Gradient estimates and domain identification for analytic Ornstein-Uhlenbeck operators

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Dedicated to Herbert Amann on the occasion of his 70th birthday

Abstract. Let P be the Ornstein-Uhlenbeck semigroup associated with the stochastic Cauchy problem

$$dU(t) = AU(t)\,dt + dW_H(t),$$

where A is the generator of a C_0-semigroup S on a Banach space E, H is a Hilbert subspace of E, and W_H is an H-cylindrical Brownian motion. Assuming that S restricts to a C_0-semigroup on H, we obtain L^p-bounds for $D_H P(t)$. We show that if P is analytic, then the invariance assumption is fulfilled. As an application we determine the L^p-domain of the generator of P explicitly in the case where S restricts to a C_0-semigroup on H which is similar to an analytic contraction semigroup. The results are applied to the 1D stochastic heat equation driven by additive space-time white noise.

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1. Introduction

Consider the stochastic Cauchy problem

$$\begin{aligned} dU(t) &= AU(t)\,dt + dW_H(t), \quad t \geqslant 0, \\ U(0) &= x. \end{aligned} \tag{SCP}$$

Here A generates a C_0-semigroup $S = (S(t))_{t\geqslant 0}$ on a real Banach space E, H is a real Hilbert subspace continuously embedded in E, W_H is an H-cylindrical Brownian motion on a probability space $(\Omega, \mathscr{F}\, P)$, and $x \in E$. A *weak solution* is

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a measurable adapted E-valued process $U^x = (U^x(t))_{t\geqslant 0}$ such that $t \mapsto U^x(t)$ is integrable almost surely and for all $t \geqslant 0$ and $x^* \in \mathsf{D}(A^*)$ one has

$$\langle U^x(t), x^*\rangle = \langle x, x^*\rangle + \int_0^t \langle U^x(s), A^*x^*\rangle\, ds + W_H(t)i^*x^* \quad \text{almost surely.}$$

Here $i : H \hookrightarrow E$ is the inclusion mapping. A necessary and sufficient condition for the existence of a weak solution is that the operators $I_t : L^2(0, t; H) \to E$,

$$I_t g := \int_0^t S(s)ig(s)\, ds,$$

are γ-radonifying for all $t \geqslant 0$. If this is the case, then $s \mapsto S(t-s)i$ is stochastically integrable on $(0, t)$ with respect to W_H and the process U^x is given by

$$U^x(t) = S(t)x + \int_0^t S(t-s)i\, dW_H(s), \quad t \geqslant 0.$$

For more information and an explanation of the terminology we refer to [30].

Assuming the existence of the solution U^x, on the Banach space $C_{\mathrm{b}}(E)$ of all bounded continuous functions $f : E \to \mathbb{R}$ one defines the *Ornstein-Uhlenbeck semigroup* $P = (P(t))_{t\geqslant 0}$ by

$$P(t)f(x) := \mathbb{E}f(U^x(t)), \quad t \geqslant 0,\ x \in E. \tag{1.1}$$

The operators $P(t)$ are linear contractions on $C_{\mathrm{b}}(E)$ and satisfy $P(0) = I$ and $P(s)P(t) = P(s+t)$ for all $s, t \geqslant 0$. For all $f \in C_{\mathrm{b}}(E)$ the mapping $(t, x) \mapsto P(t)f(x)$ is continuous, uniformly on compact subsets of $[0, \infty) \times E$.

If the operator $I_\infty : L^2(0, \infty; H) \to E$ defined by

$$I_\infty g := \int_0^\infty S(t)ig(t)\, dt$$

is γ-radonifying, then the problem (SCP) admits a unique invariant measure μ_∞. This measure is a centred Gaussian Radon measure on E, and its covariance operator equals $I_\infty I_\infty^*$. Throughout this paper we shall assume that this measure exists; if (SCP) has a solution, then this assumption is for instance fulfilled if S is uniformly exponentially stable. The reproducing kernel Hilbert space associated with μ_∞ is denoted by H_∞. The inclusion mapping $H_\infty \hookrightarrow E$ is denoted by i_∞. Recall that $Q_\infty := i_\infty i_\infty^* = I_\infty I_\infty^*$. Is is well-known that S restricts to a C_0-contraction semigroup on H_∞ [5] (the proof for Hilbert spaces E extends without change to Banach spaces E), which we shall denote by S_∞.

