Polynomially bounded operators whose spectrum on the unit circle has measure zero

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Abstract. It is known from the Sz.-Nagy–Foias theory of operators that if T is a Hilbert space contraction of class $C_{1\bullet}$ and if the unitary spectrum $\sigma(T) \cap \partial\mathbf{D}$ is of Lebesgue measure zero, then T is a singular unitary operator. We extend this statement to polynomially bounded operators acting on arbitrary Banach spaces, presenting also its local version. It is shown how the method applied provides Katznelson–Tzafriri type theorems. One-parameter semigroups of Hilbert space contractions are also considered.

0. Introduction

It is known from the Sz.-Nagy–Foias theory of operators [23] that if T is a Hilbert space contraction of class $C_{1\bullet}$ (cf. the definition below) and if the *unitary spectrum* $\sigma(T) \cap \partial\mathbf{D}$ is of Lebesgue measure zero, then T is a singular unitary operator. In the present paper we extend this statement to polynomially bounded operators acting on arbitrary Banach spaces. Recall that a bounded linear operator $T \in \mathcal{L}(\mathcal{X})$ acting on the (complex) Banach space $\mathcal{X}$ is *polynomially bounded* if there exists a constant $M > 0$ such that $\|p(T)\| \le M\|p\|$ for all polynomials p. Here, and in the sequel, $\|f\| := \sup\{|f(z)| : |z| = 1\}$ is the supremum norm for an element $f \in C(\partial\mathbf{D})$, the space of continuous functions defined on the unit circle $\partial\mathbf{D} = \partial\{z \in \mathbf{C} : |z| < 1\}$. We prove that if T is a polynomially bounded operator of class $C_{1\bullet}$ whose unitary spectrum $\sigma(T) \cap \partial\mathbf{D}$ has Lebesgue measure zero, then T is similar to an invertible isometry. We also prove a local version of this result. By the same method we further extend the Esterle–Strouse–Zouakia version of the Katznelson–Tzafriri

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theorem to polynomially bounded operators. One-parameter semigroups of Hilbert space contractions are considered as well.

1. Polynomially bounded operators

Let us consider a polynomially bounded linear operator $T \in \mathcal{L}(\mathcal{X})$ acting on a Banach space $\mathcal{X}$. Denoting by $A(\mathbf{D})$ the disc algebra, i.e. the closure of the set of polynomials in $C(\partial\mathbf{D})$, it is clear that T admits an $A(\mathbf{D})$-functional calculus. To be specific, this means that there exists an algebra-homomorphism $\Phi : A(\mathbf{D}) \to \mathcal{L}(\mathcal{X})$ such that $1 \mapsto I$ and $\chi \mapsto T$, where $\chi(z) := z$; moreover $\|\Phi\| \leq M$. We use the notation $u(T) := \Phi(u)$, $u \in A(\mathbf{D})$.

Let $L : \ell^\infty(\mathbf{N}) \to \mathbf{C}$ be a generalized Banach limit, i.e. a positive, bounded, linear functional on $\ell^\infty(\mathbf{N})$ such that $L(\mathbf{1}) = 1$ and $L(\{x_n\}_{n\in\mathbf{N}}) = \alpha$ whenever $\lim_{n\to\infty} x_n = \alpha$. We define a new semi-norm ψ on $\mathcal{X}$ by

$$\psi(x) := L(\{\|T^n x\|\}_{n\in\mathbf{N}})$$

and consider the linear subspace

$$\mathcal{X}_0(T) := \{x \in \mathcal{X} : \psi(x) = 0\}.$$

Then ψ induces a norm on the quotient space $\mathcal{X}/\mathcal{X}_0(T)$; let $\mathcal{Y}$ be the completion of $\mathcal{X}/\mathcal{X}_0(T)$ with respect to this norm. Since T is power bounded, that is $\sup_{n\in\mathbf{N}} \|T^n\| (\leq M) < \infty$, it follows that $\inf_{n\in\mathbf{N}} \|T^n x\| = 0$ if and only if $\lim_{n\to\infty} \|T^n x\| = 0$. Hence $\mathcal{X}_0(T)$ coincides with the set of vectors $x \in \mathcal{X}$ satisfying the condition $\inf_{n\in\mathbf{N}} \|T^n x\| = 0$. It is immediate that $X : \mathcal{X} \to \mathcal{Y}$, $x \mapsto x + \mathcal{X}_0(T)$ is a bounded, linear mapping having the properties $\ker X = \mathcal{X}_0(T)$, $\overline{\operatorname{ran} X} = \mathcal{Y}$ and $\|X\| \leq M$. Exploiting the fact that L is a generalized Banach limit we can easily infer that there exists a unique isometry $V \in \mathcal{L}(\mathcal{Y})$ such that $VX = XT$. Furthermore, given any operator C in the commutant

$$\{T\}' := \{C \in \mathcal{L}(\mathcal{X}) : CT = TC\}$$

of T there exists a unique operator $D \in \{V\}'$ such that $XC = DX$. The mapping $\gamma : \{T\}' \to \{V\}'$, $C \mapsto D$ is a contractive algebra-homomorphism, and $\sigma(C) \supset \sigma(D)$ holds, for every $C \in \{T\}'$. As a consequence, we obtain that the isometry V is also polynomially bounded, having the same bound M as T has. For more details see [22], [23, Section II.5.2], [14], [15], [16] and [24].

