Some recent results on adjoint semigroups

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In this expository paper, we discuss some recent results in the theory of adjoint semigroups.

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0. Introduction

1. Strong continuity of the adjoint semigroup

Let us start with an elementary example of a C_0-semigroup whose adjoint fails to be strongly continuous.

Example 1.1. Let $T(t)$ be the translation semigroup on $C_0(\mathbb{R})$, the Banach space of continuous functions vanishing at infinity with the sup-norm;

$$(T(t)f)(s) = f(s+t).$$

One easily checks that this is indeed a C_0-semigroup and that its adjoint on $(C_0(\mathbb{R}))^* = M(\mathbb{R})$, the space of bounded Borel measures on $\mathbb{R}$, is given by

$$(T^*(t)\mu)(F) = \mu(F-t).$$

Taking for μ a Dirac measure δ, it is clear that

$$\limsup_{t\downarrow 0} \|T^*(t)\delta - \delta\| = 2.$$

The occurence of the number 2 is no coincidence; cf. Theorem ??? below.
Although $T^*(t)$ need not be strongly continuous, the inequality

$$|\langle T^*(t)x^* - x^*, x\rangle| \leqslant \|x^*\|\ \|T(t)x - x\|$$

implies that $T^*(t)$ is weak*-continuous. Hence, if X is a reflexive Banach space, then $T(t)$ is weakly continuous. By a standard theorem of semigroup theory, weakly continuous semigroups are strongly continuous, and we obtain:

Theorem 1.2 [Ph]. *If $T(t)$ is a C_0-semigroup on a reflexive Banach space, then $X^{\odot} = X^*$.*

The converse of this theorem is not true: there are non-reflexive Banach spaces on which the adjoint of every C_0-semigroup is strongly continuous. In fact, it is a theorem of Lotz [L] that every C_0-semigroup on $L^\infty[0,1]$ is uniformly continuous, i.e.

$$\lim_{t\downarrow 0} \|T(t) - I\| = 0.$$

Of course, the adjoint of such a semigroup is uniformly continuous as well, and hence strongly continuous.

Lotz's theorem shows that there exists Banach spaces X which admit 'trivial' semigroups only. If X does admit 'sufficiently many' semigroups, then the converse of Theorem 1.2 does hold. A sequence (x_n) is called a *Schauder basis* for X if for each $x \in X$ there is a unique scalar sequence (α_n) such that

$$x = \sum_{n=1}^{\infty} \alpha_n x_n.$$

The *coordinate functionals* are the (bounded) functionals $(x_n)^*$ given by

$$\langle x_n^*, x\rangle = \alpha_n.$$

Theorem 1.3. *If X is a non-reflexive Banach space with a shauder basis, then there exists a C_0-semigroup on X whose adjoint fails to be strongly continuous.*

In fact, X has a (probably different) Schauder basis whose coordinate functionals span a *proper* closed subspace of X^*. On the other hand, on every Banach space with a Schauder basis the formula

$$T(t)x_n = e^{-nt}x_n$$

can be shown to define a C_0-semigroup such that $X^{\odot}$ is precisely the closed linear span of the coordinate functionals.

Theorem 1.3 shows that in general reflexivity if the only sufficient criterion that guaratees $X^{\odot} = X^*$. If one restricts oneself to special classes of Banach spaces or semigroups however, one may hope for stronger results. The most striking one is the following theorem about c_0, the Banach space of all scalar sequences which converge to 0 with the sup-norm.

