ON ANALYTIC ORNSTEIN-UHLENBECK SEMIGROUPS IN INFINITE DIMENSIONS

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Abstract. We extend to infinite dimensions an explicit formula of Chill, Fašangová, Metafune, and Pallara [2] for the optimal angle of analyticity of analytic Ornstein-Uhlenbeck semigroups. The main ingredient is an abstract representation of the Ornstein-Uhlenbeck operator in divergence form.

1. Introduction

It is well known that a uniformly elliptic operator of the form

$$Lf(x) = \tfrac{1}{2}\sum_{i,j=1}^{n} q_{ij}D_{ij}f(x) + \sum_{i=1}^{n} b_i(x)D_i f(x), \quad x \in \mathbb{R}^n, \tag{1.1}$$

where $Q = (q_{ij})$ is a real, symmetric, and strictly positive definite matrix, may fail to generate an analytic semigroup on $L^p(\mathbb{R}^n)$ for all $1 \leqslant p < \infty$ if the first order coefficients b_i are unbounded. Let us consider the simplest case of linear coefficients

$$b_i(x) = \sum_{j=1}^{n} a_{ij}x_j, \tag{1.2}$$

where $A = (a_{ij})$ is a real matrix all of whose eigenvalues lie in the open left-half plane $\{z \in \mathbb{C} : \ \text{Re } z < 0\}$. In this situation L is called the *Ornstein-Uhlenbeck operator* associated with Q and A. It has been shown recently by Metafune [11] that this operator is closable as an operator on $L^p(\mathbb{R}^n)$ with initial domain $C_c^2(\mathbb{R}^n)$ and that the spectrum of its closure, also denoted by L, equals

$$\sigma(L) = \{z \in \mathbb{C} : \ \text{Re } z \leqslant -\text{tr}(A)/p\}.$$

By standard results from semigroup theory, this prevents L from generating an analytic semigroup on $L^p(\mathbb{R}^n)$.

The assumption $\sigma(A) \subseteq \{z \in \mathbb{C} : \ \text{Re } z < 0\}$ implies the convergence of the integral

$$Q_\infty = \int_0^\infty e^{tA}Qe^{tA^*}\,dt,$$

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and the centred Gaussian measure μ_∞ on $\mathbb{R}^n$ whose covariance matrix equals Q_∞ is an invariant measure for L, in the sense that

$$\int_{\mathbb{R}^n} Lf\,d\mu_\infty = 0, \quad f \in \mathscr{D}(L).$$

The realization of L in the space $L^p(\mathbb{R}^n, \mu_\infty)$ behaves much better, at least for $1 < p < \infty$. Indeed, for these values of p it is well known [5, 10, 6] that L generates an analytic C_0-semigroup on $L^p(\mathbb{R}^n, \mu_\infty)$. In a recent paper by Chill, Fašangová, Metafune and Pallara [2], the sector of analyticity of the semigroup $P = (P(t))_{t\geqslant 0}$ generated by L was computed explicitly: it was shown that P is an analytic C_0-contraction semigroup on the sector

$$\Sigma_{\theta_p} := \{re^{i\phi} \in \mathbb{C} :\ r > 0,\ |\phi| < \theta_p\}.$$

where

$$\cot\theta_p = \frac{\sqrt{(p-2)^2 + p^2\gamma^2}}{2\sqrt{p-1}}$$

and γ is a constant depending on Q and A. Moreover, the authors proved that the above sector is optimal. An extension of this result to nonsymmetric submarkovian semigroups was subsequently obtained by the same authors [3].

The purpose of this paper is to extend the results of [2] to analytic Ornstein-Uhlenbeck semigroups in infinite dimensions and removing the nondegeneracy assumption on Q (see Theorems 3.4 and 3.5 below). As is well known, for degenerate Q the Ornstein-Uhlenbeck semigroup may fail to be analytic in $L^p(E, \mu_\infty)$ even in finite dimensions. An explicit example was given by Fuhrman [5]; see also [6, 8]. Our extension is based on a characterization of analyticity of Ornstein-Uhlenbeck semigroups obtained recently by Goldys and the second-named author [8] (Proposition 2.1). It allows us to obtain a representation of L in divergence form (Theorem 2.3), which we believe is the main new contribution of this paper. It is the key step in extending the arguments of the paper [2] to the infinite-dimensional setting which we shall describe next.

