On the asymptotic behaviour of a semigroup of linear operators

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Let $\mathbf{T} = \{T(t)\}_{t\geqslant 0}$ be a C_0-semigroup on a Banach space X. In this paper, we study the relations between the abscissa $\omega_{L^p}(\mathbf{T})$ of weak p-integrability of $\mathbf{T}$ $(1 \leqslant p < \infty)$, the abscissa $\omega_R^p(A)$ of p-boundedness of the resolvent of the generator A of $\mathbf{T}$ $(1 \leqslant p \leqslant \infty)$, and the growth bounds $\omega_\beta(\mathbf{T})$, $\beta \geqslant 0$, of $\mathbf{T}$. Our main results are as follows.

(i) Let $\mathbf{T}$ be a C_0-semigroup on a B-convex Banach space such that the resolvent of its generator is uniformly bounded in the right half plane. Then $\omega_{1-\varepsilon}(\mathbf{T}) < 0$ for some $\varepsilon > 0$.

(ii) Let $\mathbf{T}$ be a C_0-semigroup on L^p whose resolvent is uniformly bounded in the right half plane. Then $\omega_\beta(\mathbf{T}) < 0$ for all $\beta > |\frac{1}{p} - \frac{1}{p'}|$, $\frac{1}{p} + \frac{1}{p'} = 1$.

(iii) Let $1 \leqslant p \leqslant 2$ and let $\mathbf{T}$ be a weakly L^p-stable C_0-semigroup on a Banach space X. Then for all $\beta > \frac{1}{p}$ we have $\omega_\beta(\mathbf{T}) \leqslant 0$.

Further, we give sufficient conditions in terms of $\omega_R^q(A)$ for the existence of L^p-solutions and $W^{1,p}$-solutions $(1 \leqslant p \leqslant \infty)$ of the abstract Cauchy problem for a general class of operators A on X.

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0. Introduction

In this paper, we study the asymptotic behaviour of a C_0-semigroup $\mathbf{T} = \{T(t)\}_{t\geqslant 0}$ on a complex Banach space X. For $x \in X$ and $n \in \mathbb{N}$, the *growth bounds* $\omega(x)$ and $\omega_n(\mathbf{T})$ are defined by

$$\omega(x) := \inf\{\omega \in \mathbb{R} : \|T(t)x\| \leqslant Me^{\omega t} \text{ for some } M = M_x \text{ and all } t \geqslant 0\};$$
$$\omega_n(\mathbf{T}) := \sup\{\omega(x) : \ x \in D(A^n)\}.$$

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Here, A is the generator of $\mathbf{T}$ and $D(A^0)$ is understood to be X. More generally, if the fractional powers of $-A$ are defined, for $\beta \geqslant 0$ we put

$$\omega_\beta(\mathbf{T}) := \sup\{\omega(x) : \ x \in D((-A)^\beta)\}.$$

Of particular interest are the uniform growth bound of $\mathbf{T}$, that is $\omega_0(\mathbf{T})$, and the growth bound $\omega_1(\mathbf{T})$ which characterizes the growth of the classical solutions of the corresponding abstract Cauchy problem. The growth bound $\omega_1(\mathbf{T})$ equals the abscissa of improper convergence of the Laplace transform of $\mathbf{T}$ [Ne]:

$$\omega_1(\mathbf{T}) = \inf\Big\{\omega \in \mathbb{R} : \lim_{t\to\infty} \int_0^t e^{-\omega s} T(s)x \, ds \text{ exists for all } x \in X\Big\}.$$

Similarly, $\omega_0(\mathbf{T})$ is equal to the abscissa of absolute convergence of the Laplace transform of $\mathbf{T}$; this is a special case of the Datko-Pazy theorem [Pa, Thm. 4.4.1].

The *spectral bound* $s(A)$ is defined by

$$s(A) = \sup\{\mathrm{Re}\lambda : \ \lambda \in \sigma(A)\}.$$

We always have $s(A) \leqslant \omega_1(\mathbf{T}) \leqslant \omega_0(\mathbf{T})$ [Na, Cor. A.IV.1.5]; for positive C_0-semigroups on a Banach lattice there is the equality $s(A) = \omega_1(\mathbf{T})$ [Na, Thm. C.IV.1.3]. In general, however, both inequalities may be strict; see e.g. [Na, Ex. A.IV.1.6] or [Ne]. Therefore, in order to be able to say something about the asymptotic behaviour of $\mathbf{T}$, it is not enough to know the location of the spectrum of A. For this reason, several other quantities have been studied.