Theorem 1.4. *Let $T(t)$ be a C_0-semigroup on c_0. If there exist $M < 2$ and $\omega \in \mathbb{R}$ such that $\|T(t)\| \leqslant Me^{\omega t}$, then $c_0^{\odot} = c_0^*$.*

Recall that for every C_0-semigroup there are constants $M \geqslant 1$ and $\omega \in \mathbb{R}$ such that $\|T(t)\| \leqslant Me^{\omega t}$; the point is that M should be less than 2. At first sight, the role of the number 2 is quite mysterious. Here is a full explanation. For a closed subspace Y of a dual Banach space X^* define the *characteristic* $\rho(Y)$ of Y by

$$\rho(Y) := \inf_{x\in X, \|x\|=1} \|x\|_Y,$$

where $\|x\|_Y := \sup_{y\in B_Y} |\langle y, x\rangle|$. In other words, we norm X with Y and ask how bad this norm is. If $\alpha\|\cdot\| \leqslant \|\cdot\|_Y \leqslant \|\cdot\|$, then by definition $\rho(Y) \geqslant \alpha$. Also, if Y

is not weak*-dense, then $\rho(Y) = 0$. Now $X := c_0$ can be shown [GS] to have the following property: *If Y is any proper closed subspace of X^*, then $\rho(Y) \leqslant \frac{1}{2}$.* If $\|T(t)\| \leqslant Me^{\omega t}$ with $M < 2$, then for all $x \in c_0$ we have

$$\frac{1}{2}\|x\| < \frac{1}{M}\|x\| \leqslant \|x\|' = \|x\|_{c_0^\odot}$$

and it follows immediately that $c_0^\odot$ cannot be a *proper* subspace of l^1. An entirely elementary proof of Theorem 1.4 is given in [Ne].

The constant 2 is optimal, as is shows by the following example.

Example 1.5. (The summing semigroup) Let $x_n = (0, 0, ..., 0, 1, 0, ...)$ the nth unit vector of c_0 and put

$$y_n = \sum_{k=1}^{n} x_n = (1, 1, ..., 1, 0, 0, ...).$$

The sequence (y_n) can be shown to be a Schauder basis for c_0 (the so-called summing basis). The formula

$$T(t)y_n = e^{-nt}y_n$$

then defines a C_0-semigroup on c_0 satisfying $\|T(t)\| \leqslant 2$ for all t. Moreover, $c_0^\odot$ is the closed linear span of the coordinate functionals of (y_n), which is a co-dimension one subspace of c_0^*.

As a final example, there is the following result for C_0-groups.

Theorem 1.6. *Suppose $T(t)$ is a C_0-group on a Banach space X whose dual has the Radon-Nikodym property. Then $X^\odot = X^*$.*

If fact, if $T(t)$ is a C_0-semigroup on such a space, then it can be shown that the adjoint semigroup is strongly continuous for $t > 0$.

2. The co-dimension of $X^\odot$ in X^*

Knowing that $X^\odot$ can be a proper subspace of X^*, the question arises what can be said about its 'relative size' in X^*. We noted already in the introduction that $X^\odot$ is weak*-dense in X^*, but with respect to the norm-topology the situation is far more subtle. In that case, the natural object of study is the size of the quotient space $X^*/X^\odot$. We start with noting that there is a nice description of the quotient norm of $X^*/X^\odot$.

Theorem 2.1. *Let $T(t)$ be a C_0-semigroup on a Banach space X. Then*

$$\|qx^*\| = \limsup_{t\downarrow} \|T^*(t)x^* - x^*\|$$

defines an equivalent norm on $X^*/X^\odot$.

Example 1.5 seems to indicate that not very much can be said about the size of $X^*/X^\odot$. Indeed, for that semigroup we have

$$\dim c_0^*/c_0^\odot = 1,$$

and by taking direct sums it is possible to construct semigroups for which $X^*/X^\odot$ can have any finite dimension. Let us analyse this example more closely. Since $c_0^* = l^1$ has the Radon-Nikodym property, by the remark after Theorem 1.6 the adjoint semigroup is strongly continuous for $t > 0$. This is equivalent to saying that $T^*(t)x^* \in c_0^\odot$ for every $t > 0$ and $x^* \in l^1$. Letting $q : c_0^* \to c_0^*/c_0^\odot$ be the quotient map, this is in turn equivalent to saying that $q(T^*(t)x^*) = 0$ for all $t > 0$ and $x^* \in l^1$.

On $X^*/X^\odot$, there is a natural quotient semigroup $T_q(t)$, defined by

$$T_q(t)qx^* = q(T^*(t)x^*).$$

In the above example, all orbits of $T_q(t)$ are zero for $t > 0$. In general, one has the following result. We say that a Banach valued function is *separably valued* if its range is contained in some separable subspace.