Throughout the paper, E is a real Banach space and $Q \in \mathscr{L}(E^*, E)$ is a positive symmetric operator. That is, we assume that $\langle Qx^*, x^*\rangle \geqslant 0$ and $\langle Qx^*, y^*\rangle = \langle Qy^*, x^*\rangle$ for all $x^*, y^* \in E^*$. The reproducing kernel Hilbert space (RKHS) associated with Q will be denoted by H and the canonical inclusion mapping $H \hookrightarrow E$ by i. We refer to [12] for more details. Whenever this is convenient, we shall identify H with its image $i(H)$ in E.

If A is the generator of a C_0-semigroup $S = (S(t))_{t\geqslant 0}$ on E, for $t \geqslant 0$ we may consider the positive symmetric operators $Q_t \in \mathscr{L}(E^*, E)$ defined by

$$Q_t x^* := \int_0^t S(s)QS^*(s)x^*\,ds, \qquad x^* \in E^*.$$

The integrand is easily seen to be strongly measurable and therefore the integral is well defined as a Bochner integral in E. We shall assume that each operator Q_t is the covariance operator of a centred Gaussian Radon measure μ_t on E. Under this assumption, on the space $C_b(E)$ of bounded continuous functions $f : E \to \mathbb{R}$ we may define the operators $P(t)$ by

$$P(t)f(x) := \int_E f(S(t)x + y)\,d\mu_t(y).$$

These operators are contractions and satisfy $P(0) = I$ and $P(t) \circ P(s) = P(t+s)$ for all $t, s \geqslant 0$. Assuming furthermore that the family $(\mu_t)_{t\geqslant 0}$ is tight, by standard arguments we deduce that the weak limit

$$\mu_\infty := \lim_{t\to\infty} \mu_t$$

exists. The measure μ_∞ is a centred Radon Gaussian measure on E whose covariance operator Q_∞ equals the weak operator limit $Q_\infty = \lim_{t\to\infty} Q_t$. As is well known, the semigroup $P = (P(t))_{t\geqslant 0}$ extends in a unique way to a C_0-semigroup of contractions, also denoted by $P = (P(t))_{t\geqslant 0}$, on each of the spaces $L^p(E, \mu_\infty)$, $1 \leqslant p < \infty$. The generator of this extension will be denoted by L. As before the measure μ_∞ is invariant for L. On a suitable domain of smooth cylindrical functions (see below) we have the representation

$$Lf(x) = \tfrac{1}{2}\operatorname{tr} QD^2 f(x) + \langle x, A^* Df(x)\rangle, \tag{1.3}$$

where Df denotes the Fréchet derivative of f. For the proofs of these facts and more information we refer to [8] and the references given therein. Note that for $E = \mathbb{R}^d$ the formula (1.3) reduces to the special case (1.2) of (1.1).

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2. Analyticity of the Ornstein-Uhlenbeck semigroup

We say that a semigroup of operators $T = (T(t))_{t\geqslant 0}$ on a real Banach space X is *analytic* if its complexification $T_{\mathbb{C}} = (T_{\mathbb{C}}(t))_{t\geqslant 0}$ on $X_{\mathbb{C}}$ extends analytically to some open sector Σ containing the positive real axis. If this semigroup is contractive on (a possibly smaller sector) Σ we call T an *analytic contraction semigroup*.

Under the assumptions stated in the Introduction (which are adopted throughout this paper) and with the notations introduced there, we have the following characterization of analyticity for the Ornstein-Uhlenbeck semigroup P [8].

Proposition 2.1. *Let $1 < p < \infty$. The following assertions are equivalent.*

(1) *The Ornstein-Uhlenbeck semigroup P is analytic on $L^p(E, \mu_\infty)$;*
(2) *There exists a constant $c \geqslant 0$ such that for all $x^* \in \mathscr{D}(A^*)$ we have $Q_\infty A^* x^* \in H$ and*

$$\|Q_\infty A^* x^*\|_H \leqslant c\|i^* x^*\|_H.$$

If these equivalent conditions are fulfilled, then the semigroup P is an analytic contraction semigroup on $L^p(E, \mu_\infty)$.