The *abscissa* $\omega_{L^p}(\mathbf{T})$ $(1 \leqslant p < \infty)$ *of weak* p*-integrability of* $\mathbf{T}$ is defined as

$$\omega_{L^p}(\mathbf{T}) \ := \ \inf\Big\{\omega \in \mathbb{R} : \int_0^\infty e^{-\omega t} |\langle x^*, T(t)x\rangle|^p \, dt < \infty \text{ for all } x \in X \text{ and } x^* \in X^*\Big\}.$$

The *abscissa* $\omega_R^p(A)$ $(1 \leqslant p \leqslant \infty)$ *of* p*-boundedness of the resolvent* $R(\lambda, A) = (\lambda - A)^{-1}$ of A denotes the infimum of all $\omega \in \mathbb{R}$ for which $R(\lambda, A)x \in H^p(\{\mathrm{Re}\lambda > \omega\}, X)$ for all $x \in X$, that is,

$$\omega_R^p(A) \ := \ \inf\Big\{\omega \in \mathbb{R} : \sup\Big\{\int_{-\infty}^\infty \|R(t+is, A)x\|^p \, ds : t > \omega\Big\} < \infty \text{ for all } x \in X\Big\};$$

for $p = \infty$ we take the sup-norm along the lines $t + i\mathbb{R}$. We denote the resulting abscissa of uniform boundedness of $R(\lambda, A)$ by $\omega_R(A)$ rather than $\omega_R^\infty(A)$.

Similarly, the *abscissa* weak-$\omega_R^p(A)$ $(1 \leqslant p \leqslant \infty)$ *of weak* p*-boundedness of the resolvent* denotes the infimum of all $\omega \in \mathbb{R}$ for which $\langle x^*, R(\lambda, A)x\rangle \in H^p(\{\mathrm{Re}\lambda > \omega\})$ for all $x \in X$ and $x^* \in X^*$.

It is well-known that for a C_0-semigroup on a Hilbert space we have $\omega_R(A) = \omega_0(\mathbf{T}) = \omega_{L^p}(\mathbf{T})$ for all $1 \leqslant p < \infty$. The first equality is due to Gearhart [Ge]; an elegant short proof is given in [We]. The second equality is due to Huang and Kangsheng [HK] and, independently, Weiss [We]. In the Banach space case, several results are known about the relations between $\omega_R(A)$ and $\omega_{L^1}(\mathbf{T})$ on the one hand and the growth bounds $\omega_n(\mathbf{T})$ on the other hand:

(i) $\omega_2(\mathbf{T}) \leqslant \omega_R(A)$ [Sl]; see also [Na, Thm. A.IV.1.9];
(ii) $\omega_1(\mathbf{T}) \leqslant \omega_{L^1}(\mathbf{T})$; this follows from [GVW, Prop. 1.1] in combination with [Na, Thm. A.IV.1.4];
(iii) If X is B-convex, in particular if X is uniformly convex, then $\omega_1(\mathbf{T}) \leqslant \omega_R(A)$ [Wr];
(iv) If X is a Banach lattice and $\mathbf{T}$ is positive, then $s(A) = \omega_1(\mathbf{T}) = \omega_R(A)$ [Na, p. 107].

It can happen that $\omega_1(\mathbf{T}) < \omega_R(A)$; an example can be found in [Wr]. Huang [Hu2] claimed that $\omega_1(\mathbf{T}) \leqslant \omega_R(A) = \omega_{L^1}(\mathbf{T})$ holds for *every* C_0-semigroup on a Banach space X. His proofs, however, depend on a lemma on the extension of H^2-functions, to which we present a counterexample below. Therefore it remains an open question whether the relations $\omega_1(\mathbf{T}) \leqslant \omega_R(A)$ and/or $\omega_R(A) = \omega_{L^1}(\mathbf{T})$ hold in general.