Let us now suppose that V is an *invertible* polynomially bounded isometry. Since $V - zI$ is bounded from below and so $z \neq \partial\sigma(V)$ for all $z \in \mathbf{D}$, it follows that the spectrum $\sigma(V)$ is located on the circle $\partial\mathbf{D}$. Since V is polynomially bounded, we can extend the $A(\mathbf{D})$-functional calculus of V to a $C(\partial\mathbf{D})$-functional calculus. Indeed, for any trigonometric function $q = \sum_{n=-N}^{N} a_n\chi^n \in C(\partial\mathbf{D})$ let us define $q(V) := \sum_{n=-N}^{N} a_n V^n$. It is clear that $\|q(V)\| = \|V^N q(V)\| \leq M\|\chi^N q\| = M\|q\|$. Thus there exists an algebra-homomorphism $\Psi : C(\partial\mathbf{D}) \to \mathcal{L}(\mathcal{Y})$ such that $1 \mapsto I$ and $\chi \mapsto V$, and we have $\|\Psi\| \leq M$. We will use the notation $f(V) := \Psi(f),\ f \in C(\partial\mathbf{D})$.

The first lemma states that the $C(\partial\mathbf{D})$-functional calculus of V has the spectral mapping property and that the operator $f(V)$ is completely determined by the values of the function $f \in C(\partial\mathbf{D})$ taken on the spectrum $\sigma(V)$.

Lemma 1.1. *Let $V \in \mathcal{L}(\mathcal{Y})$ be an invertible, polynomially bounded isometry. Then:*

(i) $\sigma(f(V)) = f(\sigma(V))$ *holds for every* $f \in C(\partial\mathbf{D})$*;*

(ii) $f_1(V) = f_2(V)$ *if* $f_1|\sigma(V) = f_2|\sigma(V)$ $(f_1, f_2 \in C(\partial\mathbf{D}))$.

Proof. For the sake of reader's convenience we sketch the proof, which is standard application of Banach algebra techniques, and is patterned after the proof of [11, Theorem 2.5].

Let $\mathcal{A}$ be a maximal abelian subalgebra of $\mathcal{L}(\mathcal{Y})$ containing V and the identity I, and let Σ be the set of all non-zero complex homomorpisms of $\mathcal{A}$. The first statement is an immediate consequence of the relation $\sigma(f(V)) = \{\varphi(f(V)) : \varphi \in \Sigma\}$ and the fact that the trigonometric polynomials are dense in $C(\partial\mathbf{D})$.

If the function $f \in C(\partial\mathbf{D})$ is zero in a neighbourhood of $\sigma(V)$ then we can choose a function $g \in C(\partial\mathbf{D})$ such that $g|\sigma(V) = 1$ and $fg = 0$. Since $\sigma(g(V)) = g(\sigma(V)) = \{1\}$ it follows that $g(V)$ is invertible, and so the relation $f(V)g(V) = (fg)(V) = 0$ implies that $f(V) = 0$. If $f \in C(\partial\mathbf{D})$ vanishes only on the spectrum $\sigma(V)$, then there exists a sequence $\{f_n\}_{n\in\mathbf{N}}$ in $C(\partial\mathbf{D})$ such that $\|f_n - f\| \to 0$ and each f_n vanishes in a neighbourhood of $\sigma(V)$. Q.E.D.

We say that the polynomially bounded operator $T \in \mathcal{L}(\mathcal{X})$ is *of class* $C_{1\bullet}$ if for all non-zero $x \in \mathcal{X}$ we have

$$\inf_{n\in\mathbf{N}} \|T^n x\| > 0;$$

cf. [23, Section II.4], [6, Chapter XII] and [14]. It is clear that in that case the transformation $X \in \mathcal{L}(\mathcal{X}, \mathcal{Y})$ is injective. Since $\overline{\operatorname{ran} X} = \mathcal{Y}$ also holds, X is a *quasiaffinity* in the terminology of [23].

The following statement is a Banach space version of [1, Theorem 1] (see [17, Proposition 1] for its predecessor), and extends the Sz.-Nagy–Foias theorem mentioned earlier (see [23, Proposition II.6.7] and [23, Theorem I.3.2]). See also [2, Section 4] and [6, Proposition XII.2.1] for related results.

