# SECOND QUANTIZATION AND THE $L^p$-SPECTRUM OF NONSYMMETRIC ORNSTEIN-UHLENBECK OPERATORS

J.M.A.M. VAN NEERVEN

Abstract. The spectra of the second quantization and the symmetric second quantization of a strict Hilbert space contraction are computed explicitly and shown to coincide. As an application, we compute the spectrum of the nonsymmetric Ornstein-Uhlenbeck operator $L$ associated with the infinite-dimensional Langevin equation

$$dU(t) = AU(t)\,dt + dW(t)$$

where $A$ is the generator of a strongly continuous semigroup on a Banach space $E$ and $W$ is a cylindrical Wiener process in $E$. Assuming the existence of an invariant measure $\mu$ for $L$, under suitable assumptions on $A$ we show that the spectrum of $L$ in the space $L^p(E,\mu)$ $(1 < p < \infty)$ is given by

$$\sigma(L) = \overline{\Big\{\sum_{j=1}^{n} k_j z_j : \ k_j \in \mathbb{N}, \ z_j \in \sigma(A_\mu); \ j = 1, \dots, n; \ n \geqslant 1\Big\}},$$

where $A_\mu$ is the generator of a Hilbert space contraction semigroup canonically associated with $A$ and $\mu$. We prove that the assumptions on $A$ are always satisfied in the strong Feller case and in the finite-dimensional case. In the latter case we recover the recent Metafune-Pallara-Priola formula for $\sigma(L)$.

## 1. Introduction

There has been a considerable recent interest in the spectral theory of second order elliptic operators $L$ of the form

$$L\phi(x) = \tfrac{1}{2}\mathrm{Tr}\,(Q(x)D^2\phi(x)) + \langle A(x), D\phi(x)\rangle \qquad (x \in E) \tag{1.1}$$

with unbounded drift term $A$ on a finite or infinite dimensional space $E$; see for example [14, 22, 23, 24, 25]. In many situations $L$ admits a unique invariant measure $\mu$, in which case it is natural to consider the realization of $L$ in the space $L^p(E,\mu)$; see [1, 4, 8, 11] and the references cited there. Even in space dimension one this class of operators is not completely understood at present.

In this paper we consider the case of Ornstein-Uhlenbeck operators, i.e., the special case of (1.1) where $Q(x) = Q$ (the 'diffusion') is a fixed positive symmetric

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operator from $E^*$ into $E$ and $A(x) = Ax$ with $A$ (the 'drift') an infinitesimal generator of a strongly continuous semigroup of operators on $E$. The state space $E$ is allowed to be an arbitrary real Banach space, the operator $Q$ is not assumed to have finite trace and the operator $A$ may be unbounded. We do not assume that $L$ is symmetric. Nonsymmetric Ornstein-Uhlenbeck operators arise naturally as the infinitesimal generators of transition semigroups associated with stochastic partial differential equations and have been applied for example to optimal control problems and to interest rate models; see [10, 11, 12, 16] and the references cited therein. In finite dimensions, nonsymmetric Ornstein-Uhlenbeck operators have been recently applied in the area of nonequilibrium statistical physics [5].

Assuming the existence of an invariant measure $\mu$ for $L$, our aim is to determine the spectrum of $L$ in $L^p(E,\mu)$ for $p \in (1,\infty)$. Let us recall that in $E = \mathbb{R}^d$, the 'classical' Ornstein-Uhlenbeck operator with $Q = I$ and $A = -I$,

$$L\phi(x) = \tfrac{1}{2}\Delta\phi(x) - \langle x, \nabla\phi(x)\rangle \qquad (x \in \mathbb{R}^d)$$

which arises in quantum field theory as the boson number operator, has a unique Gaussian invariant measure $\mu$ and the spectrum of $L$ in $L^p(\mathbb{R}^d,\mu)$ is given by

$$\sigma(L) = \{-n : \ n \in \mathbb{N}\}$$

where $\mathbb{N} = \{0, 1, 2, \dots\}$. This formula is an easy consequence of the description of $L$ as a second quantized operator [21, 27, 30] and can be extended without much difficulty to the case where $E$ is a Hilbert space and $A$ is selfadjoint with compact resolvent. The nonsymmetric case is considerably more difficult, however, even for $E = \mathbb{R}^d$. Under suitable nondegeneracy assumptions on $A$ and $Q$ it was shown by Metafune, Pallara and Priola [24] that the spectrum of $L$ in $L^p(\mathbb{R}^d,\mu)$ is given by

$$\sigma(L) = \Big\{\sum_{i=1}^{n} k_j z_j : \ k_j \in \mathbb{N}, \ z_j \in \sigma(A); \ j = 1, \dots, n; \ n \geqslant 1\Big\}. \tag{1.2}$$

In particular the spectrum is independent of $p \in (1,\infty)$. On the other hand it was shown by Metafune [22] that the spectrum of $L$ in $L^p(\mathbb{R}^d)$ is $p$-dependent. This contrasts well-known results on spectral $p$-independence in $L^p(\mathbb{R}^d)$ of second-order elliptic operators under various different assumptions; see for example [13, 19] and the references cited therein.

The proof of (1.2) in [24] depends on a careful analysis of the smoothing effects of the transition semigroup $P = \{P(t)\}_{t\geqslant 0}$ generated by $L$. In this paper we will give a completely different proof of an infinite dimensional version of (1.2) which instead exploits the fact that the transition semigroup can be represented as the symmetric second quantization of the adjoint of an appropriate nonsymmetric contraction semigroup $S_\mu = \{S_\mu(t)\}_{t\geqslant 0}$ acting on the reproducing kernel Hilbert space of the invariant measure $\mu$. The crucial step in this approach is to obtain a formula for the spectrum of the symmetric second quantization of Hilbert space contractions $T$. For strict contractions $T$ this problem is solved completely. The main difficulty consists of showing that the spectra of the $n$-fold tensor product and the symmetric $n$-fold tensor product of $T$ coincide and are given by

$$\sigma(T^{\circledS n}) = \sigma(T^{\otimes n}) = \Big\{\prod_{j=1}^n z_j :\ z_j \in \sigma(T);\ j = 1, \dots, n\Big\},$$

the second equality being a classical result due to Brown and Pearcy [6]. This easily implies equality of the spectra of the second quantization and the symmetric second quantization of $T$:

$$\sigma(\Gamma^{\circledS}(T)) = \sigma(\Gamma(T)) = \{1\} \cup \overline{\bigcup_{n\geqslant 1}\Big\{\prod_{j=1}^n z_j :\ z_j \in \sigma(T);\ j = 1, \dots, n\Big\}}.$$

As a result we are able to compute the spectra of the operators $P(t)$ in $L^2(E, \mu)$ under the assumption that $S_\mu$ is a semigroup of strict contractions. By combining these arguments with standard hypercontractivity results we obtain the spectra of $P(t)$ in $L^p(E, \mu)$ for all $p \in (1, \infty)$. The spectrum of $L$ in $L^p(E, \mu)$ is then obtained via spectral mapping techniques. For this step we require that, in addition to being strictly contractive, $S_\mu$ is also eventually norm continuous. Our main result asserts that under these assumptions (which are shown to be automatically satisfied in two important cases: the strong Feller case and the finite dimensional case), we have

$$\sigma(L) = \overline{\Big\{\sum_{i=1}^n k_j z_j :\ k_j \in \mathbb{N},\ z_j \in \sigma(A_\mu);\ j = 1, \dots, n;\ n \geqslant 1\Big\}}.$$

Here, $A_\mu$ denotes the generator of the semigroup $S_\mu$. If $S_\mu$ is compact (which is the case in the strong Feller case and in finite dimensions), no closure needs to be taken and we obtain

$$\sigma(L) = \Big\{\sum_{i=1}^n k_j z_j :\ k_j \in \mathbb{N},\ z_j \in \sigma(A_\mu);\ j = 1, \dots, n;\ n \geqslant 1\Big\}. \tag{1.3}$$

In finite dimensions, under a nondegeneracy assumption we have $\sigma(A_\mu) = \sigma(A)$ and (1.3) reduces to the Metafune-Pallara-Priola formula (1.2).

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## 2. Preliminaries

In this section we collect some well known results on spectral theory and reproducing kernel Hilbert spaces. For more detailed information we refer to [2, 3].

2.1. **Spectral theory.** Let $X$ be a real or complex Banach space. The spectrum of a bounded or unbounded linear operator $T$ on $X$ will be denoted by $\sigma(T)$. When $X$ is a real Banach space, the spectrum of $T$ is defined as the spectrum of its complexification. The spectral radius of a bounded operator $T$ is denoted by $r(T)$. The point spectrum, approximate point spectrum, and residual point spectrum of $T$ will be denoted by $\sigma_{\mathrm{p}}(T)$, $\sigma_{\mathrm{a}}(T)$, and $\sigma_{\mathrm{r}}(T)$ respectively; the latter is defined as the set of all $z \in \sigma(T)$ for which the range of $z - T$ is a proper, closed subspace of $X$. Recall that $\partial\sigma(T) \subseteq \sigma_{\mathrm{a}}(T)$, where $\partial\sigma(T)$ denotes the topological boundary of $\sigma(T)$, and that $\sigma_{\mathrm{r}}(T) \subseteq \sigma_{\mathrm{p}}(T^*)$. Also note that $\sigma(T) = \sigma_{\mathrm{a}}(T) \cup \sigma_{\mathrm{r}}(T)$ and that the union is disjoint.

