\infty^* h_\infty, h'_\infty]_{H_\infty}| \leq M |[A_\infty^* h_\infty, h_\infty]_{H_\infty}|^{\frac{1}{2}} \cdot |[A_\infty^* h'_\infty, h'_\infty]_{H_\infty}|^{\frac{1}{2}}$$

for all  $h_\infty, h'_\infty \in i_\infty^* \mathcal{D}(A^*)$ .

In this situation,  $\mathbf{P}^\mathbb{C}$  and  $\mathbf{S}_\infty^\mathbb{C}$  are contractive on the same sectors.

*Proof.* The proof is analogous to the corresponding result for Hilbert spaces  $E$  given in [22], so we only sketch the main steps.

The equivalences (1)  $\Leftrightarrow$  (2)  $\Leftrightarrow$  (3) as well as the final statement follow from the fact that  $P^\mathbb{C}(t) = \Gamma^\mathbb{C}((S_\infty^\mathbb{C}(t))^*)$ , where  $\Gamma^\mathbb{C}$  denotes the complex second quantization functor.

(3)  $\Leftrightarrow$  (4): Since  $i_\infty^*(\mathcal{D}(A^*))$  is a core for  $\mathcal{D}(A_\infty^*)$ , the estimate in condition (4) holds for all  $h_\infty, h'_\infty \in \mathcal{D}(A_\infty^*)$ . Hence by [28, Proposition I.2.17], condition (4) holds if and only if there exists  $b > 0$  such that

$$[(A_\infty^\mathbb{C})^* h^\mathbb{C}, h^\mathbb{C}]_{H_\infty^\mathbb{C}} \in \{z \in \mathbb{C} : |\operatorname{Im} z| \leq b \operatorname{Re} z\}$$

for all  $h^\mathbb{C} \in \mathcal{D}((A_\infty^\mathbb{C})^*)$ . Since  $1 \in \varrho(A_\infty^\mathbb{C})$  (recall that  $\mathbf{S}_\infty$ , hence also  $\mathbf{S}_\infty^\mathbb{C}$ , is a contraction semigroup), by [21, Theorem 1.5.9] this condition is in turn equivalent to condition (3).  $\blacksquare$

*Remark 8.2.* Using the terminology of [28], condition (4) says that  $A_\infty$  satisfies a strong sector condition.

We will develop Theorem 8.1 a little further.

**Theorem 8.3** ( $\mathbf{H}\mu_\infty$ ). *The following assertions are equivalent:*

- (1)  $\mathbf{P}^{\mathbb{C}}$  extends to a analytic semigroup on  $L_2^{\mathbb{C}}(E, \mu_\infty)$ ;
- (2) For all  $x^* \in \mathcal{D}(A^*)$  we have  $AQ_\infty x^* \in H$ , and there is a constant  $C \geq 0$  such that

$$\|AQ_\infty x^*\|_H \leq C \|i^* x^*\|_H, \quad x^* \in \mathcal{D}(A^*).$$

- (3) For all  $x^* \in \mathcal{D}(A^*)$  we have  $Q_\infty(A^* x^*) \in H$ , and there exists a constant  $C > 0$  such that

$$\|Q_\infty A^* x^*\|_H \leq C \|i^* x^*\|_H, \quad x^* \in \mathcal{D}(A^*).$$

*Proof.* We start by noting that for all  $x^* \in \mathcal{D}(A^*)$  and  $y^* \in \mathcal{D}(A^*)$  we have  $i_\infty^* x^* \in \mathcal{D}(A_\infty^*)$ ,  $i_\infty^* y^* \in \mathcal{D}(A_\infty^*)$ ,

$$\langle AQ_\infty x^*, y^* \rangle = \langle Q_\infty A^* y^*, x^* \rangle = [i_\infty^* A^* y^*, i_\infty^* x^*]_{H_\infty} = [A_\infty^* i_\infty^* y^*, i_\infty^* x^*]_{H_\infty},$$

and

$$\langle Qx^*, x^* \rangle = -2\langle Q_\infty A^* x^*, x^* \rangle = -2[A_\infty^* i_\infty^* x^*, i_\infty^* x^*]_{H_\infty}.$$

Hence by Theorem 8.1,  $\mathbf{P}^{\mathbb{C}}$  is analytic if and only if there exists a constant  $M \geq 0$  such that

$$(8.1) \quad |\langle AQ_\infty x^*, y^* \rangle| \leq M \langle Qx^*, x^* \rangle^{\frac{1}{2}} \langle Qy^*, y^* \rangle^{\frac{1}{2}}, \quad x^*, y^* \in \mathcal{D}(A^*).$$

(1)  $\Rightarrow$  (2): By Theorem 8.1, for all  $x^*, y^* \in \mathcal{D}(A^*)$  we have

$$(8.2) \quad |\langle AQ_\infty x^*, y^* \rangle| \leq M \|i^* x^*\|_H \|i^* y^*\|_H.$$

It follows that the map  $i^* y^* \mapsto \langle AQ_\infty x^*, y^* \rangle$  is well defined and can be extended to a bounded linear form on  $H$  of norm  $\leq \|i^* x^*\|_H$ . Therefore by the Riesz representation theorem we can identify  $AQ_\infty x^*$  with an element of  $H$  of norm  $\leq \|i^* x^*\|_H$ . This gives (2), with  $C = M$ .

(2)  $\Rightarrow$  (1): For all  $x^*, y^* \in \mathcal{D}(A^*)$  we have

$$\begin{aligned} |\langle AQ_\infty x^*, y^* \rangle| &= |[AQ_\infty x^*, i^* y^*]_H| \\ &\leq C \|i^* x^*\|_H \|i^* y^*\|_H = C \langle Qx^*, x^* \rangle^{\frac{1}{2}} \langle Qy^*, y^* \rangle^{\frac{1}{2}} \end{aligned}$$

and  $\mathbf{P}^{\mathbb{C}}$  has an analytic extension.