For the rest of this paper *it will be a standing assumption that P is analytic on $L^p(E, \mu_\infty)$ for some (and hence all) $1 < p < \infty$*. Since i^* is weak*-to-weakly continuous, it maps $\mathscr{D}(A^*)$ onto a dense subspace of H and therefore Proposition 2.1 implies that there exists a unique bounded operator $B \in \mathscr{L}(H)$ which satisfies

$$Bi^* x^* = Q_\infty A^* x^*, \quad x^* \in \mathscr{D}(A^*). \tag{2.1}$$

Moreover, $\|B\| \leqslant c$.

Lemma 2.2. *We have $B + B^* = -I$ and $[Bh, h]_H = -\frac{1}{2}\|h\|_H^2$ for all $h \in H$.*

Proof. For $x^* \in \mathscr{D}(A^*)$ we have $Q_\infty x^* \in \mathscr{D}(A^*)$ and $AQ_\infty x^* + Q_\infty A^* x^* = -Qx^*$ [8, Proposition 4.1]. Hence, using (2.1) it follows that $iB^*i^*x^* + iBi^*x^* = -ii^*x^*$. Since i is injective this gives $B^*i^*x^* + Bi^*x^* = -i^*x^*$. The second identity follows from $[Bh,h]_H = \frac{1}{2}[(B+B^*)h,h]_H = -\frac{1}{2}\|h\|_H^2$. □

Let $\mathscr{F}C_c^{k,l}(E)$ denote the linear subspace of $C_b(E)$ of all functions f of the form

$$f(x) = \phi(\langle x, x_1^*\rangle, \dots, \langle x, x_n^*\rangle), \tag{2.2}$$

where $x_j^* \in \mathscr{D}(A^{*l})$ for all $j = 1, \dots, n$ and $\phi \in C_b^k(\mathbb{R}^n)$ has compact support. Here A^{*l} is the l-th power of the adjoint of A. We write $\mathscr{F}C_c^k(E) = \mathscr{F}C_c^{k,0}(E)$. It follows from [8, Theorem 6.6] that $\mathscr{F}C_c^{2,1}(E)$ is a core for L in $L^p(E,\mu_\infty)$.

For functions $f \in \mathscr{F}C_c^1(E)$ of the form (2.2) we define the Fréchet derivative in the direction of H by

$$D_H f(x) := \sum_{j=1}^n \frac{\partial \phi}{\partial x_j}(\langle x, x_1^*\rangle, \dots, \langle x, x_n^*\rangle)\, i^*x_j^*.$$

The analyticity of the Ornstein-Uhlenbeck semigroup P implies that for all $1 \leqslant p < \infty$, D_H is closable as an operator from $L^p(E,\mu_\infty)$ to $L^p(E,\mu_\infty;H)$ [8, Proposition 8.7]. In what follows we shall denote its closure again by D_H. We write $W_H^{1,p}(E,\mu_\infty)$ for its domain, which is a Banach space with respect to its graph norm.

Let H_∞ denote the RKHS associated with Q_∞ and let $i_\infty : H_\infty \hookrightarrow E$ be the natural inclusion mapping. The mapping

$$\phi(i_\infty^* x^*) := \langle x, x^*\rangle, \quad x^* \in E^*, \tag{2.3}$$

extends uniquely to an isometry ϕ from H_∞ onto a closed subspace of $L^2(E,\mu_\infty)$. For $h \in H_\infty$ we write $\phi_h := \phi(h)$.

The next theorem generalizes results which were proved by Fuhrman [5], and Bogachev, Röckner and Schmuland [1] in a Hilbert space setting.