By a standard application of Jensen's inequality, the semigroup P has a unique extension to a C_0-contraction semigroup to the spaces $L^p(E, \mu_\infty)$, $1 \leqslant p < \infty$. By slight abuse of notation we shall denote this semigroup by P again. Its infinitesimal generator will be denoted by L. In order to give an explicit expression for L it is useful to introduce, for integers $k, l \geqslant 0$, the space $\mathscr{F}C_{\mathrm{b}}^{k,l}(E)$ consisting of all functions $f \in C_{\mathrm{b}}(E)$ of the form

$$f(x) = \varphi(\langle x, x_1^*\rangle, \ldots, \langle x, x_N^*\rangle)$$

with $f \in C_{\rm b}^k(\mathbb{R}^N)$ and $x_1^*, \dots, x_N^* \in \mathsf{D}(A^{*l})$. With this notation one has that $\mathscr{F}C_{\rm b}^{2,1}(E)$ is a core for L, and on this core one has

$$Lf(x) = \frac{1}{2}\,\mathrm{tr}\, D_H^2 f(x) + \langle x, A^* Df(x)\rangle.$$

Here,

$$D_H f(x) = \sum_{n=1}^N \frac{\partial\varphi}{\partial x_n}(\langle x, x_1^*\rangle, \dots, \langle x, x_N^*\rangle) \otimes i^* x_n^*,$$

$$Df(x) = \sum_{n=1}^N \frac{\partial\varphi}{\partial x_n}(\langle x, x_1^*\rangle, \dots, \langle x, x_N^*\rangle) \otimes x_n^*,$$

denote the Fréchet derivatives into the directions of H and E, respectively.

2. Gradient estimates: the H-invariant case

Our first result gives a pointwise gradient bound for P under the assumption that S restricts to a C_0-semigroup on H which will be denoted by S_H. As has been shown in [17, Corollary 5.6], under this assumption the operator D_H is closable as a densely defined operator from $L^p(E, \mu_\infty)$ to $L^p(E, \mu; H)$ for all $1 \leqslant p < \infty$. The domain of its closure is denoted by $\mathsf{D}_p(D_H)$.

Proposition 2.1 (Pointwise gradient bounds). *If S restricts to a C_0-semigroup on H, then for all $1 < p < \infty$ there exists a constant $C \geqslant 0$ such that for all $t > 0$ and $f \in \mathscr{F}C_{\rm b}^{1,0}(E)$ we have*

$$\sqrt{t}|D_H P(t) f(x)| \leqslant C\kappa(t)(P(t)|f|^p(x))^{1/p},$$

where $\kappa(t) := \sup_{s\in[0,t]} \|S_H(s)\|_{\mathscr{L}(H)}$.

Proof. The proof follows the lines of [25, Theorem 8.10] and is inspired by the proof of [10, Theorem 6.2.2], where the null controllable case was considered.

The distribution μ_t of the random variable $U^0(t)$ is a centred Gaussian Radon measure on E. Let H_t denote its RKHS and let $i_t : H_t \hookrightarrow E$ be the inclusion mapping. As is well known and easy to prove, cf. [9, Appendix B] one has

$$H_t = \Big\{ \int_0^t S(t-s) i g(s)\, ds :\ g \in L^2(0,t;H) \Big\}$$

with

$$\|h\|_{H_t} = \inf \Big\{ \|g\|_{L^2(0,t;H)} :\ h = \int_0^t S(t-s) i g(s)\, ds \Big\}.$$

The mapping

$$\phi^{\mu_t} : i_t^* x^* \mapsto \langle \cdot, x^*\rangle, \quad x^* \in E^*,$$

defines an isometry from H_t onto a closed subspace of $L^2(E, \mu_t)$. For $h \in H_t$ we shall write $\phi_h^{\mu_t}(x) := (\phi^{\mu_t} h)(x)$.

Fix $h \in H$. Since S restricts to a C_0-semigroup S_H on H we may consider the function $g \in L^2(0,t;H)$ given by $g(s) = \frac{1}{t}S(s)h$. From the identity $S(t)h = \int_0^t S(t-s)g(s)\,ds$ we deduce that $S(t)h \in H_t$ and

$$\|S(t)h\|_{H_t}^2 \leqslant \|g\|_{L^2(0,t;H)}^2 = \frac{1}{t^2}\int_0^t \|S(s)h\|_H^2\,ds \leqslant \frac{1}{t}\kappa(t)^2\|h\|_H^2. \tag{2.1}$$

Fix a function $f \in \mathscr{F}C_{\rm b}^{1,0}(E)$, that is, $f(x) = \varphi(\langle x, x_1^*\rangle, \dots, \langle x, x_N^*\rangle)$ with $\varphi \in C_{\rm b}^1(\mathbb{R}^N)$ and $x_1^*, \dots, x_N^* \in E^*$. It is easily checked that for all $t > 0$ we have $P(t)f \in \mathscr{F}C_{\rm b}^{1,0}(E)$; in particular this implies that $P(t)f \in \mathsf{D}_p(D_H)$. By the Cameron-Martin formula [3],

$$\begin{aligned}\frac{1}{\varepsilon}\big(P(t)f(x+\varepsilon h) - P(t)f(x)\big) &= \frac{1}{\varepsilon}\int_E \big(f(S(t)(x+\varepsilon h)+y) - f(S(t)x+y)\big)\,d\mu_t(y)\\ &= \int_E \frac{1}{\varepsilon}(E_{\varepsilon S(t)h} - 1)f(S(t)x+y)\,d\mu_t(y),\end{aligned}$$