(1)  $\Rightarrow$  (3): By the proof of (1)  $\Rightarrow$  (2), for all  $x^* \in \mathcal{D}(A^*)$  we have  $Q_\infty A^* x^* = -Qx^* - AQ_\infty^* x^* \in H$ , and there is a constant  $c$  such that

$$\|Q_\infty A^* x^*\|_H \leq \|Qx^*\|_H + \|AQ_\infty^* x^*\|_H \leq (1+c) \|i^* x^*\|_H$$

for all  $x^* \in \mathcal{D}(A^*)$ .

(3)  $\Rightarrow$  (1): For all  $x^*, y^* \in \mathcal{D}(A^*)$  we have

$$\begin{aligned} |\langle AQ_\infty x^*, y^* \rangle| &= |\langle Q_\infty A^* y^*, x^* \rangle| = |[Q_\infty A^* y^*, i^* x^*]_H| \\ &\leq C \|i^* x^*\|_H \|i^* y^*\|_H = C \langle Qx^*, x^* \rangle^{\frac{1}{2}} \langle Qy^*, y^* \rangle^{\frac{1}{2}}. \end{aligned}$$

■

This leads to the following concise *necessary* condition for analyticity of the Ornstein-Uhlenbeck semigroup:

**Corollary 8.4** ( $\mathbf{H}\mu_\infty$ ). *If  $\mathbf{P}^{\mathbb{C}}$  extends to an analytic semigroup on  $L_2^{\mathbb{C}}(E, \mu_\infty)$ , then for all  $x^* \in \mathcal{D}(A^*)$  with  $Qx^* = 0$  we have  $QA^*x^* = 0$ .*

*Proof.* If  $Qx^* = 0$ , then  $i^*x^* = 0$  and hence  $Q_\infty A^*x^* = 0$ . This we combine with the simple observation that  $\ker Q_\infty \subseteq \ker Q$ , cf. [23, Lemma 5.2].  $\blacksquare$

*Example 8.5.* Let  $E = \mathbb{R}^2$  and let  $Q$  and  $\mathbf{S}$  be as in Example 5.5. Since Hypothesis ( $\mathbf{H}Q_\infty$ ) holds and  $E$  is finite-dimensional, Hypothesis ( $\mathbf{H}\mu_\infty$ ) trivially holds as well. Denote the centred Gaussian measure associated with  $Q_\infty$  by  $\mu_\infty$ . By direct computations, Fuhrman [20] showed that the associated Ornstein-Uhlenbeck semigroup  $\mathbf{P}^{\mathbb{C}}$  fails to be analytic on  $L_2^{\mathbb{C}}(E, \mu_\infty)$ . We derive this from Corollary 8.4, by noting that

$$Q \begin{pmatrix} 1 \\ 0 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix}, \quad QA^* \begin{pmatrix} 1 \\ 0 \end{pmatrix} = Q \begin{pmatrix} -1 \\ 1 \end{pmatrix} = \begin{pmatrix} 0 \\ 1 \end{pmatrix}.$$

Notice that  $\mathbf{S}_\infty$  is both contractive and analytic (its generator being bounded). Hence the same is true for its complexification  $\mathbf{S}_\infty^{\mathbb{C}}$ . This does not contradict Theorem 8.1; the point is that  $\mathbf{S}_\infty^{\mathbb{C}}$  fails to be an analytic contraction semigroup in the sense of the definition given at the beginning of this section. This can be verified explicitly by extending the computation in Example 5.5 to complex time. By doing so we obtain

$$\|\mathbf{S}_\infty^{\mathbb{C}}(z)\| = e^{-\operatorname{Re}z} (|z| + \sqrt{|z|^2 + 1}).$$

Let  $z = re^{i\theta}$  for a certain  $\theta \in (-\frac{\pi}{2}, \frac{\pi}{2}) \setminus \{0\}$ . Then

$$\|\mathbf{S}_\infty^{\mathbb{C}}(z)\| = e^{-r \cos \theta} (r + \sqrt{r^2 + 1}).$$

We claim that for any  $\theta \in (0, \frac{\pi}{2})$  we have

$$e^{-r \cos \theta} (r + \sqrt{r^2 + 1}) > 1 \tag{4.1}$$

for all sufficiently small  $r > 0$ . Indeed, (4.1) holds if and only if

$$f(r) = r + \sqrt{r^2 + 1} > e^{r \cos \theta} =: g_\theta(r)$$

for some  $r > 0$ . This is clearly true for small  $r > 0$  because  $f(0) = g_\theta(0) = 1$  and  $f'(0) = 1 > g'_\theta(0) = \cos \theta$ .

In the next corollary, which is a minor extension of [22, Corollary 2.5], we specialise Theorem 8.3 to Hilbert spaces  $E$ . We identify  $E$  and its dual in the usual way.

**Corollary 8.6** ( $\mathbf{H}\mu_\infty$ ). *Suppose  $E$  is a Hilbert space and  $Q \in \mathcal{L}(E)$  has a bounded inverse. Then the following assertions are equivalent:*

- (1) *The semigroup  $\mathbf{P}^{\mathbb{C}}$  extends to a analytic semigroup on  $L_2^{\mathbb{C}}(E, \mu_\infty)$ ;*
- (2) *The operator  $AQ_\infty$  extends to a bounded operator on  $H$ ;*
- (3) *The operator  $Q_\infty A^*$  extends to a bounded operator on  $H$ .*

The final result of this section is closely related to [28, Proposition 3.3].

**Proposition 8.7** ( $\mathbf{H}\mu_\infty$ ). *If the transition semigroup  $\mathbf{P}^\mathbb{C}$  is analytic, then  $D_H$  is closable.*

*Proof.* We introduce a densely defined operator  $(V, \mathcal{D}(V))$  from  $H_\infty$  to  $H$ ,

$$\begin{aligned}\mathcal{D}(V) &:= \{i_\infty^* x^* : x^* \in E^*\}, \\ V(i_\infty^* x^*) &:= i^* x^*, \quad x^* \in E^*.\end{aligned}$$

It was shown in [23]  $D_H$  is closable in  $L^2(E, \mu_\infty)$  if and only  $V$  is closable.