If $S$ and $T$ are bounded operators on $X$ satisfying $ST = TS$, then

$$\delta(\sigma(S), \sigma(T)) \leqslant r(S - T), \tag{2.1}$$

where $\delta(K, L)$ denotes the Hausdorff distance between the compact sets $K$ and $L$ [2, Theorem 3.4.1]. We will apply this result in the following situation. Let $(X_n)_{n\geqslant 0}$ be a sequence of nonzero complemented subspaces of $X$ such that $X_n \cap X_m = \{0\}$ for all $n, m \geqslant 0$ with $n \neq m$. Let $(\pi_n)_{n\geqslant 0}$ be a corresponding sequence of projections. For each $n \geqslant 0$ let $P_n := \bigoplus_{j=0}^n \pi_j$. Let $T$ be a bounded operator on $X$ which commutes with each $P_n$. We define operators $T_n$ on $X$ and $S_n$ on $X_n$ by $T_n := T \circ P_n$ and $S_n := T|_{X_n}$.

**Proposition 2.1.** *Under the above assumptions, if*

$$\lim_{n\to\infty} \|T - T_n\| = 0, \tag{2.2}$$

*then*

$$\sigma(T) = \overline{\bigcup_{n=0}^{\infty} \sigma(T_n)} = \overline{\bigcup_{n=0}^{\infty} \sigma(S_n)}. \tag{2.3}$$

*Proof.* Let $Y_n := \ker P_n = (I - P_n)X$. Then we have a direct sum decomposition $X = X_0 \oplus \cdots \oplus X_n \oplus Y_n$, relative to which we have $T_n = S_0 \oplus \cdots \oplus S_n \oplus 0$. From this we easily infer that $\sigma(T_n) = \{0\} \cup \bigcup_{j=0}^n \sigma(S_j)$. Moreover, from $\lim_{n\to\infty} \|S_n\| = \lim_{n\to\infty} \|T_n - T_{n-1}\| = 0$ we obtain that $0 \in \overline{\bigcup_{n\geqslant 0} \sigma(S_n)}$. The second identity in (2.3) immediately follows. The inclusion '$\subseteq$' in the first identity follows from (2.2) and (2.1), while the inclusion '$\supseteq$' follows from the obvious inclusions $\sigma(T_n) \subseteq \sigma(T)$. $\square$

Let $X$ and $Y$ be Banach spaces and let $X \widehat{\otimes} Y$ denote the completion of $X \otimes Y$ with respect to a uniform cross norm. If $S$ and $T$ are bounded operators on $X$ and $Y$ respectively, then the operator $S \otimes T$ defined on $X \otimes Y$

$$(S \otimes T)(x \otimes y) := (Sx \otimes Ty)$$

uniquely extends to a bounded operator $S \widehat{\otimes} T$ on $X \widehat{\otimes} Y$ of norm $\|S \widehat{\otimes} T\| \leqslant \|S\|\,\|T\|$. The spectrum of $S \widehat{\otimes} T$ is given by the following identity due to Schechter [29]:

$$\sigma(S \widehat{\otimes} T) = \{\eta \cdot \zeta : \ \eta \in \sigma(S),\ \zeta \in \sigma(T)\}. \tag{2.4}$$

We will only need the Hilbert space version, which was obtained earlier by Brown and Pearcy [6].

2.2. **Reproducing kernel Hilbert spaces.** The material of this subsection is needed in Section 5. Let $E$ be a *real* Banach space with dual $E^*$ and let $Q \in \mathscr{L}(E^*, E)$ be a positive and symmetric linear operator, i.e., we have $\langle Qx^*, x^*\rangle \geqslant 0$ for all $x^* \in E^*$ and $\langle Qx^*, y^*\rangle = \langle Qy^*, x^*\rangle$ for all $x^*, y^* \in E^*$. On the range of $Q$, the bilinear mapping $(Qx^*, Qy^*) \mapsto \langle Qx^*, y^*\rangle$ defines an inner product. The completion of the range of $Q$ with respect to this inner product is a real Hilbert space $H_Q$, the *reproducing kernel Hilbert space* associated with $Q$. The inclusion mapping from the range of $Q$ into $E$ extends to a continuous inclusion mapping $i_Q : H_Q \hookrightarrow E$. Upon identifying $H_Q$ and its dual in the canonical way we have the operator identity

$$Q = i_Q \circ i_Q^*. \tag{2.5}$$

If $Q_1, Q_2 \in \mathscr{L}(E^*, E)$ are positive and symmetric operators, then we have $H_{Q_1} \subseteq H_{Q_2}$ as subsets of $E$ if and only if there exists a constant $K \geqslant 0$ such that

$$\langle Q_1 x^*, x^*\rangle \leqslant K\langle Q_2 x^*, x^*\rangle \qquad \text{for all } x^* \in E^* \tag{2.6}$$

in which case the inclusion mapping $H_{Q_1} \hookrightarrow H_{Q_2}$ is continuous.

If $i : H \hookrightarrow E$ is a continuous embedding of a real Hilbert space $H$ into $E$, then $Q := i \circ i^*$ is positive and symmetric and its reproducing kernel space $H_Q$ equals $H$. More precisely, the mapping $i^*x^* \mapsto i_Q^* x^*$ defines an isometry from $H$ onto $H_Q$ and from $i \circ i^* = Q = i_Q \circ i_Q^*$ we have $H = H_Q$ as subsets of $E$.

Examples of positive symmetric operators arise naturally in the theory of Gaussian distributions. Recall that if $\gamma$ is a centered Gaussian Radon measure on $E$, then there exists a unique positive and symmetric operator $Q_\gamma \in \mathscr{L}(E^*, E)$, the *covariance operator* of $\gamma$, such that the Fourier transform of $\gamma$ is given by

$$\widehat{\gamma}(x^*) := \int_E \exp(-i\langle x, x^*\rangle)\, d\gamma(x) = \exp(-\tfrac{1}{2}\langle Q_\gamma x^*, x^*\rangle) \qquad \text{for all } x^* \in E^*.$$

In this situation the reproducing kernel Hilbert space $H_\gamma := H_{Q_\gamma}$ is separable, the embedding $i_\gamma : H_\gamma \hookrightarrow E$ is compact, and we have $\gamma(\overline{H_\gamma}) = 1$, the closure being taken with respect to the norm of $E$.

## 3. The spectrum of second quantized operators

Let $H$ be a nonzero complex Hilbert space. For $n \geqslant 0$ we let $H^{\otimes n} = H \widehat{\otimes} \cdots \widehat{\otimes} H$ be the $n$-fold Hilbert tensor product of $H$, with the understanding that $H^{\otimes 0} = \mathbb{C}$. The Hilbert space direct sum

$$\Gamma(H) := \bigoplus_{n \geqslant 0} H^{\otimes n}$$

is called the *Fock space* over $H$. The theory of Fock spaces is developed systematically in [27].

Given a bounded operator $T \in \mathscr{L}(H)$, we use the notation $T^{\otimes n} = T \widehat{\otimes} \cdots \widehat{\otimes} T \in \mathscr{L}(H^{\otimes n})$, with the understanding that $T^{\otimes 0} = I$. For later use we note that for all $S, T \in \mathscr{L}(H)$ and all $n \geqslant 1$ we have

$$\|T^{\otimes n}\|_{\mathscr{L}(H^{\otimes n})} = \|T\|^n \tag{3.1}$$

and, by a simple telescoping argument,

$$\|T^{\otimes n} - S^{\otimes n}\|_{\mathscr{L}(H^{\otimes n})} \leqslant \|T - S\| \cdot \sum_{j=0}^{n-1} \|S\|^j \|T\|^{n-1-j}. \tag{3.2}$$

If $T$ is a contraction, the direct sum operator

$$\Gamma(T) := \bigoplus\nolimits_{n \geqslant 0} T^{\otimes n}$$

is well defined and defines a contraction on $\Gamma(H)$. This operator is called the *second quantization* of $T$. We have the following algebraic relations:

$$\Gamma(I) = I, \qquad \Gamma(T_1 T_2) = \Gamma(T_1)\Gamma(T_2), \qquad \Gamma(T^*) = (\Gamma(T))^*. \tag{3.3}$$

**Proposition 3.1.** *If $\|T\| < 1$, then*

$$\sigma(\Gamma(T)) = \{1\} \cup \overline{\bigcup_{n \geqslant 1} \Big\{ \prod_{j=1}^n z_j : \ z_j \in \sigma(T);\ j = 1, \dots, n \Big\}}.$$

