On the orbits of an operator with spectral radius one

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Recently, V. Müller proved the following result. Let T be a bounded operator on a complex Banach space X with $r(T) = 1$. Then for all $0 < \epsilon < 1$ and $(\alpha_n) \in c_0$ of norm one there is a norm one vector $x \in X$ such that

$$\|T^k x\| \geq (1-\epsilon)|\alpha_k|, \quad \forall k = 0, 1, 2, ...$$

In this note, we give a completely elementary proof for the power bounded case, which works in the real case as well. Also, we give some analogous results for the weak orbits $\langle x^*, T^n x\rangle$.

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0. Introduction

In [M], V. Müller proved the following theorem.

Theorem. *Let T be a bounded operator on a complex Banach space X with spectral radius $r(T) = 1$. Then for all $0 < \epsilon < 1$ and $(\alpha_n) \in c_0$ of norm one there is a norm one vector $x \in X$ such that*

$$\|T^k x\| \geq (1-\epsilon)|\alpha_k|, \quad \forall k = 0, 1, 2, ...$$

For bounded operators on a Hilbert space, the above result was proved by Beauzamy [B, Thm. III.2.A.1]. He also shows that if there is no point spectrum on $\{|z| = 1\}$, such an x can be found in any ball of radius one.

For an application of the Theorem to stability theory of semigroups of operators, see [N].

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The proof given in [M] relies on results from Fredholm theory. In fact, in case $r_e(T) < r(T) = 1$, where $r_e(T)$ is the essential spectral radius, there is an unimodular eigenvalue, and the theorem is trivial. The actual proof therefore concentrates on the case $r_e(T) = r(T)$.

For power bounded operators T, we will give a completely elementary proof of the Theorem. We do not use spectral theory, and our method works for both real and complex Banach spaces. In the case of a real Banach space, we define $r(T) = r(T_{\mathbb{C}})$, where $T_{\mathbb{C}}$ is the complexification of T; cf. [Ru].

As usual, c_0 denotes the Banach space of all sequences $\alpha = (\alpha_n)_{n=0}^{\infty}$ that converge to zero, with norm $\|\alpha\| = \sup_n |\alpha_n|$.

1. Proof of the Theorem for power bounded operators

Lemma 1. *Suppose T is a bounded operator on a real or complex Banach space X with $r(T) = 1$. Then there exists a constant $C > 0$ with the following property. For each sequence $\alpha \in c_0$ of norm ≤ 1 there exists a norm one vector $x \in X$ and a subsequence (n_k) such that*

$$\|T^{n_k} x\| \geq C|\alpha_k|, \quad \forall k = 0, 1, 2, ..$$

Proof: First note that we may assume without loss of generality that $T^n \to 0$ strongly. In particular, by the uniform boundedness theorem there is a constant M such that $\sup_n \|T^n\| = M < \infty$. Let $\alpha \in c_0$ be of norm one. Fix $0 < c < \frac{1}{2}M^{-1}$, fix $0 < \delta < c$ and choose m so large that

$$2^{-m+1} + M \sum_{i=1}^{\infty} 2^{-mi} < \delta \quad \text{and} \quad \sum_{i=0}^{\infty} 2^{-mi} < 1 + \delta.$$

(In fact, the second is implied by the first).

Put $N_{-1} = -1$, $M_{-1} = -1$. Choose $N_0 \geq 0$ such that $|\alpha_i| \leq 2^{-m}$, $\forall i \geq N_0$.

In the complex case, $r(T) \geq 1$ implies that $\|T^n\| \geq 1$ for all $n \in \mathbb{N}$. In the real case, we use that $\|T_{\mathbb{C}}\| \leq 2\|T\|$ to conclude that $r(T) \geq 1$ implies $\|T^n\| \geq \frac{1}{2}$ for all $n \in \mathbb{N}$. In either case, the choice of c implies that there is a norm one vector $x_0 \in X$ such that $\|T^{N_0} x_0\| \geq cM$. For all $n = 0, 1, ..., N_0$ we have $\|T^n x_0\| \geq M^{-1}\|T^{N_0} x_0\| \geq c$. Put $n_j := j$, $j = 0, ..., N_0$. Since $\lim_n \|T^n x_0\| = 0$, we may choose M_0 such that $\|T^n x_0\| \leq 2^{-m}$, for all $n \geq M_0$.