Theorem 1.2. *Let $T \in \mathcal{L}(\mathcal{X})$ be a polynomially bounded operator, with bound M, of class $C_{1\bullet}$, and assume that the Lebesgue measure of $\sigma(T) \cap \partial\mathbf{D}$ is zero. Then for the operators $V \in \mathcal{L}(\mathcal{Y}), X \in \mathcal{L}(\mathcal{X}, \mathcal{Y})$ introduced before we have ran $V = \mathcal{Y}$ and*

$$M^{-1}\|x\| \leq \|Xx\| \leq M\|x\| \ \text{ for every } \ x \in \mathcal{X}.$$

Thus, X is a topological isomorphism from $\mathcal{X}$ onto $\mathcal{Y}$ and T is similar to the invertible isometry V. If $M = 1$, then X is a surjective isometry from $\mathcal{X}$ onto $\mathcal{Y}$ and T is an invertible isometry.

Proof. We know that $\sigma(V)$ is contained in $\sigma(T)$. Since $\sigma(V)$ does not coincide with the closed disc, it follows that $\sigma(V)$ is located on $\partial\mathbf{D}$.

By the Fatou–Rudin Theorem (see e.g. [13, Chapter 6]), for each $n \in \mathbf{N}$ there exists a function $f_n \in A(\mathbf{D})$ such that $f_n|\sigma(V) = \chi^{-n}|\sigma(V)$ and $\|f_n\| = 1$. The relation $XT = VX$ implies that $Xf_n(T) = f_n(V)X$. On the other hand, on account of Lemma 1.1 we infer that $f_n(V) = \chi^{-n}(V) = V^{-n}$. Hence $X = V^nXf_n(T) = XT^nf_n(T)$, and since X is injective we obtain that T is invertible and $f_n(T) = T^{-n}$. Therefore $\|T^{-n}\| = \|f_n(T)\| \leq M$ for every $n \in \mathbf{N}$. Let us recall that $\|Xx\|$ is a Banach limit of the sequence $\{\|T^nx\|\}_{n=1}^{\infty}$. Since $\|x\| \leq \|T^{-n}\|\|T^nx\| \leq M\|T^nx\|$, it follows that $\|Xx\| \geq M^{-1}\|x\|$. Q.E.D.

Remarks. 1. The preceding proof works also if we only assume that $\sigma(V) \cap \partial\mathbf{D}$ is of measure zero.

2. If T is a Hilbert space contraction, then T is polynomially bounded with bound $M = 1$ by the von Neumann inequality. Consequently the result above includes the Sz.-Nagy–Foias theorem.

3. If T is any Banach space contraction of class $C_{1\bullet}$ such that the unitary spectrum $\sigma(T) \cap \partial\mathbf{D}$ is *countable*, then T is an invertible isometry. This result was proved recently by Batty, Brzeźniak and Greenfield [3].

We proceed with a local version of Theorem 1.2. Recall that the *local spectrum* $\sigma(T, x)$ of the operator $T \in \mathcal{L}(\mathcal{X})$ at the vector $x \in \mathcal{X}$ is the complement of the union of all open subsets Ω of $\mathbf{C}$ for which there exists an analytic function $F : \Omega \to \mathcal{X}$ such that $(zI - T)F(z) = x$ for all $z \in \Omega$. This object is most tractable if T has the *single-valued extension property* (SVEP), i.e. if $(zI - T)G(z) = 0\,(z \in \Omega)$ implies $G(z) = 0\,(z \in \Omega)$, for every open set $\Omega \subset \mathbf{C}$ and analytic function $G : \Omega \to \mathcal{X}$. In that case, $\sigma(T, x)$ is a non-empty compact set if $x \neq 0$ and there exists a global resolvent function defined on $\mathbf{C}\backslash\sigma(T, x)$. For more details see [8], [9] or [10].

It is immediate that the isometry V possesses the (SVEP). Furthermore, the relation $X(zI-T)F(z) = (zI-V)XF(z)$ readily implies that every polynomially bounded operator T of class $C_{1\bullet}$ has the (SVEP), and $\sigma(T, x) \supset \sigma(V, Xx)\,(x \in \mathcal{X})$ holds for the associated isometry $V \in \mathcal{L}(\mathcal{Y})$ and the intertwining transformation $X \in \mathcal{L}(\mathcal{X}, \mathcal{Y})$.

We shall need the following lemma, part of which is a modification of [4, Theorem 2.2] and [18, Lemma 5.1.7], and whose proof contains a boundedness argument originated in [21]. Given an isometry $V \in \mathcal{L}(\mathcal{Y})$ and a vector $y_0 \in \mathcal{Y}$, we consider the closed invariant subspaces $\mathcal{Y}_0 = \overline{\{V^n y_0 :\ n = 0, 1, 2, ...\}}$ and $\mathcal{Y}_1 = \overline{\{Dy_0 : D \in \{V\}'\}}$.