*Proof.* Clearly, $\sigma(T^{\otimes 0}) = \{1\}$, whereas for $n \geqslant 1$ by repeated application of (2.4) we have

$$\sigma(T^{\otimes n}) = \Big\{ \prod_{j=1}^n z_j : \ z_j \in \sigma(T);\ j = 1, \dots, n \Big\}. \tag{3.4}$$

The result now follows from a straightforward application of Proposition 2.1. □

For our applications in the next sections we will be interested in the *symmetric Fock space* over $H$. This is the Hilbert space direct sum

$$\Gamma^{ⓢ}(H) := \bigoplus\nolimits_{n \geqslant 0} H^{ⓢ n},$$

where $H^{ⓢ n}$ denotes the closed subspace of $H^{\otimes n}$ spanned by all symmetric $n$-tensors, again with the understanding that $H^{ⓢ 0} = \mathbb{C}$. If $T$ is a bounded operator on $H$, then $T^{\otimes n}$ maps $H^{ⓢ n}$ into itself. The restriction of $T^{\otimes n}$ to $H^{ⓢ n}$ will be denoted by $T^{ⓢ n}$. If $T$ is a contraction, we define the *symmetric second quantization* of $T$ by

$$\Gamma^{ⓢ}(T) := \bigoplus\nolimits_{n \geqslant 0} T^{ⓢ n}.$$

Of course, $\Gamma^{ⓢ}(T)$ is just the restriction of $\Gamma(T)$ to $\Gamma^{ⓢ}(H)$. The algebraic relations (3.3) carry over in the obvious way.

Let $S_n$ denote the permutation group on $n$ elements. Given an element $h \in H$, the *creation operators* $a_n^\dagger(h) : H^{ⓢ n} \to H^{ⓢ (n+1)}$ are defined by

$$\begin{aligned} &a_n^\dagger(h) \sum_{\sigma \in S_n} g_{\sigma(1)} \otimes \cdots \otimes g_{\sigma(n)} \\ &:= \frac{1}{\sqrt{n+1}} \sum_{\sigma \in S_n} \sum_{m=1}^{n+1} g_{\sigma(1)} \otimes \cdots \otimes g_{\sigma(m-1)} \otimes h \otimes g_{\sigma(m)} \otimes \cdots \otimes g_{\sigma(n)}, \end{aligned}$$

and the *annihilation operators* $a_{n+1}(h) : H^{ⓢ(n+1)} \to H^{ⓢn}$ by

$$a_{n+1}(h) \sum_{\sigma\in S_{n+1}} g_{\sigma(1)} \otimes \cdots \otimes g_{\sigma(n+1)}$$
$$:= \frac{1}{\sqrt{n+1}} \sum_{\sigma\in S_{n+1}} \sum_{m=1}^{n+1} [g_{\sigma(m)}, h]_H \; g_{\sigma(1)} \otimes \cdots \otimes g_{\sigma(m-1)} \otimes g_{\sigma(m+1)} \otimes \cdots \otimes g_{\sigma(n+1)}.$$

These operators are well defined and bounded, and their operator norms are bounded by

$$\|a_n^\dagger\|_{\mathscr{L}(H^{ⓢn}, H^{ⓢ(n+1)})} = \|a_{n+1}(h)\|_{\mathscr{L}(H^{ⓢ(n+1)}, H^{ⓢn})} \leqslant C_n\|h\| \tag{3.5}$$

with constants $C_n$ depending on $n$ only. The first equality follows from the duality relations

$$a_n^{\dagger *}(h) = a_{n+1}(h). \tag{3.6}$$

Furthermore, we have the commutation relations

$$a_{n+2}(h)a_{n+1}^\dagger(h) - a_n^\dagger(h)a_{n+1}(h) = \|h\|^2 I. \tag{3.7}$$

For the proofs we refer to [27]. An obvious consequence of (3.6) and (3.7) is the lower bound

$$\|a_n^\dagger(h)g\|^2_{H^{ⓢ(n+1)}} \geqslant \|g\|^2_{H^{ⓢn}}\|h\|^2. \tag{3.8}$$

We will need the following result from the theory of several complex variables, known as Hartog's theorem [18, page 106], cf. also [28, Lemma 2]:

**Lemma 3.2.** *Let $K$ be a bounded set in $\mathbb{C}^n$, $n \geqslant 2$, and suppose that $f$ is analytic in a neighbourhood of $K$. If $f(p) = z$ for some point $p \in K$, then there exists a point $p' \in \partial K$, the topological boundary of $K$, such that $f(p') = z$.*

We are now in a position to prove the following result.

**Theorem 3.3.** *For all $n \geqslant 0$ we have*

$$\sigma(T^{ⓢn}) = \sigma(T^{\otimes n}) = \Big\{\prod_{j=1}^n z_j : \; z_j \in \sigma(T); \; j = 1, \dots, n\Big\}.$$

In the proof below, and in the rest of the paper, we economize on brackets; for instance, $T^{ⓢn*}$ means $(T^{ⓢn})^*$ and $T^{*ⓢn}$ means $(T^*)^{ⓢn}$.

*Proof.* The second equality has already been noted in (3.4), so we concentrate on the proof that $\sigma(T^{ⓢn}) = \sigma(T^{\otimes n})$. For $n = 0$ this is trivial, so we fix $n \geqslant 1$.

In order to prove the inclusion $\sigma(T^{ⓢn}) \subseteq \sigma(T^{\otimes n})$ it suffices to check that $T^{\otimes n}$ maps $(H^{ⓢn})^\perp$ into itself. For any elementary symmetric tensor $h \in H^{ⓢn}$, say $h = \sum_{\sigma\in S_n} h_{\sigma(1)} \otimes \cdots \otimes h_{\sigma(n)}$, and any element $x \in H^{\otimes n}$ we have

$$[h, T^{\otimes n}x]_{H^{\otimes n}} = \Big[\sum_{\sigma\in S_n} T^*h_{\sigma(1)} \otimes \cdots \otimes T^*h_{\sigma(n)}, x\Big]_{H^{\otimes n}} = [T^{*\otimes n}h, x]_{H^{\otimes n}}. \tag{3.9}$$

Clearly, $H^{ⓢn}$ is invariant under $T^{*\otimes n}$, and therefore for $x \in (H^{ⓢn})^\perp$ we obtain $[h, T^{\otimes n}x]_{H^{\otimes n}} = [T^{*\otimes n}h, x]_{H^{\otimes n}} = 0$. Thus, $T^{\otimes n}x \in (H^{ⓢn})^\perp$.

Next we prove the inclusion $\sigma(T^{\otimes n}) \subseteq \sigma(T^{\circledS n})$. We proceed by induction on $n$, the case $n=1$ being trivial since $T^{\otimes 1} = T^{\circledS 1} = T$. Assume that we already know that $\sigma(T^{\otimes n}) \subseteq \sigma(T^{\circledS n})$ and fix $z \in \sigma(T^{\otimes (n+1)})$. We have to show that $z \in \sigma(T^{\circledS (n+1)})$.

Noting that $T^{\otimes (n+1)} = T^{\otimes n}\widehat{\otimes}T$, by (2.4) we have $z = \zeta\cdot\eta$ with $\zeta \in \sigma(T^{\otimes n}) \subseteq \sigma(T^{\circledS n})$ and $\eta\in\sigma(T)$. By Lemma 3.2 we may assume that $(\zeta,\eta)\in\partial(\sigma(T^{\circledS n})\times\sigma(T))$, so $\zeta\in\partial\sigma(T^{\circledS n})$ or $\eta\in\partial\sigma(T)$.

*Case 1:* Assume that $\zeta\in\partial\sigma(T^{\circledS n})$ and $\eta\in\sigma_{\rm a}(T)$. Since boundary spectrum belongs to the approximate point spectrum we have $\zeta\in\sigma_{\rm a}(T^{\circledS n})$. Let $(g_k)_{k\geqslant 1}$ and $(h_k)_{k\geqslant 1}$ be corresponding approximate eigenvectors for $T^{\circledS n}$ and $T$, respectively. From $T^{\circledS (n+1)}a_n^\dagger(h_k)g_k = a_n^\dagger(Th_k)T^{\circledS n}g_k$ we have

$$
\begin{aligned}
&\|T^{\circledS (n+1)}a_n^\dagger(h_k)g_k - \zeta\eta\, a_n^\dagger(h_k)g_k\|_{H^{\circledS (n+1)}}\\
&\qquad\leqslant \|a_n^\dagger(Th_k)(T^{\circledS n}g_k-\zeta g_k)\|_{H^{\circledS (n+1)}} + |\zeta|\,\|a_n^\dagger(Th_k-\eta h_k)g_k\|_{H^{\circledS (n+1)}}.
\end{aligned}
$$

Hence, by (3.5) and since by assumption we have $\|g_k\|_{H^{\circledS n}} = \|h_k\| = 1$,

$$\lim_{k\to\infty}\|T^{\circledS (n+1)}a_n^\dagger(h_k)g_k - \zeta\eta\, a_n^\dagger(h_k)g_k\|_{H^{\circledS (n+1)}} = 0.$$

Moreover, by (3.8),

$$\|a_n^\dagger(h_k)g_k\|_{H^{\circledS (n+1)}} \geqslant \|g_k\|_{H^{\circledS n}}\|h_k\| = 1.$$

Since by (3.5) the sequence $(a_n^\dagger(h_k)g_k)_{k\geqslant 1}$ is bounded, upon normalizing we obtain an approximate eigenvector for $T^{\circledS (n+1)}$ with approximate eigenvalue $z=\zeta\cdot\eta$.

