MEAN SQUARE CONTINUITY OF ORNSTEIN-UHLENBECK PROCESSES IN BANACH SPACES

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ABSTRACT. Let $X(t, x_0)$ denote the weak solution of the stochastic abstract Cauchy problem

$$\begin{aligned} dX(t) &= AX(t)\,dt + B\,dW_H(t), \quad t \geq 0, \\ X(0) &= x_0. \end{aligned}$$

Here A generates a C_0-semigroup on a separable real Banach space E, $\{W_H(t)\}_{t\geq 0}$ is a cylindrical Wiener process with Cameron-Martin space H, $B \in \mathcal{L}(H, E)$ is a bounded linear operator, and $x_0 \in E$ is a given intitial value. We prove that for all $p \in [1, \infty)$ and $t \geq 0$,

$$\lim_{s\to t} \mathbb{E}\left(\|X(t, x_0) - X(s, x_0)\|^p\right) = 0.$$

We consider the following stochastic abstract Cauchy problem:

$$\begin{aligned} dX(t) &= AX(t)\,dt + B\,dW_H(t), \quad t \geq 0, \\ X(0) &= x_0, \end{aligned} \tag{1.1}$$

where A is the generator of a C_0-semigroup $\mathbf{S} = \{S(t)\}_{t\geq 0}$ on a separable real Banach space E, B is a bounded linear operator from a separable real Hilbert space H into E, and $\{W_H(t)\}_{t\geq 0}$ is a cylindrical Wiener process with Cameron-Martin space H. For the precise definition of this concept we refer to [3].

It has been shown in [3] that the problem (1.1) admits a weak solution $\{X(t, x_0)\}_{t\geq 0}$ if and only if for each $t > 0$ the operator $Q_t \in \mathcal{L}(E^*, E)$ defined by

$$Q_t x^* := \int_0^t S(s)BB^*S^*(s)x^*\,ds, \qquad x^* \in E^*,$$

is the covariance of a centred Gaussian measure μ_t on E. In this case, μ_t is the distribution of the random variable $X(t, 0)$, and the solution can be represented as a stochastic convolution as follows:

$$\langle X(t, x_0), x^*\rangle = \langle S(t)x_0, x^*\rangle + \int_0^t \langle S(t-s)B\,dW_H(s), x^*\rangle, \qquad x^* \in E^*.$$

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We will prove that the process $\{X(t,x_0)\}_{t\ge 0}$ is mean continuous in all moments. In particular this solves the problem, left open in [3], whether $\{X(t,x_0)\}_{t\ge 0}$ is mean square continuous.

Let $C_b(E)$ denote the space of all bounded continuous real functions on E.

Lemma 1. *Let (t_n) be a sequence of nonnegative real numbers in the interval $[0,T]$ with $\lim_{n\to\infty} t_n = t$. Let $g:[0,\infty)\to\mathbb{R}$ be nondecreasing and convex with $g(\|\cdot\|)\in L^1(E,\mu_T)$. Then for all $f\in C_b(E)$ we have*

$$\lim_{n\to\infty}\int_E f(x)\,g(\|x\|)\,d\mu_{t_n}(x) = \int_E f(x)\,g(\|x\|)\,d\mu_t(x).$$

Proof. For $r>0$ let

$$B_r = \{x\in E:\ g(\|x\|)\le r\}.$$

This set is symmetric and convex. Symmetry is obvious, and convexity follows from

$$g(\|\alpha x+(1-\alpha)y\|)\le g(\alpha\|x\|+(1-\alpha)\|y\|)\le \alpha g(\|x\|)+(1-\alpha)g(\|y\|),$$

where $\alpha\in[0,1]$. In view of

$$\langle Q_{t_n}x^*,x^*\rangle \le \langle Q_T x^*,x^*\rangle, \qquad x^*\in E^*,$$