In Section 1, we show that the inequalities $\omega_1(\mathbf{T}) \leqslant \omega_{L^p}(\mathbf{T})$ and $\omega_R(A) \leqslant \omega_{L^p}(\mathbf{T})$ hold for each $1 \leqslant p < \infty$.

Section 2 gives some basic definitions concerning fractional powers of unbounded operators.

Section 3 is devoted to some Paley-Wiener type lemmas concerning $\omega_R^p(A)$ and weak-$\omega_R^p(A)$.

The main result in Section 4 is as follows. If the resolvent of a C_0-semigroup $\mathbf{T}$ on X is uniformly bounded in the right half plane, then $\omega_\beta(\mathbf{T}) < 0$ for all $\beta > \frac{1}{p} - \frac{1}{p'}$ ($\frac{1}{p} + \frac{1}{p'} = 1$); here p is the so-called Fourier type of the Banach space X. In particular, we somewhat improve Slemrod's theorem (statement (i) above) by showing that $\omega_{1+\varepsilon}(\mathbf{T}) < 0$ if the resolvent is uniformly bounded in the right half plane. Also, if $\mathbf{T}$ is a C_0-semigroup on L^p, $1 \leqslant p < \infty$, with uniformly bounded resolvent in the right half plane, we obtain that $\omega_\beta(\mathbf{T}) < 0$ for all $\beta > |\frac{1}{p} - \frac{1}{p'}|$, ($\frac{1}{p} + \frac{1}{p'} = 1$). It $\mathbf{T}$ is a *positive* C_0-semigroup on L^p and the spectral bound $s(A)$ of its generator is negative, then the resolvent is automatically bounded in the right half plane. For this case, the third named author has recently proved [W] that $s(A) = \omega_0(\mathbf{T})$.

In Section 5 we study the abscissa weak-$\omega_R^p(A)$. It is shown that $\omega_1(A) \leqslant$ weak-$\omega_R^p(A)$ for all $1 \leqslant p < \infty$. Also, if the resolvent is uniformly bounded in the right half plane, then $\langle x^*, T(\cdot)x\rangle \in L^p(\mathbb{R}_+)$ for all $x \in D(A)$ and $x^* \in X^*$ and all $2 \leqslant p < \infty$. Note that the cases $p = \infty$ in these two results would imply one of the claims of Huang.

In the final Section 6, we extend some of the techniques of Section 3 in order to study the abstract Cauchy problem $u'(t) = Au(t)$ $(t \geqslant 0)$, $u(0) = x$, for a general class of operators A. Extending definitions of types of solutions given by Beals [Be] and Pazy [Pa], we discuss the existence of unique L^p-solutions and $W^{1,p}$-solutions in relation to the abscissae $\omega_R^q(A)$.

We conclude this introduction with a counterexample to Huang's lemma. We need the following terminology. For $w \in \mathbb{R}$ and $1 \leqslant p \leqslant \infty$, let $H^p(w)$ denote the Banach space of analytic functions in the open halfplane $\{\mathrm{Re}\lambda > w\}$ such that

$$\sup_{t>w}\Big(\int_{-\infty}^{\infty} |f(t+is)|^p \, ds\Big) < \infty.$$

We will write H^p for $H^p(0)$. Huang's lemma can be formulated as follows. Suppose $\delta > 0$ and f is an analytic function in the halfplane $\{\mathrm{Re}\lambda > -\delta\}$. If moreover $f \in H^2(w)$ for some $w > 0$ and

$$\lim_{r\to\infty}\Big(\sup_{|\theta|\leqslant\frac{\pi}{2}} |f(re^{i\theta})|\Big) = 0, \tag{0.1}$$

then $f \in H^2$. The counterexample below was constructed by A. Poltoratski and I. Binder.