*Case 2:* Assume that $\zeta\in\partial\sigma(T^{\circledS n})$ and $\eta\in\sigma_{\rm r}(T)$. Then also $\zeta\in\partial\sigma(T^{\circledS n*})$ and hence $\zeta\in\sigma_{\rm a}(T^{\circledS n*}) = \sigma_{\rm a}(T^{*\circledS n})$. Also, $\eta\in\sigma_{\rm a}(T^*)$, and therefore $z\in\sigma_{\rm a}(T^{*\circledS (n+1)}) = \sigma_{\rm a}(T^{\circledS (n+1)*}) \subseteq \sigma(T^{\circledS (n+1)})$ as in Case 1. Here we used that for all $k\geqslant 1$ we have $T^{\circledS k*} = T^{*\circledS k}$ by (3.9).

*Case 3:* If $\zeta\in\sigma_{\rm a}(T^{\circledS n})$ and $\eta\in\partial\sigma(T)$ we proceed as in Case 1.

*Case 4:* If $\zeta\in\sigma_{\rm r}(T^{\circledS n})$ and $\eta\in\partial\sigma(T)$ we proceed as in Case 2. □

Arguing as in the proof of Proposition 3.1 we now obtain:

**Theorem 3.4.** *If $\|T\|<1$, then*

$$\sigma(\Gamma^{\circledS}(T)) = \sigma(\Gamma(T)) = \{1\}\cup\overline{\bigcup_{n\geqslant 1}\Big\{\prod_{j=1}^n z_j:\ z_j\in\sigma(T);\ j=1,\dots,n\Big\}}.$$

If $T$ is compact, this result simplifies as follows:

**Corollary 3.5.** *If $T$ is compact and satisfies $\|T\|<1$, then*

$$\sigma(\Gamma^{\circledS}(T)) = \sigma(\Gamma(T)) = \{0\}\cup\{1\}\cup\bigcup_{n\geqslant 1}\Big\{\prod_{j=1}^n z_j:\ z_j\in\sigma(T);\ j=1,\dots,n\Big\}.$$

*Proof.* We have already seen that $\sigma(\Gamma^{\circledS}(T)) = \sigma(\Gamma(T))$; for the second identity we have to show that $0\in\sigma(\Gamma(T))$ and to prove the inclusion '$\subseteq$'.

From $\|T\|<1$ and $\sigma(T)\neq\varnothing$ we see that 0 is in the closure of $\bigcup_{n\geqslant 1}\big\{\prod_{j=1}^n z_j : z_j\in\sigma(T);\ j=1,\dots,n\big\}$. Hence, $0\in\sigma(\Gamma(T))$ by Proposition 3.1.

From (3.1) and the compactness of $T^{\otimes n}$ we see that $\Gamma(T)$ is compact. Hence, $\sigma(\Gamma(T)) \setminus \{0\} = \sigma_{\rm p}(\Gamma(T)) \setminus \{0\}$. Let $z \in \sigma(\Gamma(T)) \setminus \{0\}$ be arbitrary and choose an eigenvector $x$ for $z$. Then for all $n \geqslant 1$ we have $T^{\otimes n} P_n x = P_n \Gamma(T) x = z\, P_n x$, where $P_n$ denotes the orthogonal projection in $\Gamma(H)$ onto $\bigoplus_{0 \leqslant j \leqslant n} H^{\otimes j}$. For sufficiently large $n$ we have $P_n x \neq 0$, and for those $n$ we conclude that $z \in \sigma_{\rm p}(T^{\otimes n})$. In particular we see that $z = \prod_{j=1}^n z_j$ for certain $z_j \in \sigma(T)$. $\square$

## 4. The $L^p$-spectrum of second quantized contraction semigroup generators

Throughout this section we fix an arbitrary real Banach space $E$ and a centered Gaussian Radon measure $\gamma$ on $E$. Let $H_\gamma$ denote the reproducing kernel Hilbert space of $\gamma$ and let $i_\gamma : H_\gamma \hookrightarrow E$ be the associated embedding. Since $\gamma(\overline{H_\gamma}) = 1$, when considering the spaces $L^p(E,\gamma)$ there will be no loss of generality in assuming that $\gamma$ is *nondegenerate*, by which we mean that $H_\gamma$ is dense in $E$.

Let $H := H_{\gamma,\mathbb{C}}$ be the complexification of $H_\gamma$. It is well known that the complex Hilbert space $L^2(E,\gamma)$ is canonically isometrically isomorphic to the symmetric Fock space $\Gamma^{ⓢ}(H)$. We will describe this isometry briefly here; for a more detailed discussion we refer to [21]. Each element $h \in H_\gamma$ of the form $h = i_\gamma^* x^*$ defines a real-valued function $\phi_h \in L^2(E,\gamma)$ by $\phi_h(x) := \langle x, x^* \rangle$ and we have

$$\|\phi_h\|^2_{L^2(E,\gamma)} = \int_E \langle x, x^* \rangle^2 \, d\gamma(x) = \|i^* x^*\|^2_{H_\gamma}.$$

Since $i_\gamma^*$ has dense range in $H_\gamma$, the mapping $h \mapsto \phi_h$ uniquely extends to an isometry from $H_\gamma$ into the real part of $L^2(E,\gamma)$. By complexification we obtain an isometry $h \mapsto \phi_h$ from $H$ into $L^2(E,\gamma)$. Using this isometry, for each $n \geqslant 1$ we define $\mathscr{H}_{\leqslant n}$ as the closed subspace of $L^2(E,\gamma)$ spanned by the constant one function $\mathbf{1}$ and all products $\phi_{h_1} \cdot \ldots \cdot \phi_{h_m}$ of order $1 \leqslant m \leqslant n$, where $h_1, \ldots, h_n \in H$. We then let $\mathscr{H}_0 := \mathbb{C}\mathbf{1}$ and define, for $n \geqslant 1$, the space $\mathscr{H}_n$ as the orthogonal complement of $\mathscr{H}_{\leqslant n-1}$ in $\mathscr{H}_{\leqslant n}$. The complex form of the Wiener-Itô decomposition theorem asserts that we have an orthogonal direct sum decomposition

$$L^2(E,\gamma) = \bigoplus_{n \geqslant 0} \mathscr{H}_n.$$

The space $\mathscr{H}_n$ is usually referred to as the $n$-th *Wiener-Itô chaos*. Denoting by $I_n$ the orthogonal projection in $L^2(E,\gamma)$ onto $\mathscr{H}_n$, it is not difficult to show that

$$\langle I_n(\phi_{h_1} \ldots \phi_{h_n}), I_n(\phi_{k_1} \ldots \phi_{k_n}) \rangle = \sum_{\sigma \in S_n} [h_1, k_{\sigma(1)}]_H \ldots [h_n, k_{\sigma(n)}]_H.$$

This shows that $\mathscr{H}_n$ is canonically isometric to $H^{ⓢ n}$ as a Hilbert space, the isometry being given explicitly by

$$I_n(\phi_{h_1} \cdot \cdots \cdot \phi_{h_n}) \mapsto \frac{1}{\sqrt{n!}} \sum_{\sigma \in S_n} h_{\sigma(1)} \otimes \cdots \otimes h_{\sigma(n)}.$$

Thus the Wiener-Itô decomposition induces a canonical isometry of $L^2(E,\gamma)$ and the symmetric Fock space $\Gamma^{ⓢ}(H)$.

Let us now assume that $A$ is the infinitesimal generator of a strongly continuous semigroup of contractions $S = \{S(t)\}_{t\geqslant 0}$ on $H$. We denote by $P_2 = \{P_2(t)\}_{t\geqslant 0}$ its symmetric second quantization:

$$P_2(t) = \Gamma^{ⓢ}(S(t)).$$

By the isometry just described, $P_2$ induces a semigroup of contractions, also denoted by $P_2$, on $L^2(E,\gamma)$. Since $P_2$ is strongly continuous on each $\mathscr{H}_n$, it follows that $P_2$ is strongly continuous on $L^2(E,\gamma)$. In fact, $P_2$ is doubly Markovian and therefore $P_2$ extends uniquely to a strongly continuous semigroup of contractions on $L^p(E,\gamma)$ for every $p \in [1,\infty)$; cf. [30].