Inductively, suppose norm one vectors $x_0, x_1, ..., x_{l-1} \in X$, and numbers $N_0 < N_1 < ... < N_{l-1}$ and $n_1 < n_2 < .... < n_{N_{l-1}}$ and $M_0, ..., M_{l-1}$ have been chosen subject to the following conditions:

(a) $|\alpha_i| \leq 2^{-m(j+1)}$, $\forall i \geq N_j$; $j = 0, 1, ..., l-1$;
(b) $n_{N_{j-1}+1} \geq M_{j-1}$, $\forall j = 0, 1, ..., l-1$;

(c) $\|T^{n_k}x_j\| \geq c$, $\forall k = N_{j-1}+1, ..., N_j$; $j = 0, ..., l-1$.
(d) $\|T^n x_i\| \leq 2^{-m(j+2)}$, $\forall 0 \leq i \leq j$ and $n \geq M_j$; $j = 0, 1, ..., l-1$.

Choose $N_l \geq N_{l-1}+1$ such that $|\alpha_i| \leq 2^{-m(l+1)}$, $\forall i \geq N_l$. Then (a) holds for the induction variable l. Choose a norm one vector $x_l \in X$ and numbers $n_{N_{l-1}+1} < ... < n_{N_l}$ such that $n_{N_{l-1}+1} > n_{N_{l-1}}$, $n_{N_{l-1}+1} \geq M_{l-1}$ (this is (b)) and

$$\|T^{n_k}x_l\| \geq c, \quad k = N_{l-1}+1, ..., N_l.$$

Then (c) is satisfied. Finally, choose M_l such that

$$\|T^n x_i\| \leq 2^{-m(l+2)}, \quad \forall 0 \leq i \leq l \text{ and } n \geq M_l.$$

Then again (a)-(d) hold for the value l. Continue this process by induction. Put

$$x := \sum_{j=0}^{\infty} 2^{-mj} x_j.$$

Now let k be a fixed integer and choose $j \geq 0$ such that $N_{j-1}+1 \leq k \leq N_j$. If $j \geq 1$, then by (a) and the fact that $k \geq N_{j-1}$ we have,

$$2^{-mj} = 2^{-m[(j-1)+1]} \geq |\alpha_k|.$$

In case $j = 0$, note that this inequality holds trivially. By (b) we have $n_k \geq n_{N_{j-1}+1} \geq M_{j-1}$ and consequently, by (d), for all $0 \leq i \leq j-1$ we have $\|T^{n_k}x_i\| \leq 2^{-m(j+1)}$. Therefore,

$$\sum_{i=0}^{j-1} 2^{-mi}\|T^{n_k}x_i\| \leq 2^{-m(j+1)+1}.$$

Also, we have the trivial estimate

$$\sum_{i=j+1}^{\infty} 2^{-mi}\|T^{n_k}x_i\| \leq 2^{-mj} M \sum_{i=1}^{\infty} 2^{-mi}.$$

Therefore,

$$\|T^{n_k}x\| \geq 2^{-mj}\left(c - 2^{-m+1} - M\sum_{i=1}^{\infty} 2^{-mi}\right) \geq 2^{-mj}(c-\delta) \geq |\alpha_k|(c-\delta).$$

Finally, observe that x has norm $\leq \sum_{j=0}^{\infty} 2^{-mj} \leq 1+\delta$. Hence, by rescaling x to a norm one vector, for the rescaled x we obtain

$$\|T^{n_k}x\| \geq \frac{c-\delta}{1+\delta}|\alpha_k|.$$

This proves the theorem, with $C = (c-\delta)/(1+\delta)$. ////

Theorem 1.2. *Let T be a power bounded operator on a real or complex Banach space X with $r(T) = 1$. Then for all $\epsilon > 0$ and all $\alpha \in c_0$ of norm one, there exists a norm one vector $x \in X$ such that*

$$\|T^k x\| \geq (1-\epsilon)|\alpha_k|, \quad \forall k = 0, 1, 2, ...$$

Proof: Step 1. Put $\sup_n \|T^n\| = M < \infty$. Define the equivalent norm $\|\!|\cdot|\!\|$ on X by $\|\!|x|\!\| = \sup_n \|T^n x\|$. Then $\|x\| \leq \|\!|x|\!\| \leq M\|x\|$ and $\|\!|T|\!\| \leq 1$. Let (β_n) be a norm one sequence in c_0 such that $\beta_n \downarrow 0$ and $\beta_n \geq |\alpha_n|$ for all n. By the Lemma, there exists a vector x of $\|\!|\cdot|\!\|$-norm one and a subsequence (n_k) such that $\|\!|T^{n_k} x|\!\| \geq C\beta_k$. Set $c := CM^{-1}$. We have $\|x\| \leq 1$, and for all k we have

$$\|T^k x\| \geq M^{-1}\|\!|T^k x|\!\| \geq M^{-1}\|\!|T^{n_k} x|\!\| \geq c\beta_k \geq c|\alpha_k|.$$