where for $h \in H_t$ we write

$$E_h(x) := \exp(\phi_h^{\mu_t}(x) - \tfrac{1}{2}\|h\|_{H_t}^2).$$

It is easy to see that for each $h \in H_t$ the family $\big(\frac{1}{\varepsilon}(E_{\varepsilon h} - 1)\big)_{0<\varepsilon<1}$ is uniformly bounded in $L^2(E,\mu_t)$, and therefore uniformly integrable in $L^1(E,\mu_t)$. Passage to the limit $\varepsilon \downarrow 0$ in the previous identity now gives

$$[D_H P(t)f(x), h] = \int_E f(S(t)x+y)\phi_{S(t)h}^{\mu_t}(y)\,d\mu_t(y).$$

By Hölder's inequality with $\frac{1}{r} + \frac{1}{q} = 1$ and the Kahane-Khintchine inequality, which can be applied since $\phi_{S(t)h}^{\mu_t}$ is a Gaussian random variable,

$$\begin{aligned}&|[D_H P(t)f(x), h]|\\ &\quad\leqslant \Big(\int_E |f(S(t)x+y)|^r\,d\mu_t(y)\Big)^{\frac{1}{r}}\Big(\int_E |\phi_{S(t)h}^{\mu_t}(y)|^q\,d\mu_t(y)\Big)^{\frac{1}{q}}\\ &\quad\leqslant K_q\Big(\int_E |f(S(t)x+y)|^r\,d\mu_t(y)\Big)^{\frac{1}{r}}\Big(\int_E |\phi_{S(t)h}^{\mu_t}(y)|^2\,d\mu_t(y)\Big)^{\frac{1}{2}}\\ &\quad= K_q(P(t)|f|^r(x))^{\frac{1}{r}}\|S(t)h\|_{H_t}.\end{aligned}$$

Using (2.1) we find that

$$\big|\sqrt{t}[D_H P(t)f(x), h]\big| \leqslant K_q\kappa(t)(P(t)|f|^r(x))^{\frac{1}{r}}\|h\|_H,$$

and by taking the supremum over all $h \in H$ of norm 1 we obtain the desired estimate. □

Corollary 2.2. *If S restricts to a C_0-semigroup on H, then for all $1 < p < \infty$ the operators $D_H P(t)$, $t > 0$, extend uniquely to bounded operators from $L^p(E,\mu_\infty)$ to $L^p(E,\mu_\infty;H)$, and there exists a constant $C \geqslant 0$ such that for any $t > 0$,*

$$\sqrt{t}\|D_H P(t)\|_{\mathscr{L}(L^p(E,\mu_\infty),L^p(E,\mu_\infty;H))} \leqslant C\kappa(t).$$

Proof. Integrating the inequality of the proposition and using the fact that μ_∞ is an invariant measure for P we obtain

$$\begin{aligned}\|\sqrt{t}D_H P(t)f\|^p_{L^p(E,\mu_\infty)} &\leqslant C^p\kappa(t)^p \int_E P(t)|f|^p(x)\,d\mu_\infty(x)\\ &= C^p\kappa(t)^p \int_E |f|^p(x)\,d\mu_\infty(x) = C^p\kappa(t)^p\|f\|^p_{L^p(E,\mu_\infty)}.\end{aligned}$$

□

3. Gradient estimates: the analytic case

Analyticity of the semigroup P on $L^p(E,\mu_\infty)$ has been investigated by several authors [15, 16, 18, 24]. The following result of [18] is our starting point. Recall that in the definition of an *analytic C_0-contraction semigroup*, contractivity is required on an open sector containing the positive real axis.

Proposition 3.1. *For any $1<p<\infty$ the following assertions are equivalent:*

(1) *P is an analytic C_0-semigroup on $L^p(E,\mu_\infty)$;*
(2) *P is an analytic C_0-contraction semigroup on $L^p(E,\mu_\infty)$;*
(3) *S restricts to an analytic C_0-contraction semigroup on H_∞;*
(4) *$Q_\infty A^*$ acts as a bounded operator in H.*

A more precise formulation of (4) is that there should exist a bounded operator $B : H \to H$ such that $iBi^*x^* = Q_\infty A^* x^*$ for all $x^* \in E^*$. The identity $Q_\infty A^* + AQ_\infty = -ii^*$ implies that $B + B^* = -I$.

In what follows we shall simply say that 'P is analytic' to express that the equivalent conditions of the proposition are satisfied for some (and hence for all) $1<p<\infty$.

The next result has been shown in [24] for $p=2$ and was extended to $1<p<\infty$ in [25].

Proposition 3.2. *If P is analytic, then $\mathscr{F}C_{\rm b}^{2,1}(E)$ is a core for the generator L of P in $L^p(E,\mu_\infty)$, and on this core L is given by*

$$L = D_H^* B D_H.$$

Our first aim is to show that analyticity of P implies that H is S-invariant. For self-adjoint P this was proved in [7, 18].