Lemma 1.3. *Let $V \in \mathcal{L}(\mathcal{Y})$ be an isometry on $\mathcal{Y}$ and let $y_0 \in \mathcal{Y}$ be fixed. Let $\mathcal{Y}'$ be a closed invariant subspace of V such that $\mathcal{Y}_0 \subset \mathcal{Y}' \subset \mathcal{Y}_1$. The following assertions are valid for the local spectra of y_0:*

(i) $\sigma(V_j, y_0) = \sigma(V_j)$, *where* $V_j = V|\mathcal{Y}_j$ $(j = 0, 1)$;

(ii) $\sigma(V_0, y_0) \supset \sigma(V_1, y_0) \supset \sigma(V, y_0)$;

(iii) *The unbounded components of* $\mathbf{C}\backslash\sigma(V')$ *and* $\mathbf{C}\backslash\sigma(V, y_0)$ *coincide, where* $V' = V|\mathcal{Y}'$;

(iv) *If* $\sigma(V, y_0)$ *does not cover the unit circle* $\partial\mathbf{D}$ *then* $\sigma(V') = \sigma(V, y_0) \subset \partial\mathbf{D}$.

Proof. Since $\sigma(V_1) = \cup_{y\in\mathcal{Y}_1}\sigma(V_1, y)$, we have to show that $\sigma(V_1, y) \subset \sigma(V_1, y_0) =: \sigma$ holds, for every $y \in \mathcal{Y}_1$.

Let $F_0 : \mathbf{C}\backslash\sigma \to \mathcal{Y}_1$ be the analytic function satisfying the condition $(zI - V_1)F_0(z) = y_0$ $(z \in \mathbf{C}\backslash\sigma)$. The equations $(zI - V_1)DF_0(z) = D(zI - V_1)F_0(z) = Dy_0$ $(z \in \mathbf{C}\backslash\sigma, D \in \{V\}')$ show that $\sigma(V_1, y) \subset \sigma$ is valid, for any vector y in the dense subset $\mathcal{Y}_1' = \{Dy_0 : D \in \{V\}'\}$ of $\mathcal{Y}_1$.

Let us consider now an arbitrary vector $y \in \mathcal{Y}_1$, and let $\{y_n\}_{n\in\mathbf{N}}$ be a sequence in $\mathcal{Y}_1'$ such that $\|y_n - y\| \to 0$. For every $n \in \mathbf{N}$, let $F_n : \mathbf{C}\backslash\sigma \to \mathcal{Y}_1$ be the analytic function having the property $(zI - V_1)F_n(z) = y_n$ $(z \in \mathbf{C}\backslash\sigma)$. It is sufficient to show that the sequence $\{F_n\}_{n\in\mathbf{N}}$ converges uniformly on compact subsets of $\mathbf{C}\backslash\sigma$ to a function F. Indeed, then F is necessarily analytic and the relation $(zI - V_1)F(z) = y$ $(z \in \mathbf{C}\backslash\sigma)$ is evidently true.

Given any $z \in \mathbf{C}\backslash(\sigma \cup \partial\mathbf{D})$ we have

$$\|y_m - y_n\| = \|(zI - V_1)(F_m(z) - F_n(z))\| \geq (|1 - |z|^*)\|F_m(z) - F_n(z)\|,$$

and so

$$\|F_m(z) - F_n(z)\| \leq (1 - |z|^*)^{-1}\|y_m - y_n\|$$

for every $m, n \in \mathbf{N}$; here $|z|^*$ stands for $|z|$ if $|z| < 1$ and for $1/|z|$ if $|z| > 1$. We conclude that $\{F_n\}_{n\in\mathbf{N}}$ converges uniformly on compact subsets of $\mathbf{C}\backslash(\sigma \cup \partial\mathbf{D})$.

Let us assume now that $z_0 \in (\mathbf{C}\backslash\sigma) \cap \partial\mathbf{D}$, and let $n \in \mathbf{N}$ be arbitrary. There exists a number $0 < s < 1$ such that $z_0\tau(s\overline{\mathbf{D}}) \subset \mathbf{C}\backslash\sigma$, where $\tau(z) = (1-z)/(1+z)$. If $z = se^{i\vartheta} \neq i\mathbf{R}$ then $z_0\tau(z) \in \mathbf{C}\backslash(\sigma\cup\partial\mathbf{D})$ and so $\|F_n(z_0\tau(z))\| \leq (1-|\tau(z)|^*)^{-1}\|y_n\| \leq 2\|y_n\|(s\cdot|\cos\vartheta|)^{-1}$, whence $|1 + z^2/s^2| \cdot \|F_n(z_0\tau(z))\| \leq 4\|y_n\|/s$. We infer that the last inequality holds for every $|z| \leq s$, which immediately implies that

$$\|F_n(z_0\tau(z))\| \leq \frac{8\|y_n\|}{s} \quad \text{whenever} \quad |z| \leq \frac{s}{2}.$$

Since $F_n(z_0\tau(z)) = (z_0\tau(z)I - V_1)^{-1}y_n \to (z_0\tau(z)I - V_1)^{-1}y$ holds if $|z| \leq s/2$ and $\operatorname{Re} z < 0$, an application of Vitali's theorem (see [12, Theorem 3.14.1]) shows that the sequence $\{F_n\}_{n\in\mathbf{N}}$ converges uniformly on compact subsets of $z_0\tau((s/2)\mathbf{D})$.