We will show that  $V$  is closable if  $\mathbf{P}^\mathbb{C}$  is analytic. The proof uses the trick from [28, Theorem 2.15]. Let  $i_\infty^* x_n^* \rightarrow 0$  in  $H_\infty$  and  $V(i_\infty^* x_n^*) = i^* x_n^* \rightarrow g$  in  $H$ ; we have to prove that  $g = 0$ . Fix  $\varepsilon > 0$  arbitrary and choose an index  $N$  large enough such that  $\|V(i_\infty^* x_n^* - i_\infty^* x_N^*)\|_H \leq \varepsilon$  and  $\|i_\infty^* x_n^*\| \leq \varepsilon$  for all  $n \geq N$ . Then for all  $n \geq N$  we have

$$\begin{aligned}\frac{1}{2}\|V(i_\infty^* x_n^*)\|_H^2 &= \frac{1}{2}\langle Qx_n^*, x_n^* \rangle = |\langle Q_\infty A^* x_n^*, x_n^* \rangle| \\ &\leq |\langle Q_\infty A^*(x_n^* - x_N^*), (x_n^* - x_N^*) \rangle| \\ &\quad + |\langle Q_\infty A^*(x_n^* - x_N^*), x_N^* \rangle| + |\langle Q_\infty A^* x_N^*, x_n^* \rangle| \\ &= \frac{1}{2}\|V(i_\infty^* x_n^* - i_\infty^* x_N^*)\|^2 \\ &\quad + |[Q_\infty A^*(x_n^* - x_N^*), i^* x_N^*]_H| + |[i_\infty^* A^* x_N^*, i_\infty^* x_n^*]_{H_\infty}| \\ &\leq \frac{1}{2}\|V(i_\infty^* x_n^* - i_\infty^* x_N^*)\|^2 \\ &\quad + C\|i^*(x_n^* - x_N^*)\|_H \|i^* x_N^*\|_H + \|i_\infty^* A^* x_N^*\|_{H_\infty} \|i_\infty^* x_n^*\|_{H_\infty} \\ &= \frac{1}{2}\|V(i_\infty^* x_n^* - i_\infty^* x_N^*)\|^2 \\ &\quad + C\|V(i_\infty^* x_n^* - i_\infty^* x_N^*)\|_H \|i^* x_N^*\|_H + \|i_\infty^* A^* x_N^*\|_{H_\infty} \|i_\infty^* x_n^*\|_{H_\infty} \\ &\leq \frac{1}{2}\varepsilon^2 + C\varepsilon M + \|i_\infty^* A^* x_N^*\|_{H_\infty} \|i_\infty^* x_n^*\|_{H_\infty},\end{aligned}$$

where  $C$  is the constant from Theorem 8.3(3) and  $M := \sup_n \|i^* x_n^*\|_H$  is finite since  $\lim_{n \rightarrow \infty} i^* x_n^* = g$ . Upon letting  $n \rightarrow \infty$ , it follows that

$$\limsup_{n \rightarrow \infty} \frac{1}{2}\|V(i_\infty^* x_n^*)\|_H^2 \leq \frac{1}{2}\varepsilon^2 + C\varepsilon M.$$

Since  $\varepsilon > 0$  was arbitrary, we conclude that  $g = \lim_{n \rightarrow \infty} V(i_\infty^* x_n^*) = 0$ . ■

## 9. ANALYTICITY AND INVARIANCE OF $H$

It turns out that there is a close relationship between analyticity of the Ornstein-Uhlenbeck semigroup and invariance of  $H$ . This will be the topic of the present section.

We start with a necessary condition for analyticity:

**Theorem 9.1** ( $\mathbf{H}\mu_\infty$ ). *If  $H_\infty \subseteq H$  and  $\mathbf{P}^\mathbb{C}$  is analytic, then  $\mathbf{S}^\mathbb{C}$  restricts to a bounded analytic  $C_0$ -semigroup on  $H^\mathbb{C}$ .*

*Proof.* <sup>2</sup> By Proposition 3.1 it suffices to check that there exists a constant  $M$  such that for all  $t \geq 0$  and  $x^* \in E^*$  we have

$$\|i^* S^*(t)x^*\|_H \leq M \|i^* x^*\|_H.$$

---

<sup>2</sup> The proof in the published version of this paper contains a mistake which has been corrected here.

Since  $A_\infty^*$  generates an analytic  $C_0$ -contraction semigroup on  $H_\infty$ , the form

$$\mathcal{E}(g, h) := -[A_\infty^* g, h]_{H_\infty}, \quad g, h \in \mathcal{D}(A_\infty^*),$$

is sectorial.

By Proposition 8.7 the operator  $V$  is closable as a densely defined operator from  $H_\infty$  to  $H$ . Let  $\bar{V}$  be its closure. Its domain,  $\mathcal{D}(\bar{V})$ , is a Banach space with respect to the graph norm  $\|h\|_{\mathcal{D}(\bar{V})}^2 := \|h\|_{H_\infty}^2 + \|\bar{V}h\|_H^2$ . Taking  $h = i_\infty^* x^*$  and using (2.1) we have two-sided estimate

$$\begin{aligned} \|\bar{V}i_\infty^* x^*\|_H^2 &\leq \|i_\infty^* x^*\|_{\mathcal{D}(\bar{V})}^2 = \|i_\infty^* x^*\|_{H_\infty}^2 + \|\bar{V}i_\infty^* x^*\|_H^2 \\ &\leq K^2 \|i_\infty^* x^*\|_H^2 + \|\bar{V}i_\infty^* x^*\|_H^2 = (K^2 + 1) \|\bar{V}i_\infty^* x^*\|_H^2, \end{aligned}$$

where  $K$  is the norm of the embedding  $H_\infty \hookrightarrow H$ . This shows that

$$\|h\|_{\mathcal{D}(\bar{V})} := \|\bar{V}h\|_H, \quad h \in \mathcal{D}(\bar{V}),$$

defines an equivalent norm on  $\mathcal{D}(\bar{V})$ .