Theorem 3.3. *If P is analytic, then S restricts to a bounded analytic C_0-semigroup S_H on H.*

Proof. Consider the linear mapping

$$V : i_\infty^* x^* \mapsto i^* x^*, \quad x^* \in E^*. \tag{3.1}$$

It is shown in [17] that $i_\infty^* x^* = 0$ implies $i^* x^* = 0$, so that this mapping is well-defined, and that the closability of D_H implies the closability of V as a densely defined operator from H_∞ to H. With slight abuse of notation we denote its closure by V again and let $\mathsf{D}(V)$ the domain of the closure.

By [1, Proposition 7.1], the operator $-VV^*B$ is sectorial of angle $< \frac{\pi}{2}$, and therefore $G := VV^*B$ generates a bounded analytic C_0-semigroup $(T(t))_{t\geq 0}$ on H. To prove the theorem, by uniqueness of analytic continuation and duality it suffices to show that $T(t)\circ i^* = i^*\circ S^*(t)$ for all $t \geqslant 0$.

For all $x^* \in \mathsf{D}(A^*)$ we have $Bi^*x^* \in \mathsf{D}(V^*)$ and $V^*Bi^*x^* = i_\infty^* A^*x^*$. Indeed, for $y^* \in E^*$ one has

$$[Bi^*x^*, Vi_\infty^* y^*] = \langle i_\infty^* A^*x^*, i_\infty^* y^*\rangle,$$

which implies the claim. By applying the operator V to this identity we obtain $i^*x^* \in \mathsf{D}(G)$ and $G\,i^*x^* = i^*A^*x^*$, from which it follows that $T(t)i^*x^* = i^*S^*(t)x^*$. This proves the theorem, with $S_H = T^*$. □

This result should be compared with [18, Theorem 9.2], where it is shown that if S restricts to an analytic C_0-semigroup on H which is contractive in some equivalent Hilbert space norm, then P is analytic on $L^p(E,\mu_\infty)$.

Under the assumption that P is analytic on $L^p(E,\mu_\infty)$, the gradient estimates of the previous section can be improved as follows. Recall that a collection of bounded operators $\mathscr{T}$ between Banach spaces X and Y is said to be *R-bounded* if there exists a constant C such that for any finite subset $T_1,\dots,T_n \subset \mathscr{T}$ and any $x_1,\dots,x_n \in X$ we have

$$\mathbb{E}\Big\|\sum_{j=1}^n r_jT_jx_j\Big\|^2 \leqslant C^2\mathbb{E}\Big\|\sum_{j=1}^n r_jx_j\Big\|^2,$$

where $(r_j)_{j\geqslant 1}$ is an independent collection of Rademacher random variables. The notion of R-boundedness plays an important role in recent advances in the theory of evolution equations (see [12, 21]).

Theorem 3.4. *If P is analytic, then for all $1 < p < \infty$ the set*

$$\{\sqrt{t}D_HP(t):\ t>0\}$$

is R-bounded in $\mathscr{L}(L^p(E,\mu_\infty), L^p(E,\mu_\infty;H))$ and we have the square function estimate

$$\Big\|\Big(\int_0^t \|D_HP(t)f\|_H^2\,dt\Big)^{1/2}\Big\|_{L^p(E,\mu_\infty)} \lesssim \|f\|_{L^p(E,\mu_\infty)}$$

with implied constant independent of $f \in L^p(E,\mu_\infty)$.

Proof. By Proposition 3.2 and Theorem 3.3, the theorem is a special case of [25, Theorem 2.2]. □

The above result plays a crucial role in our recent paper [25] in which L^p-domain characterisations for the operator L and its square root have been obtained. Before stating the result, let us informally sketch how Theorem 3.4 enters the argument. In order to prove a domain characterisation for the operator L, we first aim to obtain two-sided estimates for $\|\sqrt{-L}f\|_{L^p(E,\mu_\infty)}$ in terms of suitable Sobolev norms. For this purpose we consider a variant of an operator theoretic framework introduced in [2] in the analysis of the famous Kato square root problem. The idea behind this framework is that the second order operator L can be naturally studied through the first order Hodge-Dirac-type operator

$$\Pi = \begin{bmatrix} 0 & -D_H^* B \\ D_H & 0 \end{bmatrix} \text{ on } L^p(E,\mu_\infty) \oplus L^p(E,\mu_\infty;H).$$

This operator is bisectorial and its square is the sectorial operator given by

$$-\Pi^2 = \begin{bmatrix} D_V^* B D_V & 0 \\ 0 & D_V D_V^* B \end{bmatrix} = \begin{bmatrix} L & 0 \\ 0 & \underline{L} \end{bmatrix},$$

where $\underline{L} := D_V D_V^* B$. The approach in [25] consists of proving estimates for $\sqrt{-L}f$ along the lines of the following formal calculation:

$$\|D_H f\|_p = \|\Pi(f,0)\|_p \leqslant \|\Pi/\sqrt{\Pi^2}\|_p \, \|\sqrt{\Pi^2}(f,0)\|_p = \|\Pi/\sqrt{\Pi^2}\|_p \, \|\sqrt{L}f\|_p.$$

Oversimplifying things considerably, the proof consists of turning this calculation into rigourous mathematics. This can be done once we know that the operator $\Pi/\sqrt{\Pi^2}$ is bounded. Since the function $z \mapsto z/\sqrt{z^2}$ is a bounded analytic function on each bisector around the real axis, it suffices to show that Π has a bounded H^∞-functional calculus. This in turn will follow if we show that

1. the resolvent set $\{(it-\Pi)^{-1}\}_{t\in\mathbb{R}\setminus\{0\}}$ is R-bounded;
2. the operator Π^2 admits a bounded functional calculus.