We have proved that $\sigma(V_1, y_0) = \sigma(V_1)$. The equation $\sigma(V_0, y_0) = \sigma(V_0)$ can be verified by obvious modifications of the previous proof. The inclusions in (ii) are immediate consequences of the relations $\mathcal{Y}_0 \subset \mathcal{Y}_1 \subset \mathcal{Y}$.

Let ω_0, ω_1 and ω stand for the unbounded components of the open sets $\mathbf{C}\backslash\sigma(V_0, y_0)$, $\mathbf{C}\backslash\sigma(V_1, y_0)$, and $\mathbf{C}\backslash\sigma(V, y_0)$, respectively. In view of (ii) we know that $\omega \supset \omega_1 \supset \omega_0 \supset \mathbf{C}\backslash\overline{\mathbf{D}}$. Let $F : \omega \to \mathcal{Y}$ be the analytic function with the property $(zI - V)F(z) = y_0$ $(z \in \omega)$. Let $\pi_0 : \mathcal{Y} \to \mathcal{Y}/\mathcal{Y}_0, y \mapsto y + \mathcal{Y}_0$ be the quotient mapping and let us consider the analytic function $\pi_0 F : \omega \to \mathcal{Y}/\mathcal{Y}_0$. For every $|z| > 1$ the vector $F(z) = (zI - V)^{-1}y_0 = z^{-1}\sum_{n=0}^{\infty} z^{-n}V^n y_0$ belongs to the subspace $\mathcal{Y}_0$, whence $\pi_0 F(z) = 0$. Taking into account that $\pi_0 F$ is analytic and ω is connected, it follows that $\pi_0 F(z) = 0$ and so $F(z) \in \mathcal{Y}_0$, for every $z \in \omega$. Therefore, we obtain that $\omega_0 = \omega_1 = \omega$.

Let ω' and ω'_0 be the unbounded components of $\mathbf{C}\backslash\sigma(V')$ and $\mathbf{C}\backslash\sigma(V', y_0)$, respectively; clearly $\omega' \subset \omega'_0$. Since $\sigma(V_0, y_0) \supset \sigma(V', y_0) \supset \sigma(V_1, y_0)$ and $\omega_0 = \omega_1 = \omega$, we infer that $\omega'_0 = \omega$, whence $\omega' \subset \omega$ follows. For any $y \in \mathcal{Y}'$ let us consider the analytic function $F_y : \omega \to \mathcal{Y}$ satisfying the condition $(zI - V)F_y(z) = y$ $(z \in \omega)$. By the use of the quotient mapping $\pi' : \mathcal{Y} \to \mathcal{Y}/\mathcal{Y}'$ we can show as before that $F_y(\omega) \subset \mathcal{Y}'$, and so $\omega \subset \mathbf{C}\backslash\sigma(V', y)$. Since $y \in \mathcal{Y}'$ was arbitrary, we obtain that $\omega \subset \mathbf{C}\backslash\sigma(V')$, whence $\omega' = \omega$ follows.

If $\sigma(V, y_0)$ does not cover the whole unit circle $\partial\mathbf{D}$ then $\omega' = \omega$ contains a point in $\mathbf{D}$. Thus $\sigma(V')$ is a proper subset of $\partial\mathbf{D}$, and so $\omega' = \mathbf{C}\backslash\sigma(V')$ and $\omega = \mathbf{C}\backslash\sigma(V, y_0)$, whence the relations $\sigma(V') = \sigma(V, y_0) \subset \partial\mathbf{D}$ readily follow. Q.E.D.

Remark. We remind the reader that if V is the bilateral shift on $\ell^2(\mathbf{Z})$ and $y_0(n) = \delta_{0n}$ $(n \in \mathbf{Z})$, then $\sigma(V_0) = \overline{\mathbf{D}}$ and $\sigma(V_1) = \partial\mathbf{D}$.

Now we are ready to state our second theorem.

Theorem 1.4. *Let $T \in \mathcal{L}(\mathcal{X})$ be a polynomially bounded operator (with bound M) of class $C_{1\bullet}$. Suppose that the unitary local spectrum $\sigma(T, x_0) \cap \partial\mathbf{D}$ of T at $x_0 \in \mathcal{X}$ is of measure zero. Then the restriction $T|\mathcal{X}_{x_0}$ of T to the hypercyclic subspace $\mathcal{X}_{x_0} = \overline{\{Cx_0 : C \in \{T\}'\}}$ is similar to a surjective isometry, whose spectrum is located on the circle $\partial\mathbf{D}$ and is of measure zero. Furthermore, the similarity can be implemented by an affinity with an upper bound M and lower bound M^{-1}.*

Proof. Let $V \in \mathcal{L}(\mathcal{Y})$ and $X \in \mathcal{L}(\mathcal{X}, \mathcal{Y})$ be the associated isometry and the intertwining transformation, respectively. We have seen that $\sigma(V, y_0) \subset \sigma(T, x_0)$, where $y_0 = Xx_0$.