Theorem 2.3 (L in divergence form)**.** *For all $f \in \mathscr{F}C_c^{2,1}(E)$ we have $BD_Hf \in \mathscr{D}(D_H^*)$ and*

$$Lf = D_H^* B D_H f.$$

Proof. Define the operator V with initial domain $\mathscr{D}(V) := i_\infty^* E^*$ from H_∞ to H by $Vi_\infty^* x^* := i^*x^*$. By [7, Theorem 3.5], the closability of D_H implies the closability of V; its closure will be denoted by V as well. For all $x^* \in \mathscr{D}(A^*)$ and $y^* \in E^*$ we have

$$[Bi^*x^*, Vi_\infty^* y^*]_H = [Bi^*x^*, i^*y^*]_H = \langle Q_\infty A^* x^*, y^*\rangle = [i_\infty^* A^* x^*, i_\infty^* y^*]_{H_\infty}.$$

Hence, $Bi^*x^* \in \mathscr{D}(V^*)$ and $V^*Bi^*x^* = i_\infty^* A^* x^*$.

¿From [7, Theorem 3.5] we know that for all $g \in \mathscr{F}C_b^1(E)$ and $h \in \mathscr{D}(V^*)$ we have $g \otimes h \in \mathscr{D}(D_H^*)$ and

$$D_H^*(g \otimes h) = \phi_{V^*h}\, g - [D_H g, h]_H. \tag{2.4}$$

Fix $x_1^*,\dots,x_n^*\in\mathscr{D}(A^*)$ and define $T:E\to\mathbb{R}^n$ by $Tx:=(\langle x,x_1^*\rangle,\dots,\langle x,x_n^*\rangle)$. Using the identity $B+B^*=-I$ we obtain, for $f\in\mathscr{F}C_c^{2,1}(E)$ as in (2.2), that

$$\begin{aligned}
&\sum_{j=1}^n\sum_{k=1}^n[i^*x_k^*,Bi^*x_j^*]_H\frac{\partial^2\phi}{\partial x_j\partial x_k}\circ T\\
&=\tfrac12\sum_{j=1}^n\sum_{k=1}^n\big([i^*x_k^*,Bi^*x_j^*]_H+[i^*x_j^*,Bi^*x_k^*]_H\big)\frac{\partial^2\phi}{\partial x_j\partial x_k}\circ T\\
&=-\tfrac12\sum_{j=1}^n\sum_{k=1}^n[i^*x_k^*,i^*x_j^*]_H\frac{\partial^2\phi}{\partial x_j\partial x_k}\circ T\\
&=-\tfrac12\mathrm{tr}\,D_H^2f.
\end{aligned}\tag{2.5}$$

Combining (2.4) (applied with $g=\frac{\partial\phi}{\partial x_j}\circ T$) and (2.5) we obtain

$$\begin{aligned}
D_H^*BD_Hf&=\sum_{j=1}^n\phi_{V^*Bi^*x_j^*}\big(\frac{\partial\phi}{\partial x_j}\circ T\big)-\big[D_H\big(\frac{\partial\phi}{\partial x_j}\circ T\big),Bi^*x_j^*\big]_H\\
&=\sum_{j=1}^n\langle\,\cdot\,,A^*x_j^*\rangle\big(\frac{\partial\phi}{\partial x_j}\circ T\big)-\sum_{k=1}^n\sum_{j=1}^n[i^*x_k^*,Bi^*x_j^*]_H\frac{\partial^2\phi}{\partial x_k\partial x_j}\circ T\\
&=\langle\,\cdot\,,A^*Df\rangle+\tfrac12\mathrm{tr}\,D_H^2f=Lf.
\end{aligned}$$

□

This result allows us to study the properties of L in $L^2(E,\mu_\infty)$ with form methods. Let ℓ be the densely defined form with domain $\mathscr{D}(\ell)=W_H^{1,2}(E,\mu_\infty)$ defined by

$$\ell(f,g):=\langle BD_Hf,D_Hg\rangle.$$

In this formula, the brackets refer to the inner product of $L^2(E,\mu_\infty;H)$.