Example 0.1. For $t \geqslant 0$, let

$$f_0(t) = \begin{cases} e^n, & \text{if } t \in [n, n+e^{-2n}]; \quad n = 1, 2, ...; \\ 0, & \text{otherwise.} \end{cases}$$

Then $e^{(1-\varepsilon)t} f_0(t) \in L^1(\mathbb{R}_+)$ and $e^{-\varepsilon t} f_0(t) \in L^2(\mathbb{R}_+)$ for each $\varepsilon > 0$, but $f_0 \notin L^2(\mathbb{R}_+)$. Let f be the Laplace transform of f_0,

$$f(s) = \int_0^\infty e^{-st} f_0(t) \, dt.$$

Then f exists and is analytic in $\{\mathrm{Re} s > -1\}$, and, by the Paley-Wiener theorem, $f \in H^2(\varepsilon)$ for all $\varepsilon > 0$ but $f \notin H^2$. We claim that (0.1) holds. To this end, let $\eta > 0$. Then we use that

$$\lim_{r\to\infty}\Big(\sup\Big\{|f(re^{i\theta})| : \ |\theta| \leqslant \frac{\pi}{2},\ \mathrm{Re}(re^{i\theta}) > \eta\Big\}\Big) = 0$$

by the general theory of H^2 functions (e.g., [Ho, Ch. 8]); also,

$$\lim_{r\to\infty}\Big(\sup\Big\{|f(re^{i\theta})| : \ |\theta| \leqslant \frac{\pi}{2},\ 0 \leqslant \mathrm{Re}(re^{i\theta}) \leqslant \eta\Big\}\Big) = 0$$

by the Riemann-Lebesgue lemma.

1. The abscissae $\omega_{L^p}(\mathbf{T})$

Let $\mathbf{T}$ be a C_0-semigroup on a Banach space X and let $1 \leqslant p < \infty$. We say that $\mathbf{T}$ is *weakly L^p-stable* if for each $x \in X$ and $x^* \in X^*$ we have $\langle x^*, T(\cdot)x\rangle \in L^p(\mathbb{R}_+)$. We start with the observation that for a weakly L^p-stable semigroup on a Banach space X, there exists a constant M such that

$$\Big(\int_0^\infty |\langle x^*, T(t)x\rangle|^p \, dt\Big)^{\frac{1}{p}} \leqslant M\|x\| \, \|x^*\|, \quad \text{for all } x \in X, x^* \in X^*, \tag{1.1}$$

cf. [We]. In fact, we have the following more general situation. Let X, Y, and Z be Banach spaces and let $S : X \times Y \to Z$ be a separately continuous bilinear map. For $x \in X$ define $S_x : Y \to Z$, $S_x y := S(x, y)$. Then each S_x is bounded by the continuity in the Y-variable. Using the continuity in the X-variable, it is easy to see that the map $x \mapsto S_x$ is closed, and hence bounded by the closed graph theorem. It follows that

$$\|S(x, y)\| \leqslant \|S_x\| \, \|y\| \leqslant K\|x\| \, \|y\|.$$

Now let μ be a positive σ-finite Borel measure on a locally compact Hausdorff space Ω, let E be a Banach function space over (Ω, μ), and let X be a Banach space. Assume that $S : \Omega \to \mathcal{L}(X)$ is a mapping such that for each $x \in X$ and $x^* \in X^*$, the function

$$S_{x,x^*}(\omega) := \langle x^*, S(\omega)x\rangle, \quad \omega \in \Omega,$$

belongs to E. Consider the map $\mathcal{S} : X \times X^* \to E$, defined by $\mathcal{S}(x, x^*) := S_{x,x^*}$. We claim that this bilinear map is separately continuous. Indeed, fix $x^* \in X^*$. We will show that $S_{x^*} : X \to E$, defined by $S_{x^*}x := \mathcal{S}(x, x^*)$, is closed. To this end, let $x_n \to x$ in X and $S_{x^*}x_n \to f$ in E. Since Cauchy sequences in Banach function spaces have pointwise a.e. convergent subsequences [Za], and since $(S_{x^*}x_n)(\omega) \to (S_{x^*}x)(\omega)$ for all ω, it follows that $S_{x^*}x = f$, proving closedness. Therefore, each operator S_{x^*} is bounded. Similarly, each operator $S_x : X^* \to E$, $S_x x^* := \mathcal{S}(x, x^*)$, is bounded. Therefore, $\mathcal{S}$ is separately continuous as claimed.

From the above discussion we see that there is a constant K such that for all $x \in X$ and $x^* \in X^*$,

$$\|\langle x^*, S(\cdot)x\rangle\|_E \leqslant K\|x\| \, \|x^*\|. \tag{1.2}$$

Applied to $E = L^p(\mathbb{R}_+)$, we obtain (1.1).