We claim that  $\mathcal{D}(\bar{V})$  can be identified with the form domain of  $\mathcal{E}$ . By Theorem 8.3 the mapping  $B : i^* x^* \mapsto Q_\infty A^* x^*$ , defined on the dense subspace  $i^* \mathcal{D}(A^*)$  of  $H$ , takes values in  $H$  and extends to a bounded operator  $B$  on  $H$ . Moreover, for all  $x^*, y^* \in \mathcal{D}(A^*)$  we have  $i_\infty^* x^* \in \mathcal{D}(A_\infty^*)$  and

$$\mathcal{E}(i_\infty^* x^*, i_\infty^* y^*) = -[A_\infty^* i_\infty^* x^*, i_\infty^* y^*]_{H_\infty} = -[BV i_\infty^* x^*, V i_\infty^* y^*]_H.$$

Therefore, for all  $h \in \mathcal{D}(\bar{V})$ ,

$$\mathcal{E}(g, h) = -[B\bar{V}g, \bar{V}h]_H, \quad g, h \in \mathcal{D}(\bar{V}).$$

This proves the claim.

It follows from the general theory of sectorial operators that  $\mathcal{D}(\bar{V})$  is invariant under  $S_\infty^*$  and that the restriction of  $S_\infty^*$  to  $\mathcal{D}(\bar{V})$  is a bounded analytic  $C_0$ -semigroup. Therefore, for some constant  $m$  and all  $t \geq 0$  and  $x^* \in E^*$ ,

$$\begin{aligned} \|i^* S^*(t)x^*\|_H &= \|\bar{V}i_\infty^* S^*(t)x^*\|_H = \|\bar{V}S_\infty^*(t)i_\infty^* x^*\|_H \\ &= \|S_\infty^*(t)i_\infty^* x^*\|_{\mathcal{D}(\bar{V})} \leq c \|i_\infty^* x^*\|_{\mathcal{D}(\bar{V})} = c \|\bar{V}i_\infty^* x^*\|_H = c \|i^* x^*\|_H. \end{aligned}$$

This proves that  $H$  is  $S$ -invariant and that the restricted semigroup  $S_H$  is bounded. By Proposition 3.2,  $S_H$  is strongly continuous. It remains to prove that  $S_H$  is bounded analytic on  $H$ . Since  $S_\infty^*$  restricts to a bounded analytic  $C_0$ -semigroup on  $\mathcal{D}(\bar{V})$  there is a constant  $C$  such that for all  $t > 0$  and  $h \in \mathcal{D}(\bar{V})$ ,

$$\|\bar{V}A_\infty^* S_\infty^*(t)h\|_H \leq \frac{C}{t} \|\bar{V}h\|_H.$$

As above, taking  $h = i^* x^*$  this implies

$$\|i^* A^* S^*(t)x^*\|_H \leq \frac{C}{t} \|i^* x^*\|_H.$$

Therefore,  $\|A_H^* S_H^*(t)\|_H \leq \frac{C}{t}$ , which implies the result.  $\blacksquare$

We proceed with a partial converse.

**Theorem 9.2** ( $\mathbf{H}\mu_\infty$ ). <sup>3</sup> *Suppose that  $\mathbf{S}$  restricts to an analytic  $C_0$ -semigroup  $\mathbf{S}_H$  on  $H$  which is contractive in some equivalent Hilbertian norm on  $H^\mathbb{C}$ . Then  $\mathbf{P}^\mathbb{C}$  is analytic.*

*Proof.* We will show that  $\mathbf{S}_\infty^\mathbb{C}$  is an analytic contraction semigroup on  $H_\infty^\mathbb{C}$ . Once this is proved, the theorem follows by an appeal to Theorem 8.1.

Identifying  $H$  and its dual in the usual way, we define  $R_\infty \in \mathcal{L}(H)$  by

$$R_\infty h := \int_0^\infty S_H(t) S_H^* h dt \quad (h \in H).$$

Let  $i : H \hookrightarrow E$  denote the embedding; then we have  $Q_\infty = i \circ R_\infty \circ i^*$ . By an observation in Section 2, the RKHS's  $H_\infty = H_{Q_\infty}$  and  $H_{R_\infty}$  are canonically isometrically isomorphic as Hilbert spaces and identical as subsets of  $E$ . By complexifying, the same is true for their complexifications  $H_\infty^\mathbb{C}$  and  $H_{R_\infty}^\mathbb{C}$ . It follows that, in order to prove that  $\mathbf{S}_\infty^\mathbb{C}$  extends to an analytic contraction semigroup on  $H_\infty^\mathbb{C}$ , we may assume without loss of generality that  $E = H$  and  $Q = I$ . We also note that  $H_\infty = H_{R_\infty} \subseteq H$ .

Let  $\|\cdot\|$  be an equivalent Hilbertian norm on  $H^\mathbb{C}$  such that  $\|S_H^\mathbb{C}(z)\| \leq 1$  for all  $z$  in some sector containing the positive real axis, and let  $\llbracket \cdot, \cdot \rrbracket$  be the corresponding inner product. For all  $x, y \in H$  we have

$$\|x + iy\|^2 = \llbracket x + iy, x + iy \rrbracket = \llbracket x, x \rrbracket + \llbracket y, y \rrbracket = \|x\|^2 + \|y\|^2.$$

Hence  $(H^\mathbb{C}, \|\cdot\|)$  is the complexification of its real part, and we may apply the observation from Section 2 once more, this time to the isomorphism  $j : (H, \|\cdot\|) \simeq (H, \|\cdot\|)$ . It follows that the RKHS's associated with  $R_\infty$  and  $j \circ R_\infty \circ j^*$  are canonically isometrically isomorphic, and identical as subsets of  $H$ , and again the same is true for their complexifications. Thus, in order to prove that  $\mathbf{S}_\infty^\mathbb{C}$  extends to an analytic contraction semigroup on  $H_\infty^\mathbb{C}$ , it even suffices to prove this for the case where  $\mathbf{S}_H^\mathbb{C}$  extends to an analytic contraction semigroup on  $H^\mathbb{C}$ .