Proposition 2.4. *The form ℓ is closed, continuous, and dissipative. Moreover, L is the operator associated with ℓ, and $\mathscr{D}(L)$ is a core for $\mathscr{D}(\ell)$.*

Proof. To prove closedness we need to show that $\mathscr{D}(\ell)$ is complete with respect to the norm $\|f\|_\ell:=\|f\|_2-\mathrm{Re}\,\ell(f,f)$ $(=\|f\|_2-\ell(f,f)$ since we are working over the real scalars). This follows from the fact that D_H is a closed operator with domain $W_H^{1,2}(E,\mu_\infty)$. To prove continuity we need to show that there is a constant $M\geqslant 0$ such that $|\ell(f,g)|\leqslant M\|f\|_\ell\|g\|_\ell$ for all $f,g\in\mathscr{D}(\ell)$. This follows from

$$|\ell(f,g)|\leqslant\|B\|\cdot\|D_Hf\|_2\cdot\|D_Hg\|_2\leqslant 2\|B\|\cdot\|f\|_\ell\cdot\|g\|_\ell.$$

To prove dissipativity we need to show that $\ell(f,f)\leqslant 0$ for all $f\in\mathscr{D}(\ell)$. This follows from

$$\ell(f,f)=\langle BD_Hf,D_Hf\rangle=-\tfrac12\|D_Hf\|_2^2\leqslant 0.$$

The fact that L is associated with ℓ follows from Theorem 2.3; that $\mathscr{D}(L)$ is a core for $\mathscr{D}(\ell)$ follows from [13, Lemma 1.25]. □

We shall not pursue this point here and content ourselves with the observation that Proposition 2.4 implies that in $L^2(E,\mu_\infty)$ we have the domain inclusion

$$\mathscr{D}(L)\hookrightarrow W_H^{1,2}(E,\mu_\infty).$$

3. The sector of analyticity of the Ornstein-Uhlenbeck semigroup

Let X be a complex Banach space. For an element $x \in X$ we define its *duality set* by

$$\partial x := \{x^* \in X^* : \|x\| = \|x^*\| \text{ and } \langle x, x^*\rangle = \|x\|\,\|x^*\|\}.$$

By the Hahn-Banach Theorem, $\partial(x) \neq \varnothing$ for all $x \in X$.

Example 3.1. Let (M, μ) be a σ-finite measure space and let $1 \leqslant p < \infty$. With respect to the duality pairing $\langle f, g\rangle = \int_M fg\,d\mu$ (note that there is no complex conjugation), for all $f \in L^p(M)$ we have

$$\partial f = \{\|f\|_p^{2-p} f^*\},$$

where $f^* := |f|^{p-2}\overline{f}$.

Fix $\theta \in [0, \frac{\pi}{2})$ and put

$$C_\theta := \cot\theta.$$

Note that $\lambda \in \overline{\Sigma}_{\frac{\pi}{2}-\theta}$ if and only if $|\text{Im } \lambda| \leqslant C_\theta \,\text{Re } \lambda$. We will apply the following well-known criterion to show that the Ornstein-Uhlenbeck semigroup is analytic on a certain sector in the complex plane. For a proof see [9, Theorem 11.4].

Proposition 3.2. *Let $\mathscr{A}$ be a densely defined operator on X and assume that $1 \in \varrho(\mathscr{A})$. The following assertions are equivalent:*

1. *$\mathscr{A}$ generates an analytic C_0-semigroup on E which is contractive on Σ_θ;*
2. *For all $0 \neq x \in \mathscr{D}(\mathscr{A})$ and all $x^* \in \partial(x)$ we have*
$$|\text{Im } \langle \mathscr{A}x, x^*\rangle| \leqslant -C_\theta \,\text{Re } \langle \mathscr{A}x, x^*\rangle;$$
3. *For all $0 \neq x \in \mathscr{D}(\mathscr{A})$ there exists $x^* \in \partial(x)$ such that*
$$|\text{Im } \langle \mathscr{A}x, x^*\rangle| \leqslant -C_\theta \,\text{Re } \langle \mathscr{A}x, x^*\rangle.$$

After these preliminaries we return to the setting of Section 2 and leave it to the reader to check that all results proved so far can be extended to the complex case by means of complexification.