we may apply Anderson's inequality [2, Theorem 3.3.6] to obtain

$$\mu_{t_n}(B_r)\ge \mu_T(B_r).$$

In combination with the identity

$$\int_E |h(x)|\,d\nu(x) = \int_0^\infty \nu\{x\in E:\ |h(x)|>s\}\,ds,$$

we find, with $M=\sup_{x\in E}|f(x)|$,

(1.2)
$$\begin{aligned}\int_{E\setminus B_r}|f(x)|\,g(\|x\|)\,d\mu_{t_n}(x) &\le M\int_{E\setminus B_r} g(\|x\|)\,d\mu_{t_n}(x)\\ &= M\int_{g(r)}^\infty \mu_{t_n}\{x\in E:\ g(\|x\|)>s\}\,ds\\ &\le M\int_{g(r)}^\infty \mu_T\{x\in E:\ g(\|x\|)>s\}\,ds\\ &= M\int_{E\setminus B_r} g(\|x\|)\,d\mu_T(x).\end{aligned}$$

The same argument gives

$$\int_{E\setminus B_r} |f(x)|\, g(\|x\|)\, d\mu_t(x) \le M \int_{E\setminus B_r} g(\|x\|)\, d\mu_T(x). \tag{1.3}$$

It now follows easily that the family $f(X(t_n,0))g(\|X(t_n,0)\|)$ is uniformly integrable. Since $\mu_{t_n} \to \mu_t$ weakly [6], the lemma follows from [1, Theorem 5.4]. Alternatively, the weak convergence $\mu_{t_n} \to \mu_t$ implies

$$\lim_{n\to\infty} \int_E f(x)\,(g(\|x\|) \wedge g(r))\, d\mu_{t_n}(x) = \int_E f(x)\,(g(\|x\|) \wedge g(r))\, d\mu_t(x)$$

for all $r > 0$. Choosing r so large that $\int_{E\setminus B_r} g(\|x\|)\, d\mu_T(x) < \varepsilon/M$, by (1.2) and (1.3) both truncation errors are at most ε, and again the lemma follows. ∎

If $f : E \to \mathbb{R}$ is a bounded Borel function, then for all $t \ge 0$ we have

$$\mathbb{E}\big(f(X(t,0))\big) = \int_E f(y)\, d\mu_t(y).$$

By an easy approximation argument, for $t \ge 0$ fixed this identity extends to all functions $f \in L^1(E,\mu_t)$.

Theorem 2. *Let $g : [0,\infty) \to \mathbb{R}$ be a nondecreasing convex function with $g(0) = 0$ such that*

$$g(c\|\cdot\|) \in L^1(E,\mu_T)$$

for some $c > M+2$, where $M = \limsup_{u\downarrow 0} \|S(u)\|$. Then for all $x \in E$ and $t \in [0,T]$ we have

$$\lim_{s\to t} \mathbb{E}\, g(\|X(t,x) - X(s,x)\|) = 0.$$

Proof. The assumption $c > M+2$ enables us to choose $\alpha,\beta,\gamma \in (0,1)$ with $\alpha+\beta+\gamma = 1$ subject to the following two conditions:

- $\beta c > M+1$;
- $\gamma c \ge 1$.

Step 1 - First we note that for all $\tau \in [0,T]$ and $0 \le c' \le c$,

$$\begin{aligned}
\int_E g(c'\|y\|)\, d\mu_\tau(y) &= \int_0^\infty \mu_\tau\{x \in E:\ g(c'\|x\|) > s\}\, ds \\
&\le \int_0^\infty \mu_T\{x \in E:\ g(c'\|x\|) > s\}\, ds \\
&= \int_E g(c'\|y\|)\, d\mu_T(y) \\
&\le \int_E g(c\|y\|)\, d\mu_T(y) < \infty.
\end{aligned}$$

Hence by the condition $1/\gamma \le c$ and the remark preceding the theorem,

$$\mathbb{E}\ g\big(\tfrac{1}{\gamma}\|X(\tau,0)\|\big) = \int_E g(\tfrac{1}{\gamma}\|y\|)\,d\mu_\tau(y),$$

and therefore by Lemma 1,

$$\lim_{\tau\downarrow 0}\mathbb{E}\ g\big(\tfrac{1}{\gamma}\|X(\tau,0)\|\big) = \lim_{\tau\downarrow 0}\int_E g(\tfrac{1}{\gamma}\|y\|)\,d\mu_\tau(y) = 0.$$