It is well known that

$$H_\infty = \left\{ \int_0^\infty S_H(t) f(t) dt : f \in L^2(\mathbb{R}_+; H) \right\}$$

with norm given by

$$\|h_\infty\|_{H_\infty} = \inf \left\{ \|f\|_{L^2(\mathbb{R}_+; H)} : h_\infty = \int_0^\infty S_H(t) f(t) dt \right\},$$

cf. [15, Appendix B]. Upon complexifying we see that

$$H_\infty^\mathbb{C} = \left\{ \int_0^\infty S_H^\mathbb{C}(t) f(t) dt : f \in L^2(\mathbb{R}_+; H^\mathbb{C}) \right\}$$

with norm given by

$$(9.1) \quad \|h_\infty\|_{H_\infty^\mathbb{C}} = \inf \left\{ \|f\|_{L^2(\mathbb{R}_+; H^\mathbb{C})} : h_\infty = \int_0^\infty S_H^\mathbb{C}(t) f(t) dt \right\}.$$

<sup>3</sup> In the published version of the paper, the word ‘analytic’ was missing in the first line of the statement of the result.

Indeed, the representation of  $H_\infty^{\mathbb{C}}$  follows immediately by considering real and imaginary parts of elements in  $H_\infty^{\mathbb{C}}$  separately, and the expression (9.1) for the complexified norm is proved as follows. Denote the infimum on the right hand side of (9.1) by  $I^{\mathbb{C}}$ . Fix  $h_\infty \in H_\infty^{\mathbb{C}}$  and write  $h_\infty = a_\infty + ib_\infty$  with  $a_\infty, b_\infty \in H_\infty$ . Fix  $\varepsilon > 0$  arbitrary and choose  $f, g \in L^2(\mathbb{R}_+, H)$  representing  $a_\infty, b_\infty$  such that

$$\|f\|_{L^2(\mathbb{R}_+, H)}^2 \leq \|a_\infty\|_{H_\infty}^2 + \varepsilon, \quad \|g\|_{L^2(\mathbb{R}_+, H)}^2 \leq \|b_\infty\|_{H_\infty}^2 + \varepsilon.$$

Then,

$$\begin{aligned} \|f + ig\|_{L^2(\mathbb{R}_+, H^{\mathbb{C}})}^2 &= \|f\|_{L^2(\mathbb{R}_+, H)}^2 + \|g\|_{L^2(\mathbb{R}_+, H)}^2 \\ &\leq (\|a_\infty\|_{H_\infty}^2 + \varepsilon) + (\|b_\infty\|_{H_\infty}^2 + \varepsilon) = \|h_\infty\|_{H_\infty^{\mathbb{C}}}^2 + 2\varepsilon, \end{aligned}$$

which shows that  $I^{\mathbb{C}} \leq \|h_\infty\|_{H_\infty^{\mathbb{C}}}$ . On the other hand, if  $f, g \in L^2(\mathbb{R}_+, H)$  are arbitrary functions representing  $a_\infty, b_\infty$ , then

$$\|h_\infty\|_{H_\infty^{\mathbb{C}}}^2 = \|a_\infty\|_{H_\infty}^2 + \|b_\infty\|_{H_\infty}^2 \leq \|f\|_{L^2(\mathbb{R}_+, H)}^2 + \|g\|_{L^2(\mathbb{R}_+, H)}^2 = \|f + ig\|_{L^2(\mathbb{R}_+, H^{\mathbb{C}})}^2,$$

which gives the converse inequality  $\|h_\infty\|_{H_\infty^{\mathbb{C}}} \leq I^{\mathbb{C}}$ . This proves (9.1).

Now it is easy to finish the proof. Given  $h_\infty \in H_\infty^{\mathbb{C}}$ , choose an arbitrary  $f \in L^2(\mathbb{R}_+; H^{\mathbb{C}})$  representing  $h_\infty$ :

$$h_\infty = \int_0^\infty S_H^{\mathbb{C}}(t)f(t) dt.$$

Then, for any  $z$  in the sector where  $\mathbf{S}_H^{\mathbb{C}}$  is contractive, we have

$$S_H^{\mathbb{C}}(z)h_\infty = S_H^{\mathbb{C}}(z) \int_0^\infty S_H^{\mathbb{C}}(t)f(t) dt = \int_0^\infty S_H^{\mathbb{C}}(t)[S_H^{\mathbb{C}}(z)f(t)] dt.$$

It follows that  $S_H^{\mathbb{C}}(z)h_\infty \in H_\infty^{\mathbb{C}}$ , with norm

$$\|S_H^{\mathbb{C}}(z)h_\infty\|_{H_\infty^{\mathbb{C}}} \leq \|S_H^{\mathbb{C}}(z)f(\cdot)\|_{L^2(\mathbb{R}_+; H^{\mathbb{C}})} \leq \|f(\cdot)\|_{L^2(\mathbb{R}_+; H^{\mathbb{C}})}.$$

Taking the infimum over all representing functions  $f$ , we obtain

$$\|S_H^{\mathbb{C}}(z)h_\infty\|_{H_\infty^{\mathbb{C}}} \leq \|h_\infty\|_{H_\infty^{\mathbb{C}}}.$$

It follows that the operators  $S_H^{\mathbb{C}}(z)$  restrict to a contractions on  $H_\infty^{\mathbb{C}}$ . The restriction of  $\mathbf{S}_H^{\mathbb{C}}$  to  $H_\infty^{\mathbb{C}}$  agrees with  $\mathbf{S}_\infty^{\mathbb{C}}$  for real time, and it is routine to check that it is strongly continuous and analytic.  $\blacksquare$

Notice that there is only a small gap between Theorems 9.1 and 9.2. The assumption  $H_\infty \subseteq H$  in Theorem 9.1 implies that  $\mathbf{S}_H$  is uniformly exponentially stable, and conversely the assumption in Theorem 9.2 that  $\mathbf{S}_H$  is uniformly exponentially stable implies that  $H_\infty \subseteq H$ .