**Lemma 4.1.** *If $t \mapsto S(t)$ is norm continuous for $t > t_0$ and $S(t)$ is a strict contraction for $t > s_0$, then for every $p \in (1,\infty)$, $t \mapsto P_p(t)$ is norm continuous for $t > \max\{t_0, s_0\}$.*

*Proof.* First we consider the case $p = 2$. By (3.2), for each $n \geqslant 1$ the restriction $P_{2,n}$ of $P_2$ to $\mathscr{H}_n$ is contractive and norm continuous for $t > t_0$. For $t > s_0$, by (3.1) we have $P_2(t) = \sum_{n\geqslant 0} P_{2,n}(t)$ with convergence in the operator norm, uniformly on $[s,\infty)$ for every $s > s_0$. Hence, for $t > \max\{t_0, s_0\}$ the function $t \mapsto P_2(t)$ is norm continuous, since on this interval it is the locally uniform limit of a sequence of norm continuous functions.

Next we take $p \in (1,2)$ and use the fact that $\|P_1(t) - P_1(s)\|_{\mathscr{L}(L^1(E,\gamma))} \leqslant 2$ and the Riesz-Thorin interpolation theorem to find that

$$\|P_p(t) - P_p(s)\|_{\mathscr{L}(L_p(E,\gamma))} \leqslant 2^{1-\theta_p} \|P_2(t) - P_2(s)\|^{\theta_p}_{\mathscr{L}(L^2(E,\gamma))}$$

where $(1-\theta_p) + \frac{1}{2}\theta_p = \frac{1}{p}$. For $p \in (2,\infty)$ we proceed similarly, this time interpolating between $L^2(E,\gamma)$ and $L^{p'}(E,\gamma)$ with $p' \in (p,\infty)$. $\square$

In the next lemma we need some further results about the spaces $L^p(E,\gamma)$. We refer to [21] for the proofs, which are based on standard hypercontractivity arguments. For all $p \in (1,\infty)$ and $n \geqslant 0$ we have $\mathscr{H}_n \subseteq L^p(E,\gamma)$ and the restrictions of the $L^2(E,\gamma)$-norm and the $L^p(E,\gamma)$-norm are equivalent on $\mathscr{H}_n$. Furthermore, the orthogonal projections $I_n$ in $L^2(E,\gamma)$ onto $\mathscr{H}_n$ extend uniquely to projections $I_{p,n}$ in $L^p(E,\gamma)$ onto $\mathscr{H}_n$. As a subspace of $L^p(E,\gamma)$, $\mathscr{H}_n$ will be denoted by $\mathscr{H}_{p,n}$. By the observations just made, each $\mathscr{H}_{p,n}$ is complemented in $L^p(E,\gamma)$.

**Lemma 4.2.** *For all $p \in (1,\infty)$ there is a constant $\theta_p \in (0,1]$ such that for all $n \geqslant 1$ and $t \geqslant 0$ we have*

$$\|P_{p,n}(t)\|_{\mathscr{L}(\mathscr{H}_{p,n})} \leqslant \|S(t)\|^{n\theta_p}.$$

*As a consequence we have $P_p(t) = \sum_{n\geqslant 0} P_{p,n}(t)$, the convergence being in the operator norm, uniformly on $[t_0,\infty)$ for all $t_0 > 0$.*

*Proof.* Fix $t \geqslant 0$. For $p = 2$ we may take $\theta_p = 1$. For $p \in (1,2)$, choose $p' \in (1,p)$ and recall that $P_{p'}$ is a contraction semigroup on $L^{p'}(E,\gamma)$. In particular, by taking restrictions, we see that $\|P_{p',n}(t)\|_{\mathscr{L}(\mathscr{H}_{p',n})} \leqslant 1$ for all $n \geqslant 1$. Since we also have $\|P_{2,n}(t)\|_{\mathscr{L}(\mathscr{H}_n)} = \|S(t)\|^n$, for $p \in (1,2)$ the result now follows by interpolation;

notice that $(\mathscr{H}_n, \mathscr{H}_{p',n})_{\theta_p} = \mathscr{H}_{p,n}$ with $\frac{1}{p'}(1-\theta_p) + \frac{1}{2}\theta_p = \frac{1}{p}$. For $p \in (2,\infty)$ we proceed similarly, this time interpolation between $\mathscr{H}_n$ and $\mathscr{H}_{p',n}$ with $p' \in (p,\infty)$.

By the estimate just proved, the series $\sum_{n\geqslant 0} P_{p,n}(t)$ converges in the operator norm, uniformly on $[t_0,\infty)$ for every $t_0 > 0$. Moreover, on the dense subspace spanned by the spaces $\mathscr{H}_{p,n}$ the sum equals $P_p(t)$. This completes the proof. $\square$

The infinitesimal generators of the semigroups $P_p$ and $P_{p,n}$ will be denoted by $L_p$ and $L_{p,n}$, respectively.

In the proof of the main result of this section, Theorem 4.3 below, we shall use the well-known fact [15, Theorem II.4.18] that if $A$ is the infinitesimal generator of a strongly continuous and eventually norm continuous semigroup on $X$, then

$$\text{for every } a \in \mathbb{R} \text{ the set } \{z \in \mathbb{C} : \ z \in \sigma(A),\ \operatorname{Re} z \geqslant a\} \text{ is bounded.} \tag{4.1}$$

Let us call a semigroup $T$ *strictly contractive* if $\|T(t)\| < 1$ for all $t > 0$.

**Theorem 4.3.** *Let $p \in (1,\infty)$. If $S$ is strictly contractive and eventually norm continuous, then*

$$\sigma(L_{p,0}) = \{0\}, \quad \sigma(L_{p,n}) = \Big\{\sum_{j=1}^{n} \zeta_j : \ \zeta_j \in \sigma(A);\ j = 1,\dots,n\Big\}, \tag{4.2}$$

*and*

$$\begin{aligned} \sigma(L_p) &= \{0\} \cup \overline{\bigcup_{n\geqslant 1} \Big\{\sum_{j=1}^{n} \zeta_j : \ \zeta_j \in \sigma(A);\ j = 1,\dots,n\Big\}} \\ &= \overline{\Big\{\sum_{j=1}^{n} k_j z_j : \ \ k_j \in \mathbb{N},\ z_j \in \sigma(A);\ j = 1,\dots,n;\ n \geqslant 1\Big\}}. \end{aligned} \tag{4.3}$$

*Proof.* Let $p \in (1,\infty)$ be fixed. It is clear that $\sigma(L_{p,0}) = \{0\}$, so let us fix $n \geqslant 1$.

From Theorem 3.3, Lemma 4.1, and the spectral mapping theorem for eventually norm continuous semigroups, first applied to $P_{p,n}$ and then to $S$, for $t \geqslant 0$ we obtain

$$\begin{aligned} \exp\big(t\,\sigma(L_{p,n})\big) &= \sigma(P_{p,n}(t)) \setminus \{0\} \\ &= \Big\{\prod_{j=1}^{n} z_j : \ z_j \in \sigma(S(t)) \setminus \{0\};\ j = 1,\dots,n\Big\} \\ &= \Big\{\prod_{j=1}^{n} \exp(t\zeta_j) : \ \zeta_j \in \sigma(A);\ j = 1,\dots,n\Big\} \\ &= \Big\{\exp\Big(t\sum_{j=1}^{n}\zeta_j\Big) : \ \zeta_j \in \sigma(A);\ j = 1,\dots,n\Big\}. \end{aligned} \tag{4.4}$$

In order to obtain the corresponding equality for $\sigma(L_p)$ we first check that for all $t \geqslant 0$,

$$\sigma(P_p(t)) = \{1\} \cup \overline{\bigcup_{n\geqslant 1} \Big\{\prod_{j=1}^{n} z_j : \ z_j \in \sigma(S(t));\ j = 1,\dots,n\Big\}}.$$

Clearly this holds for $t=0$, and for fixed $t>0$ this follows from Proposition 2.1, Theorem 3.3, the expansion in Lemma 4.2, and the fact that $\mathscr{H}_n = \mathscr{H}_{p,n}$ with equivalent norms. Repeating the argument of (4.4) we obtain

$$\exp\big(t\,\sigma(L_p)\big) = \{1\} \cup \overline{\bigcup_{n\geqslant 1}\Big\{\exp\Big(t\sum_{j=1}^n \zeta_j\Big):\ \zeta_j\in\sigma(A);\ j=1,\dots,n\Big\}\setminus\{0\}}. \tag{4.5}$$

Let

$$B_n = \Big\{\sum_{j=1}^n \zeta_j:\ \zeta_j\in\sigma(A);\ j=1,\dots,n\Big\} \qquad (n\geqslant 1)$$

and

$$B := \bigcup_{n\geqslant 1}\Big\{\sum_{j=1}^n \zeta_j:\ \zeta_j\in\sigma(A);\ j=1,\dots,n\Big\}.$$

For (4.2) we have to prove the inclusions $B_n\subseteq\sigma(L_{p,n})$ and $\sigma(L_{p,n})\subseteq B_n$; for (4.3) we have to prove the inclusions $\overline{B}\subseteq\sigma(L_p)$ (the inclusion $\{0\}\subseteq\sigma(L)$ being trivial) and $\sigma(L_p)\setminus\{0\}\subseteq\overline{B}$. We shall prove the latter two; the former two are proved in the same way. We adapt an argument from [20].