The following lemma is an easy generalization of [GVW, Prop. 1.1] and [We, Prop. 2.2].

Lemma 1.1. *Let $1 \leqslant p < \infty$ and suppose $\mathbf{T}$ is a weakly L^p-stable C_0-semigroup on X.*

(i) Let $1 < p < \infty$. If $\mathbb{R}_+ = \cup_n E_n$, where $(E_n) \subset \mathbb{R}_+$ is a sequence of disjoint sets each of which is bounded, and if $f \in L^{p'}(\mathbb{R}_+)$ ($\frac{1}{p} + \frac{1}{p'} = 1$), then for all $x \in X$ the improper integral

$$\int_0^\infty f(t)T(t)x\, dt := \lim_{n\to\infty} \int_{\cup_{k=1}^n E_k} f(t)T(t)x\, dt$$

exists and is independent of the choice of (E_n). Moreover, for all $x^* \in X^*$ we have

$$\langle x^*, \int_0^\infty f(t)T(t)x\, dt\rangle = \int_0^\infty f(t)\langle x^*, T(t)x\rangle\, dt,$$

the second integral being in the Lebesgue sense.

(ii) Let $p = 1$. If $f \in L^\infty(\mathbb{R}_+)$ and $\lim_{t\to\infty} |f(t)| = 0$, then for all $x \in X$ the improper integral $\int_0^\infty f(t)T(t)x\, dt := \lim_{\tau\to\infty} \int_0^\tau f(t)T(t)x\, dt$ exists.

(iii) Let $1 \leqslant p < \infty$. For all $x \in X$ and Re$\lambda > 0$ we have $\lambda \in \varrho(A)$ and

$$R(\lambda, A)x = \int_0^\infty e^{-\lambda t} T(t)x\, dt.$$

Proof: (i) For bounded sets $E \subset \mathbb{R}_+$ define $\mu(E) := \int_E f(t)T(t)x\, dt$, the integral being Bochner. For $m \geqslant n$ define $E_{nm} := \cup_{k=n}^m E_k$ and let $\alpha_{nm} := \int_{E_{nm}} |f(t)|^{p'}\, dt$. Then we have $\lim_{n,m\to\infty} \alpha_{nm} = 0$. Therefore, for $x^* \in X^*$ be arbitrary, we have

$$\begin{aligned} |\langle x^*, \sum_{k=n}^m \mu(E_k)\rangle| = |\langle x^*, \mu(E_{nm})\rangle| &= \Big|\int_{E_{nm}} f(t)\langle x^*, T(t)x\rangle\, dt\Big| \\ &\leqslant \Big(\int_{E_{nm}} |\langle x^*, T(t)x\rangle|^p\, dt\Big)^{\frac{1}{p}} \Big(\int_{E_{nm}} |f(t)|^{p'}\, dt\Big)^{\frac{1}{p'}} \leqslant M\|x\|\,\|x^*\| \cdot \alpha_{n,m}^{\frac{1}{p'}}. \end{aligned}$$

Hence,

$$\Big\|\sum_{k=n}^m \mu(E_k)\Big\| \leqslant M\|x\| \cdot \alpha_{nm}^{\frac{1}{p'}}.$$

This proves that the series $\sum_n \mu(E_n)$ is convergent. Denoting the limit by $\mu(\mathbb{R}_+)$, by the dominated convergence theorem we have

$$\langle x^*, \mu(\mathbb{R}_+)\rangle = \sum_n \langle x^*, \mu(E_n)\rangle = \int_0^\infty f(t)\langle x^*, T(t)x\rangle\, dt.$$

This proves the second formula and shows that $\mu(\mathbb{R}_+)$ is independent of the particular choice of the (E_n).