Step 2. We will now show that the constant c can actually be replaced by $1-\epsilon$. Let $0 < \epsilon < 1$ be arbitrary and fix a norm one $(\alpha_n) \in c_0$. Fix some $\delta > 0$ such that $(1-\delta)(1+\delta)^{-1} \geq 1-\epsilon$. We start by choosing integers $0 = M_0 < M_1 < ...$ such that $|\alpha_k| \leq (1+\delta)^{-n}$ whenever $k \geq M_n$. Next, choose integers $0 = N_0 < N_1 < ...$ in such a way that $N_n \geq M_n$ for each n and $N_m + N_n \leq N_{m+n}$ for all n, m. Define the norm one element $(\beta_n) \in c_0$ by $\beta_k = (1+\delta)^{-n}$ whenever $N_n \leq k < N_{n+1}$. Note that $\beta \geq |\alpha|$.

We claim that $\beta_{m+n} \geq (1+\delta)^{-1}\beta_m\beta_n$. Indeed, choose k_m and k_n such that $N_{k_m} \leq m \leq N_{k_m+1}$ and $N_{k_n} \leq n \leq N_{k_n+1}$. Then $\beta_m = (1+\delta)^{-k_m}$ and $\beta_n = (1+\delta)^{-k_n}$, whereas from $m+n < N_{k_m+1} + N_{k_n+1} \leq N_{k_m+k_n+2}$ we have $\beta_{m+n} \geq (1+\delta)^{-k_m-k_n-1}$. This proves the claim.

Now choose a norm one vector $y \in X$ such that $\|T^k y\| \geq c\beta_k$ for all k, where c is the constant of Step 1. Let

$$\gamma := \inf_k \frac{\|T^k y\|}{\beta_k}.$$

Note that $\gamma \geq c$; moreover, for all k we have $\|T^k y\| \geq \gamma\beta_k$. Choose an index k_0 such that

$$\frac{\gamma\beta_{k_0}}{\|T^{k_0} y\|} \geq 1-\delta$$

and put $x = \|T^{k_0} y\|^{-1}\, T^{k_0} y$. Then for all n we have

$$\|T^n x\| = \frac{\|T^{k_0+n} y\|}{\|T^{k_0} y\|} \geq \frac{\gamma\beta_{k_0+n}}{\|T^{k_0} y\|} \geq (1-\delta)\frac{\beta_n}{1+\delta} \geq (1-\epsilon)|\alpha_n|.$$

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2. The weak case

In this section, we will give some partial answers as to whether every operator T with $r(T) \geq 1$ has weak orbits that converge to zero arbitrarily slowly.

Lemma 2.1. [N, Cor. 2.5] *Let X be a real or complex Banach space. Let $\beta_n \geq 0$, $n \in \mathbb{N}$, and assume that $\sum_{n=0}^{\infty} \beta_n = \infty$. If $1 \leq p < \infty$ and T is a bounded operator such that*

$$\sum_{n=0}^{\infty} \beta_n |\langle x^*, T^n x\rangle|^p < \infty, \quad \forall x \in X < x^* \in X^*,$$

then $r(T) < 1$.

Theorem 2.2. *Let T be a bounded operator on a real or compex Banach space X with $r(T) = 1$. Let $\alpha \in c_0$ be of norm one. Then each sequence (n_k) has a subsequence (n_{k_j}) with the property that there exist norm one vectors $x \in X$, $x^* \in X^*$ such that*

$$|\langle x^*, T^{n_{k_j}} x\rangle| \geq |\alpha_{k_j}|, \quad j = 0, 1, ...$$

Proof: By replacing α_n by $\sup_{k \geq n} |\alpha_k|$, we may assume that $\alpha_0 = 1$ and $\alpha_n \downarrow 0$. Put $N_0 := -1$ and for $k = 1, 2, ...$ put

$$N_k := \max\{n \in \mathbb{N} : \alpha_n \geq k^{-1}\}.$$

Then for $0 \leq n \leq N_1$ we have $\alpha_n = 1$ and for $k \geq 1$ and $N_k + 1 \leq n \leq N_{k+1}$ we have $(k+1)^{-1} \leq \alpha_n < k^{-1}$. Define the sequence (β_n) by $\beta_n = 1$, $n = 0, ..., N_1$, and

$$\beta_n := k^{-1}(N_{k+1} - N_k)^{-1}, \quad n = N_k + 1, ..., N_{k+1}; \; k = 1, 2, ...$$