Step 2 - Right continuity. Fix $t\in[0,T]$. We have, for $t\le s\le T$,

$$\begin{aligned}
\langle X(s,x),&x^*\rangle - \langle X(t,x),x^*\rangle\\
&= \langle S(s)x,x^*\rangle + \int_0^s \langle S(s-u)B\,dW_H(u),x^*\rangle\\
&\qquad - \langle S(t)x,x^*\rangle - \int_0^t \langle S(t-u)B\,dW_H(u),x^*\rangle\\
&= \langle S(s)x - S(t)x,x^*\rangle + \langle S(s-t)X(t,0) - X(t,0),x^*\rangle\\
&\qquad + \langle Y_{s,t},x^*\rangle,
\end{aligned}$$

where

$$Y_{s,t} = \int_t^s S(s-u)B\,dW_H(u).$$

Hence,

$$X(s,x)-X(t,x) = S(s)x-S(t)x+S(s-t)X(t,0)-X(t,0)+Y_{s,t}. \tag{1.4}$$

The convexity of g implies

$$\begin{aligned}
g\big(&\|X(s,x)-X(t,x)\|\big)\\
&\le \alpha g\big(\tfrac{1}{\alpha}\|S(s)x-S(t)x\|\big) + \beta g\big(\tfrac{1}{\beta}\|S(s-t)X(t,0)-X(t,0)\|\big)\\
&\qquad + \gamma g\big(\tfrac{1}{\gamma}\|Y_{s,t}\|\big).
\end{aligned}$$

Noting that g is continuous with $g(0)=0$, it follows that

$$\lim_{s\downarrow t}\mathbb{E}\ g\big(\tfrac{1}{\alpha}\|S(s)x-S(t)x\|\big) = \lim_{s\downarrow t} g\big(\tfrac{1}{\alpha}\|S(s)x-S(t)x\|\big) = 0.$$

Arguing as in Step 1 and using the condition $(M+1)/\beta < c$ we see that for $s-t$ sufficiently small,

$$\mathbb{E}\ g\big(\tfrac{1}{\beta}\|S(s-t)X(t,0)-X(t,0)\|\big) = \int_E g\big(\tfrac{1}{\beta}\|S(s-t)y-y\|\big)\,d\mu_t(y).$$

Hence by dominated convergence,

$$\begin{aligned}
&\lim_{s\downarrow t}\mathbb{E}\ g(\tfrac{1}{\beta}\|S(s-t)X(t,0)-X(t,0)\|)\\
&\qquad = \lim_{s\downarrow t}\int_E g(\tfrac{1}{\beta}\|S(s-t)y-y\|)\,d\mu_t(y) = 0.
\end{aligned}$$

Finally, noting that $Y_{s,t}$ and $X(s-t,0)$ have the same distribution, by Step 1 we have

$$\lim_{s\downarrow t}\mathbb{E}\; g(\tfrac{1}{\gamma}\|Y_{s,t}\|) = \lim_{s\downarrow t}\mathbb{E}\; g(\tfrac{1}{\gamma}\|X(s-t,0)\|) = 0.$$

Step 3 - Left continuity. Fix $t \in [0,T]$. For $0 \le s \le t$ we have, using (1.4) with the rôles of s and t reversed,

$$\begin{aligned} &g(\|X(t,x) - X(s,x)\|) \\ &\quad\le \alpha g\big(\tfrac{1}{\alpha}\|S(t)x - S(s)x\|\big) + \beta g\big(\tfrac{1}{\beta}\|S(t-s)X(s,0) - X(s,0)\|\big) \\ &\qquad + \gamma g\big(\tfrac{1}{\gamma}\|Y_{t,s}\|\big). \end{aligned}$$

As in Step 2, the expectation of the first term on the right hand side tends to 0 as $s \uparrow t$ by continuity, and the expectation of the third term tends to 0 by Step 1. It remains to prove that

$$\begin{aligned} \lim_{s\uparrow t}\;& \mathbb{E}\; g\big(\tfrac{1}{\beta}\|S(t-s)X(s,0) - X(s,0)\|\big) \\ &= \lim_{s\uparrow t}\int_E g(\tfrac{1}{\beta}\|S(t-s)y - y\|)\,d\mu_s(y) = 0. \end{aligned}$$