The assumptions of Theorem 9.2 are fulfilled when  $E$  is a Hilbert space,  $H = E$ , and  $\mathbf{S}$  satisfies an estimate of the type  $\|S(t)\| \leq e^{-\omega t}$  for some  $\omega > 0$  and all  $t \geq 0$ . In this special setting, the theorem is due to Da Prato [12], who proved it by using interpolation theory and maximal regularity.

*Remark 9.3.* The existence of an equivalent Hilbertian norm on  $H^{\mathbb{C}}$  in Theorem 9.2 is equivalent to the existence of an isomorphism  $T : H^{\mathbb{C}} \rightarrow H^{\mathbb{C}}$  such that

$$(9.2) \quad \|T^{-1}S_H^{\mathbb{C}}(z)T\| \leq 1$$

for all  $z$  in some sector containing the positive real line. The question when such an isomorphism exists is related to a famous question posed by Halmos in [25]. For bounded analytic semigroups, this question was answered recently by Le Merdy [27]. To quote his answer let us recall first that if  $\ker A = \{0\}$  and  $A$  generates a bounded analytic semigroup on  $H^{\mathbb{C}}$ , then for any  $s \in \mathbb{R}$  one can define a closed operator  $(-A)^{is}$ . We say that  $A$  has *bounded imaginary powers* (briefly,  $A \in \text{BIP}$ ) if  $(-A)^{is}$  is bounded for all  $s \in \mathbb{R}$  and the function  $s \mapsto \|(-A)^{is}\|$  is locally bounded on  $\mathbb{R}$ .

It is known that  $A \in \text{BIP}$  in the following important cases:

- (a)  $A$  is  $m$ -dissipative on  $H$ ;
- (b)  $A$  is normal and sectorial on  $H$ ;
- (c)  $A$  generates a bounded  $C_0$ -group on  $H$ .

By an example of Baillon and Clément [3], there exist analytic semigroups on Hilbert spaces which are uniformly bounded on a sector, but whose generator does not belong to BIP.

Le Merdy [27] proved that for a bounded analytic semigroup whose generator  $A$  satisfies  $\ker A = \{0\}$ , (9.2) holds if and only if  $A \in \text{BIP}$ .

In the situation of Theorem 9.2,  $\mathbf{S}_H$  is uniformly exponentially stable and therefore  $0 \notin \sigma(A_H)$ . Hence the condition  $\ker A_H = \{0\}$  is trivially fulfilled.

In the example below we consider a stochastic linear heat equation with correlated cylindrical noise in  $L^p(\Omega)$  with  $p \in [2, \infty)$ . Similar equations were considered in [14], where it is a starting point for the analysis of nonlinear stochastic differential equations with dissipative drifts.

*Example 9.4.* Let  $\Omega$  be a bounded open domain in  $\mathbb{R}^d$  with  $C^2$ -boundary, let  $2 \leq p < \infty$ , and let  $A$  be the  $L^p(\Omega)$ -realization of a uniformly elliptic differential operator of the form

$$A_0 = \sum_{i,j=1}^d a_{ij} \partial_{ij} + \sum_{i=1}^d b_i \partial_i$$

with domain  $\mathcal{D}(A) = W^{2,p}(\Omega) \cap W_0^{1,p}(\Omega)$ . We assume that the coefficients  $a_{ij} = a_{ji}$  belong to  $C^\theta(\bar{\Omega})$  for a certain  $\theta \in (0, 1)$  and that the functions  $b_i$  are bounded and measurable on  $\Omega$ . Under these assumptions it is known that  $A$  generates a uniformly exponentially stable and analytic  $C_0$ -semigroup  $\mathbf{S}$  in  $L^p(\Omega)$ ; cf. [1].

In  $E = L^p(\Omega)$  we consider a stochastic evolution equation

$$dX(t) = AX(t) dt + dW_H(t).$$

Here  $H$  is a separable Hilbert space which is continuously embedded into  $E$  and  $\{W_H(t)\}_{t \geq 0}$  is a (possibly cylindrical) Wiener process with Cameron-Martin space  $H$ .

We will consider two cases. To simplify notations, we will not distinguish between real spaces and their complexifications.

(a) We take  $p = 2$ ,  $E = L^2(\Omega)$ , and  $H = H_0^\beta(\Omega)$  with  $\beta \geq 0$  and  $\beta > \frac{d}{4} - \frac{1}{2}$ . By a result in [34]  $A \in \text{BIP}$ . In case  $\beta = 0$  we can apply Le Merdy's result to find an equivalent Hilbertian norm in which  $\mathbf{S}_H = \mathbf{S}$  is an analytic contraction semigroup. In case  $\beta > 0$ , the fact that  $A \in \text{BIP}$  implies that  $H$  equals the interpolation space  $D_A(\frac{\beta}{2}, 2)$  up to an equivalent norm. Then by interpolation theory,  $\mathbf{S}_H$  is an analytic  $C_0$ -semigroup on  $H$  which is contractive with respect to the  $D_A(\frac{\beta}{2}, 2)$  norm. From [1] we have

$$\int_0^\infty \|S(t) \circ i_\beta\|_{\mathcal{L}_2(H_0^\beta(\Omega), L^2(\Omega))}^2 dt = \int_0^\infty \|A^{-\frac{\beta}{2}} S(t)\|_{\mathcal{L}_2(L^2(\Omega))}^2 dt < \infty,$$

where  $\|\cdot\|_{\mathcal{L}_2}$  denotes the Hilbert-Schmidt norm and  $i_\beta : H_0^\beta(\Omega) \hookrightarrow L^2(\Omega)$  is the inclusion mapping. This implies that Hypothesis  $(\mathbf{H}\mu_\infty)$  is satisfied.