Repeating the computations of [2] we arrive at the following two identities:

Lemma 3.3. *Let $p \in [2, \infty)$ and $f \in \mathscr{F}C_c^{2,1}(E)$. Then,*

$$\begin{aligned} -\text{Re } [BD_H f, D_H\overline{f^*}]_H &= -\text{Re } [B^* D_H f, D_H \overline{f^*}]_H \\ &= \tfrac{1}{2}|f|^{p-4}\big((p-1)\|\text{Re } (\overline{f}D_H f)\|_H^2 + \|\text{Im } (\overline{f}D_H f)\|_H^2\big); \end{aligned}$$

and

$$\begin{aligned} \text{Im } [BD_H f, D_H\overline{f^*}]_H &= p|f|^{p-4}\big[(B + \tfrac{1}{p}I)\,\text{Im } (\overline{f}D_H f), \text{Re } (\overline{f}D_H f)\big]_H, \\ \text{Im } [B^* D_H f, D_H\overline{f^*}]_H &= p|f|^{p-4}\big[(B^* + \tfrac{1}{p}I)\,\text{Im } (\overline{f}D_H f), \text{Re } (\overline{f}D_H f)\big]_H. \end{aligned}$$

Theorem 3.4. *Assume that the Ornstein-Uhlenbeck semigroup P is analytic on $L^p(E, \mu_\infty)$ for some (and hence all) $1 < p < \infty$. Then for all $1 < p < \infty$, P is analytic and contractive on the sector Σ_{θ_p}, where*

$$\cot\theta_p := \frac{\sqrt{(p-2)^2 + p^2\gamma^2}}{2\sqrt{p-1}} \tag{3.1}$$

and $\gamma := \|B - B^*\|$.

Proof. The proof follows the arguments of [2]. First we take $p \geqslant 2$. Using that $B - B^*$ is skewadjoint it is easily checked that

$$\|B + \tfrac{1}{p}I\|^2 = \tfrac{1}{4}\gamma^2 + (\tfrac{1}{2} - \tfrac{1}{p})^2.$$

Let $f \in \mathscr{F}C_c^{2,1}(E)$ be fixed. With

$$a := \|\mathrm{Re}\,(\overline{f}D_H f)\|_H, \quad b := \|\mathrm{Im}\,(\overline{f}D_H f)\|_H$$

it follows from the first equality in Lemma 3.3 that

$$-\mathrm{Re}\,[BD_H f, D_H\overline{f^*}]_H = \tfrac{1}{2}|f|^{p-4}((p-1)a^2 + b^2).$$

By the Cauchy-Schwarz inequality and the second equality in Lemma 3.3 yields

$$\big|\mathrm{Im}\,[BD_H f, D_H\overline{f^*}]_H\big| \leqslant |f|^{p-4}abc_p\sqrt{p-1},$$

where $c_p := \sqrt{p^2\gamma^2 + (p-2)^2}/2\sqrt{p-1}$. Hence, using the inequality $2ab\sqrt{p-1} \leqslant (p-1)a^2 + b^2$,

(3.2)
$$\big|\mathrm{Im}\,[BD_H f, D_H\overline{f^*}]_H\big| \leqslant \tfrac{1}{2}|f|^{p-4}c_p((p-1)a^2 + b^2) = -c_p\,\mathrm{Re}\,[BD_H f, D_H\overline{f^*}]_H.$$

In a similar way one proves that

$$\big|\mathrm{Im}\,[B^*D_H f, D_H\overline{f^*}]_H\big| \leqslant -c_p\,\mathrm{Re}\,[B^*D_H f, D_H\overline{f^*}]_H. \tag{3.3}$$

¿From Proposition 2.4 and (3.2) we obtain

$$\begin{aligned}\Big|\mathrm{Im}\int_E Lf\cdot f^*\,d\mu_\infty\Big| &\leqslant \int_E \big|\mathrm{Im}\,[BD_H f, BD_H\overline{f^*}]_H\big|\,d\mu_\infty\\ &\leqslant \int_E -c_p\,\mathrm{Re}\,[BD_H f, D_H\overline{f^*}]_H\,d\mu_\infty = -c_p\,\mathrm{Re}\int_E Lf\cdot f^*\,d\mu_\infty.\end{aligned}$$

By approximation this inequality extends to all $f \in \mathscr{D}(L)$. Now we can apply Proposition 3.2 to obtain the desired result.