To prove (1), we observe that

$$(I-it\Pi)^{-1} = \begin{bmatrix} (1-t^2L)^{-1} & -it(I-t^2L)^{-1}D_H^*B \\ itD_H(I-t^2L)^{-1} & (I-t^2\underline{L})^{-1} \end{bmatrix}, \quad t \in \mathbb{R}\setminus\{0\}.$$

It suffices to prove R-boundedness for each of the entries separately. The diagonal entries can be dealt with using abstract results on R-boundedness for positive contraction semigroups on L^p-spaces. The R-boundedness for the off-diagonal entries can be derived using Theorem 3.4.

To prove (2) we use the fact that the semigroup generated by $\underline{L}$ equals $P \otimes S_H^*$ on the range of the gradient D_H. Here S_H denotes the restriction of the semigroup S to H (see Theorem 3.3). Therefore (2) follows, provided that the negative generator $-A_H$ of S_H has a bounded H^∞-calculus. This reduces the original question about $\sqrt{-L}$ to a question about the operator A_H, which is defined directly in terms of the data H and A of the problem. The latter question should be thought of as expressing the compatibility of the drift (represented by the operator A) and the noise (represented by the Hilbert space H). This compatibility condition is not automatically satisfied. In fact, by a result of Le Merdy [22], $-A_H$ admits a bounded H^∞-functional calculus on H if and only if S_H is an analytic C_0-*contraction* semigroup on H with respect to some equivalent Hilbert space norm. Such needs not always be the case, as is shown by well-known examples [26].

The following result summarises the informal discussion above and provides an additional equivalent condition in terms of the operator A_∞. In this result we let $\mathsf{D}_p(D_H^2)$ denote the second order Sobolev space associated with the operator D_H.

Theorem 3.5. *Let $1 < p < \infty$. If P is analytic on $L^p(E, \mu_\infty)$, then the following assertions are equivalent:*

(1) $\mathsf{D}_p(\sqrt{-L}) = \mathsf{D}_p(D_H)$ *with norm equivalence*

$$\|\sqrt{-L}f\|_{L^p(E,\mu_\infty)} \eqsim \|D_H f\|_{L^p(E,\mu_\infty;H)};$$

(2) $\mathsf{D}(\sqrt{-A_\infty}) = \mathsf{D}(V)$ *with norm equivalence*

$$\|\sqrt{-A_\infty}h\|_{H_\infty} \eqsim \|Vh\|_H;$$

(3) $-A_H$ *admits a bounded H^∞-functional calculus on H.*

If these equivalent conditions are satisfied we have

$$\mathsf{D}_p(L) = \mathsf{D}_p(D_H^2) \cap \mathsf{D}_p(A_\infty^* D),$$

where D is the Malliavin derivative in the direction of H_∞.

Proof. By Proposition 3.2 and Theorem 3.3, the theorem is a special case of [25, Theorems 2.1, 2.2] provided we replace A_∞ by A_∞^* in (2). The equivalence of (2) for A_∞ and A_∞^*, however, is well known (see also [25, Lemma 10.2]). □

The problem of identifying the domains of $\sqrt{-L}$ and L has a long and interesting history. We finish this paper by presenting three known special cases of Theorem 3.5. In each case, it is easy to verify that (3) is satisfied.

Example 1. For the classical Ornstein-Uhlenbeck operator, which corresponds to $H = E = \mathbb{R}^d$ and $A = -I$, conditions (2) and (3) of Theorem 3.5 are trivially fulfilled and (1) reduces to the classical Meyer inequalities of Malliavin calculus. For a discussion of Meyer's inequalities we refer to the book of Nualart [31].

Example 2. Meyer's inequalities were extended to infinite dimensions by Shigekawa [32], and Chojnowska-Michalik and Goldys [6, 7], who considered the case where E is a Hilbert space and A_H is self-adjoint. Both authors deduce the generalised Meyer inequalities from square functions estimates. The identification of $\mathsf{D}_p(L)$ in the self-adjoint case is due to Chojnowska-Michalik and Goldys [6, 7], who extended the case $p = 2$ obtained earlier by Da Prato [8].

So far, these examples were concerned with the selfadjoint case.