Let us consider the subspaces $\mathcal{Y}_0$ and $\mathcal{Y}_1$ introduced in Lemma 1.3. It is clear that $\mathcal{Y}' = \overline{X\mathcal{X}_{x_0}}$ is an invariant subspace for V and $\mathcal{Y}_0 \subset \mathcal{Y}' \subset \mathcal{Y}_1$. We infer by Lemma 1.3 that $\sigma(V') = \sigma(V, y_0) \subset \partial\mathbf{D}$. Applying Theorem 1.2 (and the first remark following it) to the transformation $X' : \mathcal{X}_{x_0} \to \mathcal{Y}', x \mapsto Xx$, we obtain that X' is an affinity with upper bound M and lower bound M^{-1}. Q.E.D.

If the polynomially bounded operator T is not of class $C_{1\bullet}$ then we can consider the operator $\tilde{T}\pi(x) := \pi(Tx)$ $(x \in \mathcal{X})$ acting on the quotient space $\tilde{\mathcal{X}} = \mathcal{X}/\mathcal{X}_0(T)$. (Here $\pi : \mathcal{X} \to \tilde{\mathcal{X}}$ denotes the quotient mapping, and the norm considered on $\tilde{\mathcal{X}}$ is induced by the original norm of $\mathcal{X}$.) It is easy to see that $\tilde{T}$ is polynomially bounded as well, with the same bound as T.

The transformation $\tilde{X} \in \mathcal{L}(\tilde{\mathcal{X}}, \mathcal{Y}), \tilde{X}\pi(x) := Xx$ $(x \in \mathcal{X})$ will be a quasiaffinity, and we have $\tilde{X}\tilde{T} = V\tilde{X}$. Since $\|\tilde{X}\|\|\tilde{T}^n\pi(x)\| \geq \|\tilde{X}\tilde{T}^n\pi(x)\| = \|V^n\tilde{X}\pi(x)\| = \|\tilde{X}\pi(x)\|$, it follows that $\inf_n \|\tilde{T}^n\pi(x)\| > 0$ if $\pi(x) \neq 0$; therefore $\tilde{T}$ is of class $C_{1\bullet}$.

Taking into account that $\mathcal{X}_0(T)$ is invariant for every operator commuting with T, we infer that $\sigma(T) \supset \sigma(\tilde{T})$. Furthermore, it is easy to check that $\sigma(T, x) \supset \sigma(\tilde{T}, \pi(x))$ holds, for every $x \in \mathcal{X}$.

As a consequence of Theorems 1.2 and 1.4 we obtain the following corollary.

Corollary 1.5. *Let $T \in \mathcal{L}(\mathcal{X})$ be a polynomially bounded operator (with bound M).*

(i) *If the unitary spectrum $\sigma(T) \cap \partial\mathbf{D}$ is of measure zero, then the quotient operator $\tilde{T} = T/\mathcal{X}_0(T)$ is similar to a surjective isometry.*

(ii) *If the local unitary spectrum $\sigma(T, x_0) \cap \partial\mathbf{D}$ at x_0 is of measure zero, then the restriction $\tilde{T}|\overline{\{\tilde{C}\pi(x_0) : \tilde{C} \in \{\tilde{T}\}'\}}$ is similar to a surjective isometry.*

In both cases the similarity can be implemented by an affinity with upper bound M and lower bound M^{-1}.

Remarks. 1. In the Hilbert space setting the quotient operator $\tilde{T}$ can be replaced by the compression $P_{\mathcal{H}_1}T|\mathcal{H}_1$ of T to the orthogonal complement $\mathcal{H}_1$ of $\mathcal{H}_0(T)$.
2. If T is a Hilbert space contraction and if $\mathcal{H}'$ is an invariant subspace of T such that $T|\mathcal{H}'$ is unitary, then $\mathcal{H}'$ is a reducing subspace (i.e. the orthogonal complement of $\mathcal{H}'$ is also invariant for T); see [23, Theorem I.3.2].

Using the method of [24] we can extend the Katznelson–Tzafriri-type theorem in [11, Corollary 2.2] to polynomially bounded operators acting on a Banach space.