For $p \in (1,2)$ we use a duality argument. For $f \in \mathscr{F}C_c^{2,1}(E)$ we have

$$\int_E Lf\cdot f^*\,d\mu_\infty = \int_E [B^*D_H g, D_H\overline{g^*}]_H\,d\mu_\infty,$$

where $g := \overline{f^*}$ belongs to $L^q(E,\mu_\infty)$, $\frac{1}{p} + \frac{1}{q} = 1$. The desired result now follows from the estimate (3.3) applied to g. □

This result is optimal in the following sense:

Theorem 3.5. *If, for some $1 < p < \infty$, the Ornstein-Uhlenbeck semigroup P on $L^p(E,\mu_\infty)$ is analytic and contractive on a sector Σ_θ for some $\theta \in (0,\frac{\pi}{2})$, then $\theta \leqslant \theta_p$.*

Here, of course, θ_p is the angle defined by (3.1). The proof of Theorem 3.5 follows the lines of [2], but there are some subtle differences. In particular, since we are working in infinite dimensions the diagonalization arguments used in [2] have to be avoided.

For $h \in H_\infty$ we define $K_h : E \to \mathbb{C}$ by

$$K_h(x) := \exp(\phi_h(x) - \tfrac{1}{2}[h,\overline{h}]_{H_\infty}),$$

where $\phi : H_\infty \to L^2(E,\mu_\infty)$ is defined by (2.3). Then $K_h \in L^p(E,\mu_\infty)$ for all $1 < p < \infty$, and by a second quantization argument (see [4, 12]) we see that

$$P(t)K_h = K_{S^*_\infty(t)h}, \quad h \in H_\infty,\ t \geqslant 0,$$

first in $L^2(E,\mu_\infty)$ and then also in $L^p(E,\mu_\infty)$ by consistency. By an analytic continuation argument, this implies that

$$P(z)K_h = K_{S^*_\infty(z)h}, \quad h\in H_\infty,\ z\in\Sigma_\theta, \tag{3.4}$$

where Σ_θ is as in the theorem.

Lemma 3.6. *For all $h\in H_\infty$ and $z\in\Sigma_\theta$ we have*

$$(p-1)\|\mathrm{Re}\, S^*_\infty(z)h\|^2_{H_\infty} + \|\mathrm{Im}\, S^*_\infty(z)h\|^2_{H_\infty} \leqslant (p-1)\|\mathrm{Re}\, h\|^2_{H_\infty} + \|\mathrm{Im}\, h\|^2_{H_\infty}.$$

Proof. First let $h = i^*_\infty x^*$ for some $x^*\in E^*$ and put $g(x) := \exp(\phi_h(x))$. Then

$$\int_E |g(x)|^p\, d\mu_\infty(x) = \int_E \exp(p\langle x, \mathrm{Re}\, x^*\rangle)\, d\mu_\infty(x) = \int_{\mathbb{R}} \exp(pu)\, d(\tau\mu_\infty)(u),$$

where $\tau x := \langle x, \mathrm{Re}\, x^*\rangle$ so that $\tau\mu_\infty$ is Gaussian with variance $\sigma^2 = \|\mathrm{Re}\, h\|^2_{H_\infty}$. Therefore,

$$\int_E |g(x)|^p\, d\mu_\infty(x) = \frac{1}{\sigma\sqrt{2\pi}}\int_{\mathbb{R}} \exp\big(pu - \frac{u^2}{2\sigma^2}\big)\, du = \exp\big(\frac{\sigma^2 p^2}{2}\big).$$

Following the argument of [2, Lemma 7] we obtain

$$\begin{aligned}\|K_h\|_p &= \big|\exp(-\tfrac12[h,\overline{h}]_{H_\infty})\big|\Big(\int_E |g(x)|^p\, d\mu_\infty\Big)^{1/p}\\ &= \exp\Big(\frac{\|\mathrm{Im}\, h\|^2_{H_\infty} - \|\mathrm{Re}\, h\|^2_{H_\infty}}{2}\Big)\exp\Big(\frac{p\|\mathrm{Re}\, h\|^2_{H_\infty}}{2}\Big)\\ &= \exp\Big(\tfrac12\|\mathrm{Im}\, h\|^2_{H_\infty} + \frac{p-1}{2}\|\mathrm{Re}\, h\|^2_{H_\infty}\Big).\end{aligned} \tag{3.5}$$