By Theorem 9.2, the associated Ornstein-Uhlenbeck semigroup  $\mathbf{P}$  is analytic in  $L^2(E, \mu_\infty)$ .

(b) Let  $p \in (2, \infty)$ ,  $E = L^p(\Omega)$ , and  $H = H_0^\alpha(\Omega)$  with  $\alpha > \frac{1}{2} - \frac{1}{p}$  (in dimension  $d = 1$ ) or  $\alpha > d(\frac{3}{4} - \frac{1}{p}) - \frac{1}{2}$  (in dimensions  $d \geq 2$ ). In both cases we may choose  $\beta \geq 0$  with  $\beta > \frac{d}{4} - \frac{1}{2}$  and  $\gamma > d(\frac{1}{2} - \frac{1}{p})$  such that  $\alpha > \beta + \gamma$ . By the Sobolev embedding theorem we have a continuous inclusions  $H \hookrightarrow H_0^\beta(\Omega) \hookrightarrow E$ . Then  $H$  equals the interpolation space  $D_A(\frac{\alpha}{2}, 2)$  up to an equivalent norm. Again  $H$  is invariant for  $\mathbf{S}$ , and  $\mathbf{S}_H$  is an analytic  $C_0$ -semigroup on  $H$  which is contractive with respect to the  $D_A(\frac{\alpha}{2}, 2)$  norm.

We will show next that Hypothesis  $(\mathbf{H}\mu_t)$  is satisfied in  $E$ . The argument will be somewhat informal but can easily be rewritten in a rigorous way. First recall that the realization of  $A$  in  $L^2(\Omega)$  belongs to BIP, from which it follows that  $H = \mathcal{D}((-A)^{\frac{\alpha}{2}})$  with equivalent norms. Hence,  $(-A)^{\frac{1}{2}(\alpha-\beta)}$  is an isomorphism from  $H$  onto  $H_0^\beta(\Omega)$ , and

$$W_A(t) := (-A)^{\frac{1}{2}(\alpha-\beta)} W_H(t)$$

defines a cylindrical Wiener process whose Cameron-Martin space equals  $H_0^\beta(\Omega)$  up to an equivalent norm. Then by case (b), the  $L^2(\Omega)$ -valued process

$$Y(t) := \int_0^t S(t-s) dW_A(s), \quad t \geq 0$$

solves the equation

$$dY(t) = AY(t) dt + dW_A(t)$$

with initial condition  $Y(0) = 0$  in  $L^2(\Omega)$ . Then the process  $\{X(t)\}_{t \geq 0}$  defined by

$$X(t) := (-A)^{-\frac{1}{2}(\alpha-\beta)} Y(t)$$

takes values in  $H$ , hence in  $E$ , and solves the original equation in  $E$ ,

$$dX(t) = AX(t) dt + dW_H(t)$$

with initial condition  $X(0) = 0$ . It follows from Proposition 6.1 that Hypothesis  $(\mathbf{H}\mu_t)$  is satisfied in  $E$ . This proves the claim. By [31], the uniform exponential stability of  $\mathbf{S}$  in  $E$  now implies that also  $(\mathbf{H}\mu_\infty)$  is satisfied in  $E$ .

In conclusion, Theorem 9.2 applies and we find that the Ornstein-Uhlenbeck semigroup  $\mathbf{P}$  is analytic in  $L^2(E, \mu_\infty)$ .

## REFERENCES

- [1] S.A. AGMON, “Lectures on Elliptic Boundary Value Problems”, Van Nostrand, 1965.
- [2] W. ARENDT AND N. NIKOLSKI, *Vector-valued holomorphic functions revisited*, Math. Z. **234** (2000), 777–805.
- [3] J.-B. BAILLON AND PH. CLÉMENT, *Examples of unbounded imaginary powers of operators*, J. Funct. Anal. **100** (1991), 419–434.
- [4] V. I. BOGACHEV, M. ROCKNER AND B. SCHMULAND, *Generalized Mehler semigroups and applications*, Probab. Theory Related Fields **105** (1996), 193–225
- [5] Z. BRZEŹNIAK, *Stochastic partial differential equations in  $M$ -type 2 Banach spaces*, Potential Anal. **4** (1995), 1–45.
- [6] Z. BRZEŹNIAK AND J.M.A.M. VAN NEERVEN, *Stochastic convolution in separable Banach spaces and the stochastic linear Cauchy problem*, Studia Math. **143** (2000), 43–74.
- [7] S. CERRAI AND F. GOZZI, *Strong solutions of Cauchy problems associated to weakly continuous semigroups*, Diff. Integral Eq. **8** (1995), 465–486.
- [8] A. CHOJNOWSKA-MICHALIK AND B. GOLDYS, *Nonsymmetric Ornstein-Uhlenbeck semigroup as second quantized operator*, J. Math. Kyoto Univ. **36** (1996), 481–498.
- [9] A. CHOJNOWSKA-MICHALIK AND B. GOLDYS, *Nonsymmetric Ornstein-Uhlenbeck generators*, in: “Infinite Dimensional Stochastic Analysis” (Amsterdam, 1999) Verh. Afd. Natuurkd., 1st Series, Vol. 52, Royal. Netherl. Acad. Arts Sci., Amsterdam, 2000, pp. 99–116.
- [10] A. CHOJNOWSKA-MICHALIK AND B. GOLDYS, *Symmetric Ornstein-Uhlenbeck semigroups and their generators*, Probab. Theory Relat. Fields. **124** (2002), 459–486.
- [11] K.L. CHUNG AND R.J. WILLIAMS, “Introduction to Stochastic Integration”, 2nd ed., Birkhäuser Verlag, 1990.
- [12] G. DA PRATO, *Null controllability and strong Feller property of Markov transition semigroups*, Nonlinear Anal. **25** (1995), 941–949.
- [13] G. DA PRATO, *Bounded perturbations of Ornstein-Uhlenbeck semigroups*, in: “Evolution Equations, Semigroups and Functional Analysis” (Milano, 2000), 97–114, Progr. Nonlinear Differential Equations Appl., Vol. 50, Birkhäuser, Basel, 2002.
- [14] G. DA PRATO, “Stochastic Evolution Equations by Semigroup Methods”, Centre de Recerca Matemàtica, Quaderns nùm. 11, 1998.
- [15] G. DA PRATO AND J. ZABCZYK, “Stochastic Equations in Infinite Dimensions”, Encyclopedia of Mathematics and its Applications, Cambridge University Press, Cambridge, 1992.
- [16] J. DIESTEL AND J.J. UHL, “Vector Measures”, Math. Surveys, Vol. 15, Amer. Math. Soc., Providence, R.I. (1977).
- [17] R. Engelking, “General Topology”, Revised edition, Heldermann Verlag, Berlin, 1989.
- [18] K.-J. ENGEL AND R. NAGEL, “One-Parameter Semigroups for Linear Evolution Equations”, Graduate Texts Math., Vol. 194, Springer-Verlag, 2000.
- [19] D. FREMLIN, D. GARLING AND R. HAYDON, *Bounded measures on topological spaces*, Proc. London Math. Soc. **25** (1972), 115–136
- [20] M. FUHRMAN, *Analyticity of transition semigroups and closability of bilinear forms in Hilbert spaces*, Studia Math. **115** (1995), 53–71.
- [21] J.A. GOLDSTEIN, “Semigroups of Linear Operators and Applications”, Oxford University Press, Oxford, 1985.
- [22] B. GOLDYS, *On analyticity of Ornstein-Uhlenbeck semigroups*, Atti Accad. Naz. Lincei Cl. Sci. Fis. Mat. Natur. Rend. Lincei (9) Mat. Appl. **10** (1999), 131–140.
- [23] B. GOLDYS, F. GOZZI, AND J.M.A.M. VAN NEERVEN, *On closability of directional gradients*, Potential Anal. **18** (2003), 289–310.
- [24] B. GOLDYS AND M. KOCAN, *Diffusion semigroups in spaces of continuous functions with mixed topology*, J. Differential Equations **173** (2001), 17–39.
- [25] P. HALMOS, *Ten problems in Hilbert space*, Bull. Amer. Math. Soc. **176** (1970), 887–933.