By Lemma 1, for all $s \in [0,T]$ the measure $g(c\|x\|)\,d\mu_s(x)$ is a finite Radon measure and the family

$$\big\{g(c\|x\|)\,d\mu_s(x) :\; s \in [\tfrac12 t, t]\big\}$$

is tight. Fix $\varepsilon > 0$ arbitrary and use Prokhorov's theorem to choose a compact set K such that

$$\int_{E\setminus K} g(c\|x\|)\,d\mu_s(x) < \varepsilon, \qquad s \in [\tfrac12 t, t].$$

Choose $0 < \tau \le \frac12 t$ so small that

$$\tfrac{1}{\beta}(\|S(u)\| + 1) \le c \quad\text{and}\quad \tfrac{1}{\beta}\|S(u)y - y\| < \varepsilon, \qquad u \in [0,\tau],\; y \in K.$$

It follows that for $s \in [t-\tau, t]$,

$$\begin{aligned} \int_E & g(\tfrac{1}{\beta}\|S(t-s)y - y\|)\,d\mu_s(y) \\ &\le \int_K g\big(\tfrac{1}{\beta}\|S(t-s)y - y\|\big)\,d\mu_s(y) \\ &\qquad + \int_{E\setminus K} g\big(\tfrac{1}{\beta}\|S(t-s)y - y\|\big)\,d\mu_s(y). \\ &\le g(\varepsilon) + \varepsilon. \end{aligned}$$

Since $\lim_{\varepsilon\downarrow 0} g(\varepsilon) = 0$, this completes the proof. ∎

Under a slightly stronger assumption on g, we can rephrase this result in terms of Orlicz norms.

If $g : [0,\infty) \to \mathbb{R}$ is a nondecreasing convex function with $g(0) = 0$, then for a strongly measurable function $\xi : (\Omega, \mathbb{P}) \to E$ we define

$$\|\xi\|_{L_g(E)} := \inf \left\{ c > 0 : \ \mathbb{E}\, g\left(\frac{\|\xi\|}{c}\right) \le 1 \right\}.$$

The set $L_g(E)$ of all ξ for which $\|\xi\|_{L_g(E)}$ is finite is a Banach space; cf. [5].

Corollary 3. *Let* $g : [0,\infty) \to \mathbb{R}$ *be a nondecreasing convex function with* $g(0) = 0$ *such that* $g(c\|\cdot\|) \in L^1(E, \mu_T)$ *for all* $c > 0$*. Then for all* $x \in E$ *and* $t \in [0,T]$ *we have*

$$\lim_{s\to t} X(s,x) = X(t,x) \quad \textit{in} \quad L_g(E).$$

Proof. Let $\varepsilon > 0$ be fixed and define $g_\varepsilon(\tau) := g(\varepsilon^{-1}\tau)$. According to Theorem 2, for $|t-s|$ sufficiently small we have

$$\mathbb{E}\, g_\varepsilon\big(\|X(t,x) - X(s,x)\|\big) \le 1.$$

Hence,

$$\mathbb{E}\, g\left(\frac{\|X(t,x) - X(s,x)\|}{\varepsilon}\right) = \mathbb{E}\, g_\varepsilon\big(\|X(t,x) - X(s,x)\|\big) \le 1,$$

which means that $\|X(t,x) - X(s,x)\|_{L_g(E)} \le \varepsilon$. ∎

By Fernique's theorem, this result applies, e.g., to the functions

$$g(\tau) = \exp(\tau^p) - 1, \qquad 1 \le p < 2,$$

and $g(\tau) = \tau^p$, $1 \le p < \infty$. In the latter case we can apply Theorem 2 directly and obtain:

Corollary 4. *For all* $x \in E$ *and* $t \ge 0$ *we have*

$$\lim_{s\to t} \mathbb{E}\left(\|X(t,x) - X(s,x)\|^p\right) = 0, \qquad p \in [1,\infty).$$

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