Theorem 2.2. *Let $T(t)$ be a C_0-semigroup on a Banach space X. If the orbit $t \mapsto T_q(t)qx^*$ is separably-valued, then $T_q(t)qx^* = 0$ for all $t > 0$.*

In particular this theorem implies that non-zero orbits of the quotient semigroup cannot be strongly continuous. An elegant elementary proof of this fact was recently obtained by Ben de Pagter.

Corollary 2.3. *If $X^*/X^\odot$ is separable, then $T^*(t)$ is strongly continuous for $t > 0$.*

Corollary 2.4. *If $T(t)$ extends to a C_0-group, then $X^*/X^\odot$ is either zero or non-separable.*

3. The adjoint of a positive semigroup

Many semigroups encountered in applications are positive, i.e. they map positive elements to positive elements. Throughout this section, we assume that $T(t)$ is a positive C_0-semigroup on a Banach lattice E.

The first question we address is whether $E^\odot$ has some nice lattice properties if $T(t)$ has. For example, one might hope that $E^\odot$ is a sublattice if $T(t)$ is positive. This was an open problem for some time and was finally solved to the negative by Grabosch and Nagel [???], who constructed a counterexample on an L^1-space E.

On the other hand, if E is for example a space of continuous functions, then $E^\odot$ is even a projection band [???]:

Theorem 3.1. *Let $T(t)$ be a positive C_0-semigroup on a Banach lattice E. If E^* has order continuous norm, then $E^\odot$ is a projection band in E^*.*

Apart from the almost trivial fact that $E^{\odot}$ is a sublattice if $T(t)$ is a positive C_0-group, and more geenrally if $T^*(t)$ is a lattice semigroup, no significant positive results about the lattice structure of $E^{\odot}$ are known.

The next two results are concerned with the behaviour of the adjoint itself.

Theorem 3.2. *Let $T(t)$ be a positive C_0-semigroup on $E = C(K)$, K compact Hausdorff. If $T^*(t)$ is weakly Borel measurable, then it is strongly continuous for $t > 0$.*

In particular, if $T(t)$ extends to a C_0-group, then $T^*(t)x^*$ is weakly Borel measurable if and only if $x^* \in E^{\odot}$.

Theorem 3.2 is highly non-trivial; it depends on a deep result result of Riddle, Saab and Uhl that a weakly Borel measurable map taking values in the dual of a separable Banach space is Pettis integrable, and on a detailed analysis of the behaviour of the second adjoint semigroup $T^{**}(t)$. One might wonder whether weak (i.e. weak Lebesgue) measurability already implies strong continuity for $t > 0$. Under certain set-theoretical assumptions, this is true, but it is an open question whether this can be proved within ZFC.

At least equally non-trivial is the following beautiful result of Talagrand [T]:

Theorem 3.3. *Let $T(t)$ be the translation group on $L^1(\mathbb{R})$. If for some $f \in L^\infty(\mathbb{R})$ the orbit $t \mapsto T^*(t)f$ is weakly measurable, then f is equal a.e. to a Riemann measurable function.*

Recall that a function is Riemann measurable if it is continuous a.e.

These results support the following conjecture of Ben de Pagter: *Let $T(t)$ be a positive C_0-group on a Banach lattice E and let $x^* \in E^*$. Then the orbit $t \mapsto T^*(t)x^*$ is weakly measurable if and only if x^* belongs to the Dedekind closure of $E^{\odot}$ in E^*.* Indeed, the Dedekind closure of $C_0(\mathbb{R})$ (which is the $\odot$-dual of $L^1(\mathbb{R})$ with respect to the translation group) is precisely the space of Riemann measurable functions. On the other hand, a projection band (for example, the $\odot$-dual of $C(K)$ with respect to any positive C_0-semigroup) is always Dedekind closed.