Proposition 1.6. *Let $T \in \mathcal{L}(\mathcal{X})$ be a polynomially bounded operator. If $f \in A(\mathbf{D})$ and $f|\sigma(T) \cap \partial\mathbf{D} = 0$ then $\lim_{n\to\infty} \|T^n f(T)\| = 0$.*

Proof. If $\sigma(T) \cap \partial\mathbf{D}$ is of positive measure then $f = 0$, hence we can assume that $\sigma(T) \cap \partial\mathbf{D}$ is of measure zero. Let $V \in \mathcal{L}(\mathcal{Y})$ and $X \in \mathcal{L}(\mathcal{X}, \mathcal{Y})$ be as before. We know that $\sigma(V) \subset \sigma(T)$, hence V is invertible and $\sigma(V) \subset \partial\mathbf{D}$. Since $f|\sigma(V) = 0$, we infer by Lemma 1.1 that $f(V) = 0$. Therefore $Xf(T) = f(V)X = 0$ and so $\lim_{n\to\infty} \|T^n f(T)x\| = 0$ holds, for every $x \in \mathcal{X}$. Applying the previous argumentation to the operator $L(T) \in \mathcal{L}(\mathcal{L}(\mathcal{X}))$, $L(T)Z := TZ$, we get the statement. Q.E.D.

We can obtain also a local version of the preceding statement, the proof is left to the reader.

Proposition 1.7. *Let $T \in \mathcal{L}(\mathcal{X})$ be a polynomially bounded operator and $x_0 \in \mathcal{X}$. If $f \in A(\mathbf{D})$ and $f|\sigma(T, x_0) \cap \partial\mathbf{D} = 0$ then $\lim_{n\to\infty} \|T^n f(T)z\| = 0$ holds, for every $z \in \mathcal{X}_{x_0} = \overline{\{Cx_0 : C \in \{T\}'\}}$.*

It is worth mentioning that *the norm-convergence* $\lim_{n\to\infty} \|T^n f(T)|\mathcal{X}_{x_0}\| = 0$ *does not hold in general* under the previous conditions. Indeed, as it was pointed out to us by

C.J.K. Batty, a counterexample can be derived from [5, Example 3.4]. Instead of relying on results in [5] concerning one-parameter semigroups, we prefer here considering the truncated backward shift operator.

Example 1.8. Let $\{e_n\}_{n=0}^\infty$ be an orthonormal basis in the Hilbert space $\mathcal{H}$, and let $T \in \mathcal{L}(\mathcal{H})$ denote the contraction defined by $Te_n := e_{n-1}$ for $n \geq 1$, and $Te_0 := 0$. Let us consider a vector $h_0 = \sum_{n=0}^\infty \xi_n e_n$ such that $\xi_n \neq 0$ for infinitely many n, and $\lim_{n\to\infty}(\sum_{k=n}^\infty |\xi_k|^2)^{1/n} = 0$.

Since $\lim_{n\to\infty} \|T^n h_0\|^{1/n} = 0$, we infer that $\sigma(T, h_0) = 0$. (Indeed, $(zI - T)F(z) = h_0$ holds for any $z \neq 0$, where $F(z) = \sum_{n=1}^\infty z^{-n}T^{n-1}h_0$.) On the other hand, the entire function $\varphi(z) = \sum_{n=0}^\infty \xi_n z^n$ is not a polynomial, hence the vector h_0 is cyclic for T by [19, Corollary II.1.4], that is $\vee\{T^n h_0\}_{n=0}^\infty = \mathcal{H}$. (The relations $\sigma(T, h_0) = \{0\}$ and $\sigma(T) = \overline{\mathbf{D}}$ can be contrasted with the equations in Lemma 1.3.(i) concerning isometries.)

It remains to observe that for any non-zero function $f \in A(\mathbf{D})$ the operator $\tilde{f}(T^*)$ is injective, where $\tilde{f}(z) = \overline{f(\overline{z})}$. (We refer to the representation of T as multiplication by χ in the Hardy space H^2.) It follows that the operator $f(T) = (\tilde{f}(T^*))^*$ has dense range, and so $\|T^n f(T)\| = \|T^n\| = 1$ is true, for any $n \in \mathbf{N}$.

2. One-parameter semigroups of Hilbert space contractions

Let $\mathbf{T} = \{T(t)\}_{t\in\mathbf{R}_+}$ be a strongly continuous semigroup of contractions acting on the Hilbert space $\mathcal{H}$ (here $\mathbf{R}_+ = \{t \in \mathbf{R} : t \geq 0\}$) and let A be the (infinitesimal) generator of T. It is well-known that A is a densely defined, closed operator and its resolvent set contains the open, right half-plane (see [20, Section 143]). The Cayley transform $T = (A + I)(A - I)^{-1}$ of A is a contraction, called the *cogenerator* of the semigroup $\mathbf{T}$. Both A and T determine the semigroup $\mathbf{T}$; for details we refer to [12, Chapter XII] and [23, Section III.8].