Example 3. A non-selfadjoint extension of Meyer's inequalities has been given for the case $E = \mathbb{R}^d$ by Metafune, Prüss, Rhandi, and Schnaubelt [27] under the non-degeneracy assumption $H = \mathbb{R}^d$. In this situation the semigroup P is analytic on $L^p(\mu_\infty)$ [15], see also [16, 18]; no symmetry assumptions need to be imposed on A. The S-invariance of H and the fact that the generator of $S = S_H$ admits a bounded H^∞-calculus are trivial. Therefore, (3) is satisfied again. Note that the domain characterisation reduces to $\mathsf{D}_p(L) = \mathsf{D}_p(D^2)$, where D is the derivative on $\mathbb{R}^d$. The techniques used in [27] to prove (1) are very different, involving diagonalisation arguments and the non-commuting Dore-Venni theorem. The identification of $\mathsf{D}_p(L) = \mathsf{D}_p(D^2)$ for $p = 2$ had been obtained previously by Lunardi [23].

Our final corollary extends the characterisations of $\mathsf{D}_p(L)$ contained in Examples 2 and 3 and lifts the non-degeneracy assumption on H in Example 3.

Corollary 3.6. *If S restricts to an analytic C_0-semigroup on H which is contractive with respect to some equivalent Hilbert space norm, then for all $1 < p < \infty$ we have*

$$\mathsf{D}_p(L) = \mathsf{D}_p(D_H^2) \cap \mathsf{D}_p(A_\infty^* D),$$

where D is the Malliavin derivative in the direction of H_∞.

Proof. As has already been mentioned in the discussion preceding Theorem 3.4, the assumptions imply that P is analytic. Moreover, since the restricted semigroup S_H is similar to an analytic contraction semigroup, its negative generator $-A_H$ admits a bounded H^∞-calculus, and the result follows from Theorem 3.5. □

Let us finally mention that the results in [25] have been proved for a more general class of elliptic operators on Wiener spaces (cf. Section 3 of that paper). In this setting the data consist of

- an arbitrary Gaussian measure μ on a separable Banach space E with reproducing kernel Hilbert space $\mathscr{H}$;
- an analytic C_0-contraction semigroup $\mathscr{S}$ on $\mathscr{H}$ with generator $\mathscr{A}$.

Given these data, the semigroup $\mathscr{P}$ is defined on $L^2(E,\mu)$ by second quantisation of the semigroup $\mathscr{S}$. Roughly speaking, this means that one uses the Wiener-Itô isometry to identify $L^2(E,\mu)$ with the symmetric Fock space over $\mathscr{H}$, i.e., the direct sum of symmetric tensor powers of $\mathscr{H}$. The semigroup $\mathscr{P}$ is then defined by applying $\mathscr{S}$ to each factor

$$\mathscr{P}(t) \sum_{\sigma \in S_n} (h_{\sigma(1)} \otimes \ldots \otimes h_{\sigma(n)}) := \sum_{\sigma \in S_n} \mathscr{S}(t) h_{\sigma(1)} \otimes \ldots \otimes \mathscr{S}(t) h_{\sigma(n)},$$

where S_n is the permutation group on $\{1, \ldots, n\}$. For the details of this construction we refer to [19]. Equivalently, the semigroup $\mathscr{P}$ can be defined via the the following generalisation of the classical Mehler formula,

$$\mathscr{P}(t) f(x) = \int_E f(\mathscr{S}(t)x + \sqrt{I - \mathscr{S}^*(t)\mathscr{S}(t)}y)\, d\mu(y),$$

which makes sense by virtue of the fact that any bounded linear operator on $\mathscr{H}$ admits a unique measurable linear extension to E [3]. The generator $\mathscr{L}$ of the semigroup $\mathscr{P}$ is the elliptic operator formally given by

$$\mathscr{L} = D^* \mathscr{A} D,$$

where D denotes the Malliavin derivative associated with μ and its adjoint D^* is the associated divergence operator. The application to Ornstein-Uhlenbeck operators described in this paper is obtained by taking $\mu \sim \mu_\infty$ and $\mathscr{A} \sim A_\infty^*$ (cf. [5, 28]).

4. An example

In this section we present an example of a Hilbert space E, a continuously embedded Hilbert subspace $H \hookrightarrow E$, and a C_0-semigroup generator A on E such that:

- the semigroup S generated by A fails to be analytic;
- the stochastic Cauchy problem
$$dU(t) = AU(t)\,dt + dW_H(t)$$
admits a unique invariant measure, which we denote by μ_∞;
- the associated Ornstein-Uhlenbeck semigroup P is analytic on $L^2(E, \mu_\infty)$.

Thus, although analyticity of P implies analyticity of S_H (Theorem 3.3), it does not imply analyticity of S.