After these 'weak implies strong' results, we turn to the lattice properties of individual orbits of $T^*(t)$. The most interesting results are concerned with the behaviuor of elements is E^* which are disjoint from $E^{\odot}$. Before turning to these, let us remark that such elements do not exist if E is σ-Dedekind complete, e.g. if $E = L^1(\mu)$ or more generally a Banach function space. Thus the results to follow are non-empty only for 'continuous functions'-like spaces, such as $C(K)$ or $C_0(\Omega)$.

Theorem 3.4. *Let $T(t)$ be a positive C_0-semigroup on a Banach lattice E. Suppose that either E has a quasi-interior point of E^* has order continuos norm. If $x^* \perp E^{\odot}$, then $T^*(t)x^* \perp x^*$ for almost all $t \geqslant 0$.*

Recall that $u \in E$ us a quasi-interior point if the ideal generated by u is norm dense in E. Every separable Banach lattice and every L^∞-space have quasi-interior points.

In the special case where $T(t)$ is the translation group on $E = C_0(\mathbb{R})$, we have $x^* \perp E^{\odot} = L^1(\mathbb{R})$ if and only if x^* is singular with respect to the Lebesgue measure, an the theorem reduces to the classical theorem of Wiener and Young that a singluar measure on $\mathbb{R}$ is disjoint to almost all of its translates.

Theorem 3.4 fails for arbitrary Banach lattices. Nevertheless we have the following result.

Theorem 3.5. *Let $T(t)$ be a positive C_0-semigroup on a Banach lattice E. If $x^* \perp E^\odot$, then*

$$\limsup_{t\downarrow 0} \|T^*(t)x^* - x^*\| \geqslant 2\|x^*\|.$$

If E has a quasi-interior point or E^* has order continuous norm, this follows readily from Theorem 3.4; the general case can be reduced to this, but can also be proved directly.

Our final result is concerned with the question when the disjoint complement of $E^\odot$ is $T^*(t)$-invariant.

Theorem 3.6. *Let $T(t)$ be a positive C_0-semigroup on a Banach lattice E. If $T^*(t)$ is a lattice semigroup, then the disjoint complement of $E^\odot$ is $T^*(t)$-invariant.*

In particular this is the case is $T(t)$ extends to a positive group.

Our final two results are concerned woth multiplication semigroups. A C_0-semigroup on a Banach lattice E is called a *multiplication semigroup* if each operator $T(t)$ is a band preserving operator. On can show that on most classical space of functions, an operator is band preserving if and only if it can be represented as multiplication with some (continuous, measurable) function. Multiplication semigroups are positive.

Theorem 3.7. *If $T(t)$ is a multiplication semigroup, then $E^\odot$ is an ideal in E^* and $T^*(t)$ is strongly continuous for $t > 0$.*

There are two trivial examples of $\odot$-reflexive multiplication semigroups: those on reflexive Banach lattices E, and multiplication semigroups of the form $T(t)x_n = e^{k_n t}x_n$, where (x_n) is an unconditional Schauder basis for E and (k_n) is a sequence of real numbers which is bounded from above. Not that in both cases, E has order continuous norm. The following therorem states that these are essentially the only exampes:

Theorem 3.8. *If E is $\odot$-reflexive with respect to a multiplication semigroup, then E has order continuous norm. Furthermore, if E does not contain a reflexive projection band, then E has an unconditional Schauder basis (x_n) and $T(t)$ is of the form $T(t)x_n = e^{k_n t}x_n$, where (k_n) is a sequence of real numbers which is bounded from above.*

In general, if E is reflexive with respect to a positive C_0-semigroup, then E need not have order continuous norm, as is shown be the rotation group on $C(\Gamma)$, Γ being the unit circle. However, the following is true:

Theorem 3.9. *If a Banach space X is $\odot$-reflexive with respect to a C_0-semigroup $T(t)$, then X does not contain a closed subspace isomorphic to l^∞. In particular, if X is a σ-Dedekind complete Banach lattice, then X must have order continuous norm.*

The second statement follows from the general result in Banach lattice theory that a σ-Dedekind complete Banach lattice has order continuous norm if it does not contain a closed subspace isomorphic to l^∞.

4. References