Then $\sum_{n=0}^{\infty} \beta_n = \infty$, and

$$\sum_{n=0}^{\infty} \alpha_n \beta_n \leq N_1 + 1 + \sum_{k=1}^{\infty} (N_{k+1} - N_k) \cdot k^{-1} \cdot k^{-1} (N_{k+1} - N_k)^{-1} < \infty.$$

Let (n_k) be any given sequence, and define $(\tilde{\beta}_n)$ by

$$\tilde{\beta}_j := \begin{cases} \beta_k, & \text{if } j = n_k \text{ for some } k; \\ 0, & \text{else.} \end{cases}$$

Then $\sum_{j=0}^{\infty} \tilde{\beta}_j = \sum_{n=0}^{\infty} \beta_n = \infty$. By Lemma 2.1, there exist $x \in X$ and $x^* \in X^*$ such that

$$\sum_{j=0}^{\infty} \tilde{\beta}_j |\langle x^*, T^j x\rangle| = \sum_{k=0}^{\infty} \beta_k |\langle x^*, T^{n_k} x\rangle| = \infty.$$

Since $\sum_{n=0}^{\infty} \alpha_n \beta_n < \infty$, there have to be infinitely many indices n_k for which

$$|\langle x^*, T^{n_k} x\rangle| \geq \alpha_k.$$

This proves the theorem. ////

In the case of a positive operator on a Banach lattice the full weak analogue of the Theorem holds. This is the content of our next result.

Theorem 2.3. *Let T be a positive operator on a real or complex Banach lattice with $r(T) = 1$. Then for each $\epsilon > 0$ and $\alpha \in c_0$ of norm one, there exist norm one vectors $0 \le x \in X$ and $0 \le x^* \in X^*$ such that*

$$\langle x^*, T^n x\rangle \ge (1-\epsilon)|\alpha_n|, \quad n = 0, 1, 2, ...$$

Proof: We may assume that $\alpha_n \downarrow 0$. Also, we may assume that X is complex. Indeed, if X is real we consider the complexification $T_{\mathbb{C}}$ on $X_{\mathbb{C}}$, and observe that positive vectors in $X_{\mathbb{C}}$ in fact belong to the real part X.

Choose $\delta > 0$ such that $(1+\delta)^{-2}(1-\delta) \ge 1-\epsilon$. By considering approximate eigenvectors, it is easy to see (cf. [N, Lemma 2.1]) that for each $N \in \mathbb{N}$, there exist norm one vectors $0 \le x_N \in X$ and $0 \le x_N^* \in X^*$ such that

$$\langle x_N^*, T^n x_N\rangle \ge 1-\delta, \quad n = 0, 1, ..., N.$$

The proof can now be given along the lines of Lemma 1.1; the positivity simplifies the argument.

Choose m such that $\sum_{n=0}^{\infty} 2^{-mn} \le 1+\delta$. For each $k = 0, 1, ...$, let

$$N_k = \max\{n \in \mathbb{N} : \alpha_n \ge 2^{-2mk}\},$$

and choose norm one vectors $0 \le x_k \in X$ and $0 \le x_k^* \in X^*$ such that

$$\langle x_k^*, T^n x_k\rangle \ge 1-\delta, \quad n = 0, 1, ..., N_{k+1}.$$

Set $x = (1+\delta)^{-1}\sum_{k=0}^{\infty} 2^{-mk}x_k$ and $x^* = (1+\delta)^{-1}\sum_{k=0}^{\infty} 2^{-mk}x_k^*$. Then both x and x^* are positive vectors of norm ≤ 1. Fix $n \in \mathbb{N}$. If $0 \le n \le N_0$, then

$$\langle x^*, T^n x\rangle \ge (1+\delta)^{-2}\langle x_0^*, T^n x_0\rangle \ge (1+\delta)^{-2}(1-\delta) \ge 1-\epsilon = (1-\epsilon)\alpha_n.$$

We used that $\alpha_n = 1$ for $n = 0, ..., N_0$. If $n \ge N_0 + 1$, say $N_j + 1 \le n \le N_{j+1}$ for some j, then $\alpha_n \le \alpha_{N_j+1} < 2^{-2mj}$ and consequently,

$$\langle x^*, T^n x\rangle \ge 2^{-2mj}(1+\delta)^{-2}\langle x_j^*, T^n x_j\rangle \ge 2^{-2mj}(1-\epsilon) \ge (1-\epsilon)\alpha_n.$$

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Theorem 2.3 fails for arbitrary operators, at least in the case of real scalars. Indeed, we have the following counterexample in $X = \mathbb{R}^2$.