- $\overline{B}\subseteq\sigma(L_p)$:

Since $\sigma(L_p)$ is closed, it suffices to prove that $B\subseteq\sigma(L_p)$. Fix an arbitrary $\zeta\in B$, say $\zeta=\sum_{j=1}^n\zeta_j$ with $n\geqslant 1$ and $\zeta_j\in\sigma(A)$ $(j=1,\dots,n)$. By (4.5), for every $t>0$ we find an element $\zeta(t)\in\sigma(L_p)$ and integer $N(t)\in\mathbb{Z}$ such that

$$\zeta = 2\pi i t^{-1}N(t)+\zeta(t). \tag{4.6}$$

From $\operatorname{Re}\zeta(t)=\operatorname{Re}\zeta$ and (4.1) we see that there is a constant $C\geqslant 0$ such that $|\operatorname{Im}\zeta(t)|\leqslant C$ for all $t>0$. Comparing imaginary parts in (4.6) and letting $t\downarrow 0$ we see that $N(t)=0$ for small enough $t$. For those $t$ we then have $\zeta=\zeta(t)\in\sigma(L_p)$.

- $\sigma(L_p)\setminus\{0\}\subseteq\overline{B}$:

The proof proceeds along the same lines, but extra care is needed to control the number of terms occurring in the sums defining the elements of $B$. Fix $z\in\sigma(L_p)\setminus\{0\}$. Since $\|S(t)\|<1$ for all $t>0$, standard arguments from semigroup theory imply that $S$ is uniformly exponentially stable. Choose $\omega>0$ and $M\geqslant 1$ such that $\|S(t)\|\leqslant Me^{-\omega t}$ for all $t\geqslant 0$. Then, $\sigma(A)\subseteq\{\lambda\in\mathbb{C}:\ \operatorname{Re}\lambda\leqslant-\omega\}$. Together with (4.5) this implies that $\operatorname{Re}z\leqslant-\omega$. By (4.1), there exists a constant $C\geqslant 1$ such that for every $w\in\sigma(A)$ satisfying $\operatorname{Re}w\geqslant 2\operatorname{Re}z$ we have $|\operatorname{Im}w|\leqslant C$. In particular,

$$|\operatorname{Im}z|\leqslant C. \tag{4.7}$$

For reasons that will become clear soon we fix $t_0>0$ subject to the condition that

$$2\pi t_0^{-1} > C\big(1+\tfrac{1}{\omega}|\operatorname{Re}z|\big).$$

Choose $0<\varepsilon<1$ so small that

$$2\pi t_0^{-1}\geqslant C\big(1+(\tfrac{1+\varepsilon}{\omega}+\varepsilon)|\operatorname{Re}z|\big). \tag{4.8}$$

Fix $0 < t < t_0$ arbitrary. By (4.5) there exists a sequence of complex numbers $(z_k(t))_{k\geqslant 1}$ such that

$$\lim_{k\to\infty} z_k(t) = z \tag{4.9}$$

with $\exp\bigl(tz_k(t)\bigr) \in \exp(tB)$ for all $k$. Most of the remaining argument is devoted to proving that $z_k(t) \in B$ for all sufficiently large $k$.

Choose $k_0(t)$ so large that

$$|z_k(t) - z| \leqslant \varepsilon|\mathrm{Re}\, z| \tag{4.10}$$

for all $k \geqslant k_0(t)$. Fix $k \geqslant k_0(t)$ and notice that, by (4.10) and the fact that $0 < \varepsilon < 1$,

$$\mathrm{Re}\, z_k(t) \geqslant 2\mathrm{Re}\, z. \tag{4.11}$$

Choose integers $n_k(t) \geqslant 1$ and $N_k(t) \in \mathbb{Z}$ such that

$$z_k(t) = 2\pi i t^{-1} N_k(t) + \sum_{j=1}^{n_k(t)} \zeta_{j,k}(t) \tag{4.12}$$

with all $\zeta_{j,k}(t)$ in $\sigma(A)$. Note that for all $j$,

$$\mathrm{Re}\, \zeta_{j,k}(t) \leqslant -\omega \tag{4.13}$$

and hence, by (4.12),

$$\mathrm{Re}\, z_k(t) \leqslant -n_k(t)\omega. \tag{4.14}$$

Also notice that from (4.11), (4.12), and (4.13),

$$\mathrm{Re}\, \zeta_{j,k}(t) \geqslant 2\mathrm{Re}\, z. \tag{4.15}$$

From (4.10) and (4.14) we deduce that

$$n_k(t)\omega \leqslant |\mathrm{Re}\, z_k(t)| \leqslant (1+\varepsilon)|\mathrm{Re}\, z|.$$

By (4.15) and the choice of $C$ we have $|\mathrm{Im}\, \zeta_{j,k}(t)| \leqslant C$ and therefore, by (4.7), (4.10), (4.12), (4.15), and the fact that $C \geqslant 1$,

$$\begin{aligned} C \geqslant |\mathrm{Im}\, z| &\geqslant |\mathrm{Im}\, z_k| - \varepsilon C|\mathrm{Re}\, z| \\ &\geqslant 2\pi t^{-1} N_k(t) - Cn_k(t) - \varepsilon C|\mathrm{Re}\, z| \\ &\geqslant 2\pi t^{-1} N_k(t) - (1+\varepsilon)C|\mathrm{Re}\, z|/\omega - \varepsilon C|\mathrm{Re}\, z|. \end{aligned} \tag{4.16}$$

If $N_k(t)$ were nonzero, then $N_k(t) \geqslant 1$ and in view of $0 < t < t_0$ the right hand side would be strictly greater than

$$2\pi t_0^{-1} - (1+\varepsilon)C|\mathrm{Re}\, z|/\omega - \varepsilon C|\mathrm{Re}\, z| \geqslant C, \tag{4.17}$$

where the inequality follows from the choice of $\varepsilon$ in (4.8). By comparing (4.16) and (4.17) we see that we have arrived at a contradiction. Thus, $N_k(t) = 0$ and therefore, $z_k(t) = \sum_{j=1}^{n_k(t)} \zeta_{j,k}(t)$. It follows that $z_k(t) \in B$.

So far, $k \geqslant k_0(t)$ was fixed. By letting $k \to \infty$ and recalling (4.9) we obtain $z \in \overline{B}$. □

If the eventual norm continuity assumption is strengthened to eventual compactness, there is no need to take the closure in (4.3). This is the content of the following semigroup analogue of Corollary 3.5:

**Corollary 4.4.** *Let $p \in (1,\infty)$. If $S$ is strictly contractive and eventually compact, then $P_p$ is eventually compact and*

$$\sigma(L_p) = \Big\{\sum_{j=1}^n k_j z_j : \quad k_j \in \mathbb{N},\ z_j \in \sigma(A);\ j = 1,\dots,n;\ n \geqslant 1\Big\}.$$

*Proof.* Let $S(t)$ be compact for $t > t_0$. Since $\mathscr{H}_n = \mathscr{H}_{p,n}$ with equivalent norms and since $T^{\circledS n}$ is compact whenever $T$ is, the operators $P_{p,n}(t)$ are compact for $t > t_0$. The expansion in Lemma 4.2 then shows that $P_p(t)$ is compact for $t > t_0$. Hence by the spectral mapping theorem for the point spectrum, $\sigma(L_p)$ consists of isolated eigenvalues and the result follows from Theorem 4.3. □

## 5. The $L^p$-spectrum of Ornstein-Uhlenbeck operators

Let $E$ be a real Banach space, let $A$ be infinitesimal generator of a strongly continuous semigroup $S = \{S(t)\}_{t\geqslant 0}$ on $E$, and let $Q \in \mathscr{L}(E^*, E)$ be a positive and symmetric operator. In this section we shall apply our abstract results to the Ornstein-Uhlenbeck operator $L$, given on a suitable core of cylindrical functions by

$$Lf(x) := \tfrac{1}{2}\mathrm{Tr}\,(QD^2 f(x)) + \langle Ax, Df\rangle \qquad (x \in E) \tag{5.1}$$

where $D$ denotes the Fréchet derivative. The operator $L$ arises as the infinitesimal generator of the transition semigroup of the Markov process $\{U_x(t)\}_{t\geqslant 0}$ that solves the stochastic linear evolution equation

$$\begin{aligned} dU(t) &= AU(t)\,dt + dW_Q(t) \qquad (t \geqslant 0) \\ U(0) &= x \end{aligned} \tag{5.2}$$

where $W_Q$ is a cylindrical $Q$-Wiener process in $E$; cf. [7, 10].

For $t > 0$ we define the positive symmetric operators $Q_t \in \mathscr{L}(E^*, E)$ by

$$Q_t x^* := \int_0^t S(s)QS^*(s)x^*\,ds \qquad (x^* \in E^*).$$

The right hand side integral is easily shown to exist as a Bochner integral in $E$. In order to give a rigorous description of the operator $L$, unless otherwise stated we shall assume in the remainder of this section that the following two hypotheses hold:

- (H$Q_\infty$) The weak operator limit $Q_\infty := \lim_{t\to\infty} Q_t$ exists in $\mathscr{L}(E^*, E)$;
- (H$\mu$) The operator $Q_\infty$ is the covariance of a centered Gaussian Radon measure $\mu$ on $E$.