Let $E = L^2(\mathbb{R}_+, e^{-x}\,dx)$ be the space of all measurable functions f on $\mathbb{R}_+$ such that
$$\|f\| := \Big(\int_0^\infty |f(x)|^2\, e^{-x} dx\Big)^{\frac{1}{2}} < \infty.$$

The rescaled left translation semigroup S,
$$S(t)f(x) := e^{-t} f(x+t), \quad f \in E,\ t > 0,\ x > 0,$$
is strongly continuous and contractive on E, and satisfies $\|S(t)\| = e^{-t/2}$. Let $H = H^2(\mathbb{C}_+)$ be the Hardy space of analytic functions g on the open right-half plane $\mathbb{C}_+ = \{z \in \mathbb{C} :\ \mathrm{Re}\, z > 0\}$ such that
$$\|g\|_H := \sup_{x>0} \Big(\int_{-\infty}^\infty |g(x+iy)|^2\,dy\Big)^{\frac{1}{2}} < \infty.$$

Since $\lim_{x\to+\infty} g(x) = 0$ for all $g \in H$, the restriction mapping $i : g \mapsto g|_{\mathbb{R}_+}$ is well-defined as a bounded operator from H to E. By uniqueness of analytic continuation, this mapping is injective. Since i factors through $L^\infty(\mathbb{R}_+, e^{-x}\,dx)$, i is Hilbert-Schmidt [29, Corollary 5.21]. As a consequence (see, e.g., [9, Chapter 11]), the Cauchy problem $dU(t) = AU(t)\,dt + dW_H(t)$ admits a unique invariant measure μ_∞.

The rescaled left translation semigroup S_H,
$$S_H(t)g(z) := e^{-t} g(z+t), \quad f \in H,\ t \geqslant 0,\ \mathrm{Re}\, z > 0,$$
is strongly continuous on H, it extends to an analytic contraction semigroup of angle $\frac{1}{2}\pi$, and satisfies $\|S_H(t)\|_H = e^{-t/2}$. Clearly, for all $t \geqslant 0$ we have $S(t) \circ i = i \circ S_H(t)$. By these observations combined with [18, Theorem 9.2], the associated Ornstein-Uhlenbeck semigroup P is analytic.

5. Application to the stochastic heat equation

In this final section we shall apply our results to the following stochastic PDE with additive space-time white noise:

$$\begin{aligned} \frac{\partial u}{\partial t}(t,y) &= \frac{\partial^2 u}{\partial y^2}(t,y) + \frac{\partial^2 W}{\partial t\,\partial y}(t,y), && t \geqslant 0,\ y \in [0,1], \\ u(t,0) &= u(t,1) = 0, && t \geqslant 0, \\ u(0,y) &= 0, && y \in [0,1]. \end{aligned} \tag{5.1}$$

This equation can be cast into the abstract form (SCP) by taking $H = E = L^2(0,1)$ and A the Dirichlet Laplacian Δ on E. The resulting equation

$$\begin{aligned} dU(t) &= AU(t)\,dt + dW(t), \\ U(0) &= 0, \end{aligned}$$

where now W denotes an H-cylindrical Brownian motion, has a unique solution U given by

$$U(t) = \int_0^t S(t-s)\,dW(s), \quad t \geqslant 0,$$

where S denotes the heat semigroup on E generated by A. Let μ_∞ denote the unique invariant measure on E associated with U, and let H_∞ denote its reproducing kernel Hilbert space. Let $i_\infty : H_\infty \hookrightarrow E$ denote the canonical embedding and let $i : H \to E$ be the identity mapping. By [17, Theorem 3.5, Corollary 5.6] the densely defined operator $V : i_\infty^* x^* \mapsto i^* x^*$ defined in (3.1) is closable from H_∞ to H.

Let L be the generator of the Ornstein-Uhlenbeck semigroup P on $L^p(E, \mu_\infty)$ associated with U. Since P is analytic, the results of Sections 2 and 3 can be applied. Noting that Δ is selfadjoint on H, condition (3) of Theorem 3.5 is satisfied and therefore

$$\mathsf{D}_p(\sqrt{-L}) = \mathsf{D}_p(D) \quad (1 < p < \infty)$$

where $D = D_H = D_E$ denotes the Fréchet derivative on $L^p(E, \mu_\infty)$.

One can go a step further by noting that the problem (5.1) is well-posed even on the space

$$\widetilde{E} := C_0[0,1] = \{f \in C[0,1] :\ f(0) = f(1) = 0\},$$

in the sense that the random variables $U(t)$ are $\widetilde{E}$-valued almost surely and that U admits has a modification $\widetilde{U}$ with continuous (in fact, even Hölder continuous) trajectories in $\widetilde{E}$. Moreover, the invariant measure μ_∞ is supported on $\widetilde{E}$. In analogy to (1.1) this allows us to define an "Ornstein-Uhlenbeck semigroup" $\widetilde{P}$ on $L^p(\widetilde{E}, \mu_\infty)$ associated with $\widetilde{U}$ by

$$\widetilde{P}(t)f(x) := \mathbb{E} f(\widetilde{U}^x(t)), \quad t \geqslant 0,\ x \in \widetilde{E},$$

where $\widetilde{U}^x(t) = \widetilde{S}(t)x + \widetilde{U}(t)$ and $\widetilde{S}$ is the heat semigroup on $\widetilde{E}$. It is important to observe that we are not in the framework considered in the previous sections, due to the fact that $H = L^2(0,1)$ is not continuously embedded in $\tilde{E}$. Let $\widetilde{L}$ denote the generator of $\widetilde{P}$. Under the natural identification

$$L^p(\widetilde{E},\mu_\infty) = L^p(E,\mu_\infty)$$

(using that the underlying measure spaces are identical up to a set of measure zero), we have $\widetilde{P}(t) = P(t)$ and $\widetilde{L} = L$, so that

$$\mathsf{D}_p(\sqrt{-\widetilde{L}}) = \mathsf{D}_p(\sqrt{-L}) = \mathsf{D}_p(D) \quad (1 < p < \infty). \tag{5.2}$$