Hence, with (3.4) and (3.5),

$$\begin{aligned}\frac{\|P(z)K_h\|_p}{\|K_h\|_p} = \exp\Big(\tfrac12\big((p-1)\|\mathrm{Re}\, S^*_\infty(z)h\|^2_{H_\infty}\\ + \|\mathrm{Im}\, S^*_\infty(z)h\|^2_{H_\infty} - (p-1)\|\mathrm{Re}\, h\|^2_{H_\infty} - \|\mathrm{Im}\, h\|^2_{H_\infty}\big)\Big).\end{aligned} \tag{3.6}$$

Since $P(z)$ is a bounded operator, the exponent in (3.6) has to remain bounded if we replace h by λh and let $\lambda\to\infty$. Therefore the exponent is nonpositive and the lemma is proved for elements $h\in H_\infty$ of the form $h = i^*_\infty x^*$. The result extends to arbitrary $h\in H_\infty$ by a density argument. □

Proof of Theorem 3.5. For $j\in\{1,2\}$ let $x^*_j\in\mathscr{D}(A^*)$, $h_j := i^*_\infty x^*_j$ and $h = h_1 + ih_2$. As in [2] we check that for all $\varphi\in(-\theta,\theta)$,

$$\begin{aligned}(p-1)\cos\varphi[A^*_\infty h_1, h_1]_{H_\infty} + \cos\varphi[A^*_\infty h_2, h_2]_{H_\infty}\\ \leqslant (p-1)\sin\varphi[A^*_\infty h_2, h_1]_{H_\infty} - \sin\varphi[A^*_\infty h_1, h_2]_{H_\infty}.\end{aligned}$$

Observe that

$$[A^*_\infty h_1, h_2]_{H_\infty} = [i^*_\infty A^* x^*_1, i^*_\infty x^*_2]_{H_\infty} = \langle Q_\infty A^* x^*_1, x^*_2\rangle = [Bi^*x^*_1, i^*x^*_2]_H.$$

Therefore

$$\begin{aligned}(p-1)[A^*_\infty h_1, h_1]_{H_\infty} + [A^*_\infty h_2, h_2]_{H_\infty} &= (p-1)[Bi^*x^*_1, i^*x^*_1]_H + [Bi^*x^*_2, i^*x^*_2]_H\\ &= -\tfrac12\big((p-1)\|i^*x^*_1\|^2_H + \|i^*x^*_2\|^2_H\big),\end{aligned}$$

and

$$\begin{aligned}(p-1)[A_\infty^* h_2, h_1]_{H_\infty} - [A_\infty^* h_1, h_2]_{H_\infty} \\ &= (p-1)[Bi^*x_2^*, i^*x_1^*]_H - [Bi^*x_1^*, i^*x_2^*]_H \\ &= (p-1)[Bi^*x_2^*, i^*x_1^*]_H + [(I+B)i^*x_2^*, i^*x_1^*]_H \\ &= [(pB+I)i^*x_2^*, i^*x_1^*]_H \\ &= \tfrac{1}{2}\big(p[(I+2B)i^*x_2^*, i^*x_1^*]_H + (2-p)[i^*x_2^*, i^*x_1^*]_H\big).\end{aligned}$$

It follows that

$$\begin{aligned}\sin\varphi\big(-p[(I+2B)i^*x_2^*, i^*x_1^*]_H + (p-2)[i^*x_2^*, i^*x_1^*]_H\big) \\ \leqslant \cos\varphi\big((p-1)\|i^*x_1^*\|_H^2 + \|i^*x_2^*\|_H^2\big).\end{aligned}$$

Now, using the fact that the operator $D := (I+2B) + (1-\frac{2}{p})I$ is normal and therefore satisfies $r(D) = \|D\|$, the proof can be finished in the same way as in [2]. □

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