It will follow from Lemma 5.1 below that there is no loss of generality in assuming that $\mu$ is nondegenerate.

Some comments are in order.

(1) By Hypothesis (H$Q_\infty$), for all $x^*, y^* \in E^*$ we have
$$\langle Q_\infty x^*, y^*\rangle = \int_0^\infty \langle S(s)QS^*(s)x^*, y^*\rangle\, ds,$$
the scalar integrals being defined in the improper sense. The positivity of $Q$ together with a polarization argument imply that these integrals are actually absolutely convergent.

(2) Hypothesis (H$\mu$) and a standard tightness argument imply that each $Q_t$ is the covariance operator of a centered Gaussian Radon measure $\mu_t$, and we have $\lim_{t\to\infty} \mu_t = \mu$ weakly.

(3) If $E$ is separable, the word 'Radon' may be replaced by 'Borel'.

(4) If $E$ is a Hilbert space and we identify $E$ and $E^*$ in the usual way, then Hypotheses (H$Q_\infty$) and (H$\mu$) hold if and only if each $Q_t$ is of trace class and $\sup_{t>0} \operatorname{Tr} Q_t < \infty$. Furthermore if Hypothesis (H$Q_\infty$) holds, then Hypothesis (H$\mu$) holds if and only if $Q_\infty$ is of trace class. In particular if $\dim E < \infty$, then Hypothesis (H$Q_\infty$) implies (H$\mu$).

As is shown in [7, 10], the existence of the measures $\mu_t$ is equivalent to the existence of a (necessarily unique) weak solution $\{U(t,x)\}_{t\geqslant 0}$ of (5.2). This solution is Gaussian; the random variable $U(t,x)$ has mean $S(t)x$ and covariance operator $Q_t$. The solution is also Markovian and its transition semigroup $P = \{P(t)\}_{t\geqslant 0}$ on $B_b(E)$ is given by
$$P(t)f(x) = \mathbb{E}\, f(U(t,x)) = \int_E f(S(t)x+y)\, d\mu_t(y) \qquad (x\in E,\ f\in B_b(E)).$$
Here $B_b(E)$ denotes the space of bounded complex-valued Borel measurable functions on $E$. This semigroup leaves $C_b(E)$, the subspace of all continuous functions in $B_b(E)$, invariant. Although the restricted semigroup generally fails to be strongly continuous in the norm topology of $C_b(E)$, it is strongly continuous in the strict topology of $C_b(E)$. This is, by definition, the finest locally convex topology $\tau$ on $C_b(E)$ that agrees with the compact-open topology on bounded sets. As a result, the infinitesimal generator $L$ of $P$ is well defined as a linear operator on the domain
$$\mathscr{D}(L) = \Big\{ f\in C_b(E):\ \tau\text{-}\lim_{t\downarrow 0} \frac{1}{t}(P(t)f - f) \text{ exists in } C_b(E)\Big\}.$$
On a suitable core of cylindrical functions, $L$ is given by (5.1). The measure $\mu$ is *invariant* for $L$ in the sense that for all $t\geqslant 0$ and $f\in B_b(E)$ we have
$$\int_E P(t)f(x)\, d\mu(x) = \int_E f(x)\, d\mu(x).$$
For more details we refer to the survey paper [17].

By standard arguments, the invariance of $\mu$ implies that $P$ extends to a strongly continuous contraction semigroup $P_p$ on $L^p(E,\mu)$ for all $p\in[1,\infty)$. The infinitesimal generator of $P_p$ will be denoted by $L_p$. In order to establish the relationship between these semigroups and the ones studied in the previous section, we describe next how $P_2$ arises as a second quantized semigroup.

Let us denote the reproducing kernel Hilbert space associated with $Q_\infty$ by $H_\mu$ and the embedding $H_\mu \hookrightarrow E$ by $i_\mu$. By (2.5) we have $Q_\infty = i_\mu \circ i_\mu^*$. The key fact is the following result, due to Chojnowska-Michalik and Goldys [9] under some additional assumptions; the present formulation was given in [26, Theorem 6.2].

**Lemma 5.1.** *Assume Hypothesis* (H$Q_\infty$). *The space $H_\mu$ is invariant under the action of $S$, and the restriction of $S$ to $H_\mu$, denoted by $S_\mu$, is a strongly continuous semigroup of contractions on $H_\mu$.*

By complexification, $S_{\mu,\mathbb{C}}$ is a strongly continuous semigroup of contractions on $H_{\mu,\mathbb{C}}$, and upon identifying $L^2(E,\mu)$ with $\Gamma^{ⓢ}(H_{\mu,\mathbb{C}})$ as explained in the previous section we have the following representation of $P_2$, again due to Chojnowska-Michalik and Goldys [9]; see also [26, Theorem 6.12]:

**Proposition 5.2.** *For all $t \geqslant 0$ we have $P_2(t) = \Gamma^{ⓢ}(S^*_{\mu,\mathbb{C}}(t))$.*

Under this identification, the semigroup $P_p$ agrees with the one introduced in the previous section. As an immediate consequence of Theorem 4.3, Corollary 4.4, and Proposition 5.2 we obtain:

**Theorem 5.3.** *If $S_\mu$ is strictly contractive and eventually norm continuous, then*

$$\sigma(L_p) = \overline{\Big\{\sum_{j=1}^{n} k_j z_j : \quad k_j \in \mathbb{N},\ z_j \in \sigma(A_\mu);\ j = 1, \dots, n;\ n \geqslant 1\Big\}}.$$

*If in addition $S_\mu$ is compact, then*

$$\sigma(L_p) = \Big\{\sum_{j=1}^{n} k_j z_j : \quad k_j \in \mathbb{N},\ z_j \in \sigma(A_\mu);\ j = 1, \dots, n;\ n \geqslant 1\Big\}.$$

Notice that $\sigma(L_p)$ depends on $\sigma(A_\mu)$ rather than on $\sigma(A)$. This can be understood by observing that the definition of $L_p$ depends not only on $A$, but also on $Q$. On a deeper level, the abstract results of the previous section show that Theorem 5.3 is in fact completely natural: Theorem 4.3 shows that it can be interpreted as saying that $\sigma(L_p)$ can be computed from the part of $L_p$ in the first Wiener chaos. Of course, this limits that practical use of Theorem 5.3 to some extent, as in general it may be difficult to compute $\sigma(A_\mu)$ from $A$ and $Q$.

In the next two subsections we prove that the assumptions on $S_\mu$ are automatically satisfied in two important cases: the strong Feller case and the finite-dimensional case. In both cases we check the strict contractivity assumption by an appeal to the following result, due to Chojnowska-Michalik and Goldys [9]; cf. also [26, Theorem 6.3].

**Lemma 5.4.** *Assume Hypothesis* (H$Q_\infty$) *and fix $t > 0$. Then $\|S_\mu(t)\|_{\mathscr{L}(H_\mu)} < 1$ if and only if $H_t = H_\mu$ with equivalent norms.*

Here $H_t$ is the reproducing kernel Hilbert space associated with the positive symmetric operator $Q_t$ introduced in the previous section.

The following example shows that a uniformly exponentially stable semigroup may fail to be strictly contractive even if $\dim E < \infty$, and that even if $S_\mu$ is strictly contractive it may happen that there exists no constant $a > 0$ such that $\|S_\mu(t)\|_{\mathscr{L}(H_\mu)} \leqslant e^{-at}$ for all $t \geqslant 0$.

*Example* 5.5. For $\omega > 0$ we consider the semigroup $S^{(\omega)}$ on $E = \mathbb{R}^2$ defined by $S^{(\omega)}(t) = e^{-\omega t}\begin{pmatrix} 1 & t \\ 0 & 1 \end{pmatrix}$. This semigroup is uniformly exponentially stable, but for each $0 < \omega < 1$ we have $\|S^{(\omega)}(t)\| > 1$ for $t > 0$ small enough.

Let us now take $Q = \begin{pmatrix} 0 & 0 \\ 0 & 1 \end{pmatrix}$. With this choice, the limit $Q_\infty^{(\omega)} = \lim_{t\to\infty} Q_t^{(\omega)}$ exists for every $\omega > 0$. Let $\mu^{(\omega)}$ be the corresponding Gaussian measures. Taking $\omega = 1$ and $\mu := \mu^{(1)}$, by the computations in [17, Example 5.5] we have

$$\|S_\mu(t)\|_{\mathscr{L}(H_\mu)} = e^{-t}\big(t + \sqrt{t^2+1}\big).$$

Thus, $S_\mu(t)$ is strictly contractive on $H_\mu$ but there exists no $a > 0$ such that $\|S_\mu(t)\|_{\mathscr{L}(H_\mu)} \leqslant e^{-at}$ for all $t \geqslant 0$.