---

<sup>4</sup>The references have been updated.

- [26] E. HILLE AND R.S. PHILLIPS, “Functional Analysis and Semi-Groups”, Amer. Math. Soc. Colloq. Publ., Vol. XXXI, Rev. Ed., Providence, R.I., 1957
- [27] C. LE MERDY, *The similarity problem for bounded analytic semigroups on Hilbert spaces*, Semigroup Forum **56** (1998), 205–224.
- [28] Z.-M. MA AND M. RÖCKNER, “Introduction to the Theory of (Non-Symmetric) Dirichlet Forms”, Springer-Verlag, Berlin, 1992.
- [29] J.M.A.M. VAN NEERVEN, *Nonsymmetric Ornstein-Uhlenbeck semigroups in Banach spaces*, J. Funct. Anal. **155** (1998), 495–535.
- [30] J.M.A.M. VAN NEERVEN, *Uniqueness of invariant measures for the stochastic Cauchy problem in Banach spaces*, in: “Recent Advances in Operator Theory and Related Topics: The Bela Szökefalvi-Nagy Memorial Volume”, Operator theory: Advances and Applications, Vol. 127, Birkhauser, 2001, pp. 491–517.
- [31] J.M.A.M. VAN NEERVEN AND L. WEIS, *Weak limits and integrals of Gaussian covariances in Banach spaces*, Probab. Math. Statist. **25** (2005), 55–74
- [32] A. PAZY, “Semigroups of Linear Operators and Applications to Partial Differential Equations”, Springer-Verlag, Berlin, 1983.
- [33] E. PRIOLA, *On a class of Markov type semigroups in spaces of uniformly continuous and bounded functions*, Studia Math. **136** (1999), 271–295.
- [34] J. PRÜSS AND H. SOHR, *Imaginary powers of elliptic second order differential operators in  $L^p$ -spaces*, Hiroshima Math. J. **23** (1993), 161–192.
- [35] F.D. SENTILLES, *Bounded continuous functions on a completely regular space*, Trans. Amer. Math. Soc. **168** (1972), 311–336.
- [36] N.N. VAKHANIA, V.I. TARIELADZE, AND S.A. CHOBANYAN, “Probability Distributions in Banach Spaces”, D. Reidel Publishing Company, Dordrecht-Boston-Lancaster-Tokyo, 1987.
- [37] VŨ QUÔC PHÔNG, *The operator equation  $AX - XB = C$  with unbounded operators  $A$  and  $B$  related to abstract Cauchy problems*, Math. Z. **208** (1991), 567–588.
- [38] R.F. WHEELER, *A survey of Baire measures and strict topologies*, Exposition. Math. **1** (1983), 97–190.
- [39] A. WIWEGER, *Linear spaces with mixed topology*, Studia Math. **20** (1961), 47–68.
- [40] S. YOSIDA, “Functional Analysis”, 6th edition, Grundlehren der Mathematischen Wissenschaften, Vol. 123, Springer-Verlag, Berlin-New York, 1980.
- [41] J. ZABCZYK, *Linear stochastic systems in Hilbert spaces; spectral properties and limit behaviour*, Banach Center Publications **41** (1985), 591–609.

SCHOOL OF MATHEMATICS, THE UNIVERSITY OF NEW SOUTH WALES, SYDNEY 2052, AUSTRALIA  
*E-mail address:* B.Goldys@unsw.edu.au

DEPARTMENT OF APPLIED MATHEMATICAL ANALYSIS, TECHNICAL UNIVERSITY OF DELFT, P.O. BOX 5031, 2600 GA DELFT, THE NETHERLANDS  
*E-mail address:* J.M.A.M.vanNeerven@tudelft.nl