This representation may seem somewhat unsatisfactory, as the right-hand side refers explicitly to the ambient space E in which $\widetilde{E}$ is embedded. An intrinsic representation of $\mathsf{D}_p(\sqrt{-\widetilde{L}})$ can be obtained as follows. For functions $F : \widetilde{E} \to \mathbb{R}$ of the form

$$F(f) = \phi\Big(\int_0^1 fg_1\,dt, \dots, \int_0^1 fg_N\,dt\Big), \quad f \in \widetilde{E},$$

with $\phi \in C_{\rm b}^2(\mathbb{R}^N)$ and $g_1, \dots, g_N \in H$, we define $\widetilde{D}F : \widetilde{E} \to H$ by

$$\widetilde{D}F(f) = \sum_{n=1}^N \frac{\partial\phi}{\partial y_n}\Big(\int_0^1 fg_1\,dt, \dots, \int_0^1 fg_N\,dt\Big)g_n, \quad f \in \widetilde{E}.$$

This operator is closable in $L^p(\widetilde{E},\mu_\infty)$ for all $1 \leqslant p < \infty$. On $L^2(\widetilde{E},\mu_\infty)$ we have the representation

$$\widetilde{L} = \widetilde{D}^*\widetilde{D}.$$

As a result we can apply [25, Theorem 2.1] directly to the operator V and obtain that

$$\mathsf{D}_p(\sqrt{-\widetilde{L}}) = \mathsf{D}_p(\widetilde{D}) \quad (1 < p < \infty). \tag{5.3}$$

This answers a question raised by Zdzisław Brzeźniak (personal communication). To make the link between the formulas (5.2) and (5.3) note that, under the identification $L^p(\widetilde{E},\mu_\infty) = L^p(E,\mu_\infty)$, one also has $\mathsf{D}_p(\widetilde{D}) = \mathsf{D}_p(D)$.

Remark 5.1. It is possible to give explicit representations for the space H_∞ and the operator V. To begin with, the covariance operator Q_∞ of μ_∞ is given by

$$Q_\infty f = \int_0^\infty S(t)S^*(t)f\,dt = \int_0^\infty S(2t)f\,dt = \tfrac{1}{2}\Delta^{-1}f, \quad f \in E.$$

It follows that the reproducing kernel Hilbert space H_∞ associated with μ_∞ equals

$$H_\infty = \mathsf{R}(\sqrt{Q_\infty}) = \mathsf{D}(\sqrt{-\Delta}) = H_0^1(0,1).$$

Noting that $Q_\infty = i_\infty \circ i_\infty^*$, we see that the operator $V : i_\infty^* x^* \mapsto i^* x^*$ is given by

$$\mathsf{D}(V) = H^2(0,1) \cap H_0^1(0,1),$$
$$Vf = 2\Delta f, \quad f \in \mathsf{D}(V).$$

Remark 5.2. Formulas for $\mathsf{D}_p(\widetilde{L})$ analogous to (5.2) and (5.3) can be deduced from Theorem 3.5 and [25, Theorem 2.2] in a similar way.

The Ornstein-Uhlenbeck operators L and $\widetilde{L}$ considered above are symmetric on $L^2(E, \mu_\infty)$, and therefore the domain identifications for their square roots could essentially be obtained from the results of [6, 32]. The above argument, however, can be applied to a large class of second order elliptic differential operators A on $L^2(0,1)$ (but explicit representations as in Remark 5.1 are only possible when A is selfadjoint).

In fact, under mild assumptions on the coefficients and under various types of boundary conditions, such operators A have a bounded H^∞-calculus on $H = E = L^2(0,1)$ (see [11, 14, 20] and there references therein). By the result of Le Merdy [22] mentioned earlier, this implies that the analytic semigroup S generated by A is contractive in some equivalent Hilbertian norm. Hence, by [18, Theorem 9.2], the associated Ornstein-Uhlenbeck semigroup is analytic. Typically, under Dirichlet boundary conditions, S is uniformly exponentially stable. This implies (see [9]) that the solution U of (SCP) admits a unique invariant measure. Finally, the analyticity of S typically implies space-time Hölder regularity of U (see [4, 13]), so that the corresponding stochastic PDE is well-posed in $\widetilde{E} = C_0[0,1]$. We plan to provide more details in a forthcoming publication.

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