5.1. **The strong Feller case.** Throughout this subsection we assume Hypotheses (H$Q_\infty$) and (H$\mu$). The transition semigroup $P$ is called *strongly Feller* if $P(t)f \in C_b(E)$ for all $f \in B_b(E)$ and $t > 0$. We have the following reproducing kernel Hilbert space characterization of this property; see [10] and [26, Corollary 2.3].

**Lemma 5.6.** *The transition semigroup $P$ is strongly Feller if and only if $S(t)E \subseteq H_t$ for all $t > 0$.*

The assumptions on $S_\mu$ in Theorem 5.3 are satisfied in the strong Feller case:

**Proposition 5.7.** *If the transition semigroup $P$ is strongly Feller, then $S_\mu$ is compact and strictly contractive.*

*Proof.* By (2.6) we have a continuous inclusion $H_t \hookrightarrow H_\mu$; the inclusion mapping will be denoted by $i_{t,\mu}$. Denoting the inclusion mapping $H_t \hookrightarrow E$ by $i_t$, we have $i_t = i_\mu \circ i_{t,\mu}$. By Lemma 5.6 we have a factorization

$$S_\mu(t) = i_{t,\mu} \circ \Sigma(t) \circ i_\mu, \tag{5.3}$$

where $\Sigma(t)$ is the operator $S(t)$, viewed as an operator from $E$ into $H_t$. Recalling from Section 2 that $i_\mu$ is compact, it follows that $S_\mu(t)$ is compact for every $t > 0$.

By Lemma 5.4 it remains to prove that for all $t > 0$ we have $H_t = H_\mu$ with equivalent norms. We have already seen that $H_t \hookrightarrow H_\mu$ with continuous inclusion. To obtain the reverse inclusion we apply $i_\mu$ on both sides of (5.3) to obtain the identity

$$S(t) \circ i_\mu = i_\mu \circ S_\mu(t) = i_t \circ \Sigma(t) \circ i_\mu.$$

Together with the identity $Q_\infty = Q_t + S(t)Q_\infty S^*(t)$, which follows directly from the definitions of $Q_t$ and $Q_\infty$, for $x^* \in E^*$ we obtain

$$\begin{aligned}
\langle Q_\infty x^*, x^*\rangle &= \langle Q_t x^*, x^*\rangle + \langle Q_\infty S^*(t)x^*, S^*(t)x^*\rangle \\
&= \langle Q_t x^*, x^*\rangle + [i_\mu^* S^*(t)x^*, i_\mu^* S^*(t)x^*]_{H_\mu} \\
&= \langle Q_t x^*, x^*\rangle + [i_\mu^* \Sigma^*(t) i_t^* x^*, i_\mu^* \Sigma^*(t) i_t^* x^*]_{H_\mu} \\
&= \langle Q_t x^*, x^*\rangle + \langle Q_\infty \Sigma^*(t) i_t^* x^*, \Sigma^*(t) i_t^* x^*\rangle \\
&\leqslant \langle Q_t x^*, x^*\rangle + \|Q_\infty\|_{\mathscr{L}(E^*,E)} \|\Sigma(t)\|^2_{\mathscr{L}(E,H_t)} \|i_t^* x^*\|^2_{H_t} \\
&= \big(1 + \|Q_\infty\|_{\mathscr{L}(E^*,E)} \|\Sigma(t)\|^2_{\mathscr{L}(E,H_t)}\big)\langle Q_t x^*, x^*\rangle.
\end{aligned}$$

The desired inclusion now follows from (2.6). □

Summarizing our discussion we have proved:

**Theorem 5.8.** *If the transition semigroup generated by $L$ is strongly Feller, then*

$$\sigma(L_p) = \Big\{ \sum_{j=1}^{n} k_j z_j : \quad k_j \in \mathbb{N},\ z_j \in \sigma(A_\mu);\ j = 1, \dots, n;\ n \geqslant 1 \Big\}.$$

5.2. **The finite-dimensional case.** We will show next that the assumptions on $S_\mu$ in Theorem 5.3 are also satisfied if $\dim E < \infty$. Since in finite dimensions every strongly continuous semigroup is compact, we only have to check the strict contractivity assumption.

In the following result we do not *a priori* assume Hypothesis (H$Q_\infty$). We identify $E$ and its dual in the natural way.

**Proposition 5.9.** *Let* $\dim E < \infty$. *The following assertions are equivalent:*

(1) $Q_\infty := \lim_{t\to\infty} Q_t$ *exists and* $Q_\infty$ *is invertible;*
(2) $Q_t$ *is invertible for all* $t > 0$ *and* $S$ *is uniformly exponentially stable.*

*In this situation we have* $H_t = H_\mu = E$ *with equivalent norms and* $S_\mu$ *is a strict contraction semigroup.*

*Proof.* (1)⇒(2): From $\operatorname{rank} Q_\infty = n$ and $\lim_{t\to\infty} Q_t = Q_\infty$ we have $\operatorname{rank} Q_t = n$ for large enough $t$. For these $t$ we have $H_t = \operatorname{range} Q_t = E$. On the other hand, since the subspaces $H_t$ increase with $t$ and since their dimensions can make only finitely many jumps, there is a time $t_0 > 0$ such that $H_t = H_{t_0}$ for all $0 < t \leqslant t_0$. It then follows from [26, Theorem 1.4] that $S(s)$ maps $H_{t_0} = \operatorname{range} Q_{t_0}$ into itself for all $s \geqslant 0$. The identity

$$Q_{kt_0} = Q_{t_0} + \cdots + S((k-1)t_0) Q_{t_0} S^*((k-1)t_0)$$

then implies that

$$H_{kt_0} = \operatorname{range} Q_{kt_0} \subseteq \operatorname{range} Q_{t_0} = H_{t_0} = H_t$$

for all $k \geqslant 1$. But by the observations already made, for $k$ large enough we have $H_{kt_0} = E$ and therefore $H_t = E$ for all $0 < t \leqslant t_0$. But then we have $H_t = E$ for all $t > 0$. This means that $Q_t$ is invertible for all $t > 0$.

The above arguments show that for all $t > 0$, $H_t = H_\mu = E$ with equivalent norms. By Lemma 5.4, the first of these identities implies that $S_\mu$ is a strict contraction semigroup. The second of these identities then implies that $E$ can be renormed in such a way that $S$ is a strict contraction group. In particular, $S$ is uniformly exponentially stable.

(2)⇒(1): The existence of the limit defining $Q_\infty$ is obvious from the uniform exponential stability of $S$. The invertibility of $Q_\infty$ follows the inclusion $H_t \subseteq H_\mu$: the invertibility of $Q_t$ means that $H_t = E$ and therefore $H_\mu = E$. □

It is possible (but not entirely straightforward) to give a direct proof of the strict contractivity of $S_\mu$ based on a compactness argument and some elementary spectral theory.

In the results so far we could assume without loss of generality that $\mu$ is nondegenerate, but this was nowhere essential. In our final result, the nondegeneracy is crucial:

**Theorem 5.10.** *If* $\dim E < \infty$ *and* $\mu$ *is nondegenerate, then for all* $p \in (1, \infty)$ *we have*

$$\sigma(L_p) = \Big\{ \sum_{j=1}^{n} k_j z_j : \quad k_j \in \mathbb{N},\ z_j \in \sigma(A);\ j = 1, \dots, n;\ n \geqslant 1 \Big\}.$$

*Proof.* Since $\dim E < \infty$, the nondegeneracy of $\mu$ implies the invertibility of $Q_\infty$. Consequently we have $H_\mu = E$ with equivalent norms, and therefore $\sigma(A_\mu) = \sigma(A)$. The result now follows from the second part of Theorem 5.3. □

In [24], Theorem 5.10 was proved under the assumptions that $\dim E < \infty$, $Q_t$ is invertible for all $t > 0$, and $S$ is uniformly exponentially stable. By Proposition 5.9 and the fact that in finite dimensions every positive symmetric operator is a Gaussian covariance, these assumptions are equivalent to the ones of Theorem 5.10, viz. the existence of a nondegenerate invariant measure.

It was observed in [24] that all generalized eigenvectors of $L_p$ are polynomials. This fact follows effortless from our approach. First, by Theorem 5.3 and compactness, every $z \in \sigma(L_p)$ is an eigenvalue of finite multiplicity. If $f$ is a generalized eigenvector, then for all $n$ sufficiently large, $\sum_{j=0}^{n} I_{p,n} f$ is nonzero and therefore a generalized eigenvector as well. But since the generalized eigenspace of $z$ is finite-dimensional, at most finitely many $I_{p,n} f$ are nonzero. Hence, $f = \sum_{k=1}^{m} I_{p,n_k} f$ with each $I_{p,n_k} f \in \mathscr{H}_{p,n_k}$. But each $\mathscr{H}_{p,n_k}$ is the linear span of a finite set of polynomials, and therefore $f$ is a polynomial.

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Delft Institute of Applied Mathematics, Technical University of Delft, P.O. Box 5031, 2600 GA Delft, The Netherlands

*E-mail address*: J.vanNeerven@math.tudelft.nl