Example 2.4. Let $\gamma \in [0, 2\pi)$ be a number such that $\gamma/(2\pi)$ is irrational. Let T_γ be rotation over γ in $X = \mathbb{R}^2$. Let $C > 0$ be an arbitrary real number. For $x, y \in \mathbb{R}^2$ on norm one, let $n(x, y)$ denote the first integer such that

$$|\langle T_\gamma^n x, y\rangle| < \frac{C}{2}.$$

Because the orbit $n \mapsto T_\gamma^n x$ is dense in the unit circle by the assumption on γ, the numbers $n(x,y)$ indeed exist. We claim that

$$N := \sup\{n(x,y) : \|x\| = \|y\| = 1\} < \infty.$$

Indeed, suppose not. Then for each $n \in \mathbb{N}$ there are x_n, y_n of norm one such that

$$|\langle T_\gamma^k x_n, y_n\rangle| \geq \frac{C}{2}, \quad 0 \leq k \leq n.$$

Choose a subsequence (n_j) such that $x_{n_j} \to x$ and $y_{n_j} \to y$, and fix k. Then for all j such that $n_j \geq k$ we have

$$\begin{aligned} |\langle T_\gamma^k x, y\rangle| \geq & |\langle T_\gamma^k x_{n_j}, y_{n_j}\rangle| - |\langle T_\gamma^k x_{n_j}, y_{n_j}\rangle| \\ & - |\langle T_\gamma^k (x - x_{n_j}), y\rangle| - |\langle T_\gamma^k x_{n_j}, y - y_{n_j}\rangle|. \end{aligned}$$

Letting $j \to \infty$ we obtain

$$|\langle T_\gamma^k x, y\rangle| \geq \frac{C}{2}, \quad \forall k \in \mathbb{N}.$$

This contradicts the finiteness of $n(x,y)$. Now let $\alpha \in c_0$ be the vector

$$\alpha = (1, 1, ..., 1, 0, 0, ...),$$

where $\alpha_n = 1$ for $0 \leq n \leq N$ and $\alpha_n = 0$ for $n > N$. Then for *all* norm one vectors $x, y \in \mathbb{R}^2$ there is a $k = k(x,y) \in 0, ..., N$ such that

$$|\langle T_\gamma^k x, y\rangle| < C|\alpha_k|.$$

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As it turns out, this example works because T_γ is unitary. To see why, we need some terminology Let H be a real or complex Hilbert space. An operator T on H is called an *isometry* if $\|Tx\| = \|x\|$ for all $x \in H$ or equivalently, if $T^*T = I$. The operator T is called an *unilateral shift* if there is an orthogonal decomposition $H = \oplus_{n \in \mathbb{N}} H_n$ such that $TH_n \subset H_{n+1}$ and the map $T : H_n \to H_{n+1}$ is an isometry for all $n \in \mathbb{N}$. In the following lemma we use the so-called *Wold decomposition*: If T is an isometry on a Hilbert space H, then there is an orthogonal decomposition $H = H_0 \oplus H_1$ with $TH_i \subset H_i$, $i = 0, 1$, such that $T_0 := T|_{H_0}$ is unitary and $T_1 := T|_{H_1}$ is an unilateral shift. For a proof we refer to [SF], Theorem 1.1.

Now we have the following result: *Let T be a non-unitary isometry on a real or complex Hilbert space H. Then for all $\epsilon > 0$ and $\alpha \in c_0$ of norm one, there exist norm one vectors $x \in H$, $y \in H$, such that*

$$|\langle T^n x, y\rangle| \geq (1 - \epsilon)|\alpha_n|, \quad \forall n \in \mathbb{N}. \qquad (*)$$

Indeed, let $H = H_0 \oplus H_1$ be the Wold decomposition. Since T is not unitary, H_1 is non-empty. By considering the restriction of T to H_1, we therefore may assume that T is an unilateral shift on H.

Let $H = \oplus_{n\in\mathbb{N}} H_n$ be an orthogonal decomposition of H such that $T : H_n \to H_{n+1}$ is an isometry. Fix an arbitrary norm one vector $x_0 \in H_0$ and put $x_n := T^n x_0$. The closed linear span of $\{x_n : n \in \mathbb{N}\}$ is isometric to l^2 and the restriction of T to this span acts as the shift on l^2. Therefore, we can apply Theorem 2.3.

In fact, inspecting the proof of Theorem 2.3 for the shift operator on l^2, it is not hard to see that in fact we can find an $0 \le x \in l^2$ of norm one such that $\langle T^n x, x\rangle \ge (1-\epsilon)|\alpha_n|$ for all n. This implies that one can even achieve $x = y$ in $(*)$.

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