Let us consider the set $\mathcal{H}_0(\mathbf{T}) := \{h \in \mathcal{H} : \lim_{t\to\infty} \|T(t)h\| = 0\}$ of vectors which are stable under $\mathbf{T}$. It is easy to verify that $\mathcal{H}_0(\mathbf{T})$ is a closed subspace, invariant for every operator in the commutant $\{\mathbf{T}\}' = \{C \in \mathcal{L}(\mathcal{H}) : CT(t) = T(t)C$ for every $t \in \mathbf{R}_+\}$ of $\mathbf{T}$. Let $\mathcal{H}_1(\mathbf{T})$ denote the orthogonal complement of $\mathcal{H}_0(\mathbf{T})$ in $\mathcal{H}$.

Using the previous notation, we obtain the following theorem.

Theorem 2.1. *Let $\mathbf{T} = \{T(t)\}_{t\in\mathbf{R}_+}$ be a strongly continuous contraction semigroup on the Hilbert space $\mathcal{H}$, and let $h_1 \in \mathcal{H}_1(\mathbf{T})$. If the unitary local spectrum $\sigma(A, h_1) \cap i\mathbf{R}$ of the generator A at h_1 is of zero Lebesgue measure, then the hyperinvariant subspace*

$\mathcal{H}_{h_1} = \overline{\{Ch_1 : C \in \{\mathbf{T}\}'\}}$ *is reducing for* $\mathbf{T}$, $\mathcal{H}_{h_1}$ *is included in* $\mathcal{H}_1(\mathbf{T})$, *and the restriction* $\mathbf{T}|\mathcal{H}_{h_1}$ *is a unitary semigroup.*

Proof. First of all we show that the unitary local spectrum $\sigma(T, h_1) \cap \partial\mathbf{D}$ of the cogenerator T is contained in the set $\phi(\sigma(A, h_1) \cap i\mathbf{R}) \cup \{1\}$, where $\phi(z) = (z+1)/(z-1)$, and so the (linear Lebesgue) measure of $\sigma(T) \cap \partial\mathbf{D}$ is zero, as well.

Indeed, let us consider a point $z_0 \in i\mathbf{R}\backslash\sigma(A, h_1)$, and let Ω be an open disc centered at z_0 such that there exists an analytic function $F : \Omega \to \mathcal{H}$ with the property $(zI - A)F(z) = h_1$ $(z \in \Omega)$. Then for any z in the open set $\Omega' = \phi(\Omega)$ we can write $(\phi(z)I - A)F(\phi(z)) = h_1$ (note that $\phi^{-1} = \phi$). Since $(A-I)F(\phi(z)) = (\phi(z)-1)F(\phi(z)) - h_1$ and $(A+I)F(\phi(z)) = (\phi(z)+1)F(\phi(z)) - h_1$, we infer that

$$T((\phi(z) - 1)F(\phi(z)) - h_1) = (\phi(z) + 1)F(\phi(z)) - h_1,$$

whence $(zI - T)(1 - z)^{-1}((\phi(z) - 1)F(\phi(z)) - h_1) = h_1$ follows. Therefore $\phi(z_0)$ belongs to the set $\partial\mathbf{D}\backslash\sigma(T, h_1)$.

In view of $\mathcal{H}_0(\mathbf{T}) = \mathcal{H}_0(T)$ we know that the compression $T_1 = P_{\mathcal{H}_1(\mathbf{T})}T|\mathcal{H}_1(\mathbf{T})$ is a contraction of class $C_{1\bullet}$. It is easy to see that $\sigma(T_1, h_1) \subset \sigma(T, h_1)$. Thus we obtain by Theorem 1.4 that the restriction of T_1 to the subspace $\mathcal{H}' := \overline{\{Ch_1 : C \in \{T_1\}'\}}$ is unitary. Since T is a contraction, it follows that $T_1|\mathcal{H}' = T|\mathcal{H}'$ and that $\mathcal{H}'$ is reducing for T (see the second remark after Corollary 1.5). It is clear now that $\mathcal{H}' = \overline{\{Ch_1 : C \in \{T\}'\}}$. Taking into account that $\{\mathbf{T}\}' = \{T\}'$ (see [23, Theorem III.8.1]), we infer that $\mathcal{H}' = \mathcal{H}_{h_1}$. Finally, [23, Theorem III.8.1] and [23, Proposition III.8.2] result in that $\mathcal{H}_{h_1}$ is reducing for $\mathbf{T}$ and the restriction $\mathbf{T}|\mathcal{H}_{h_1}$ is a unitary semigroup. Q.E.D.

By a similar method we can derive from Theorem 1.2 that if $\sigma(A) \cap i\mathbf{R}$ is of measure zero then the semigroup $\mathbf{T}$ splits into the orthogonal sum $\mathbf{T} = \mathbf{T}_0 \oplus \mathbf{T}_1$, where $\mathbf{T}_0$ is a stable contraction semigroup and $\mathbf{T}_1$ is a unitary semigroup. This statement is equivalent to [7, Corollary 2].

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