(ii) In this case we have the estimate

$$\Big\|\int_{\tau_0}^{\tau_1} f(t)T(t)x\, dt\Big\| \leqslant \sup_{\|x^*\|\leqslant 1} \Big|\int_{\tau_0}^{\tau_1} f(t)\langle x^*, T(t)x\rangle\, dt\Big| \leqslant M\|x\| \cdot \Big(\sup_{\tau_0\leqslant t\leqslant\tau_1} |f(t)|\Big).$$

(iii) This can be proved either directly or by an analytic continuation argument: for each x, the map $\lambda \mapsto \int_0^\infty e^{-\lambda t}T(t)x\, dt$ is analytic for Re$\lambda > 0$ and coincides with $R(\lambda, A)x$ for Re$\lambda > \omega_0(\mathbf{T})$.
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By arguing exactly as in [We], it follows from Lemma 1.1 (iii) that $\omega_R(A) < 0$. Also, we noted in the introduction that the abscissa of improper convergence of the Laplace transform of $\mathbf{T}$ coincides with $\omega_1(\mathbf{T})$. We thus recover the following result due to Weiss [We]:

Theorem 1.2. *Let $\mathbf{T}$ be a C_0-semigroup on a Banach space X and let $1 \leqslant p < \infty$.*
(i) $\omega_R(A) \leqslant \omega_{L^p}(\mathbf{T})$;
(ii) $\omega_1(\mathbf{T}) \leqslant \omega_{L^p}(\mathbf{T})$.

For certain values of p, part (ii) will be improved later on (Theorem 5.5).

Recall that a map $f : (\Omega, \mu) \to X$ is *Pettis integrable* if $\langle x^*, f(\cdot)\rangle \in L^1(\mu)$ for all $x^* \in X^*$ and for each measurable $H \subset \Omega$ there is an element $x_H \in X$ such that

$$\langle x^*, x_H\rangle = \int_H \langle x^*, f(\omega)\rangle \, d\mu(\omega), \quad \text{for all } x^* \in X^*.$$

If in Lemma 1.1 (i) and (ii) we replace $\mathbb{R}_+$ by an arbitrary measurable subset of $\mathbb{R}_+$, the argument shows that in fact the function $f(t)T(t)x$ is Pettis integrable. We have $\omega_0(\mathbf{T}) < 0$ if and only if the maps $t \mapsto f(t)T(t)x$ are *Bochner* integrable:

Proposition 1.3. *Let $\mathbf{T}$ be weakly L^p-stable for some $1 \leqslant p < \infty$. Then $\omega_0(\mathbf{T}) < 0$ if and only if $f(t)T(t)x$ is Bochner integrable for all $x \in X$ and $f \in L^{p'}(\mathbb{R}_+)$ ($\frac{1}{p} + \frac{1}{p'} = 1$).*

Proof: We only have to prove the 'if' part. Let $x \in X$ be arbitrary and fixed and assume that $f(t)T(t)x$ is Bochner integrable for all $f \in L^{p'}(\mathbb{R}_+)$. We claim that $t \mapsto T(t)x$ is in $L^p(\mathbb{R}_+; X)$. If not, then $\int_0^\infty \|T(t)x\|^p \, dt = \infty$. Define $g_n(t) := \|T(t)x\| \cdot \chi_{[0,n]}(t)$. Then (g_n) is an unbounded sequence in $L^p(\mathbb{R}_+)$. By the uniform boundedness theorem, there is a $\phi \in L^{p'}(\mathbb{R}_+)$ such that the scalar sequence

$$(\langle \phi, g_n\rangle)_n = \Big(\int_0^n \phi(t)\|T(t)x\| \, dt\Big)_n$$

is unbounded. This contradicts the Bochner integrability of $t \mapsto \phi(t)\|T(t)x\|$. The proposition now follows from the Datko-Pazy theorem. ////

We already observed that the growth bound $\omega_1(\mathbf{T})$ coincides with the abscissa of improper convergence of the Laplace transform of $\mathbf{T}$, and that $\omega_0(\mathbf{T})$ coincides with the abscissa of absolute convergence of the Laplace transform of $\mathbf{T}$. Therefore it is of some interest to consider the abscissa of Pettis integrability. For an arbitrary C_0-semigroup $\mathbf{T}$ on a Banach space X and $1 \leqslant p < \infty$ we always have

$$\omega_1(\mathbf{T}) \leqslant \omega_{Pettis}(\mathbf{T}) \leqslant \omega_{L^p}(\mathbf{T}) \leqslant \omega_0(\mathbf{T});$$

the second inequality follows by applying Lemma 1.1 to the function $f(t) = e^{-\lambda t}$, $\lambda > 0$, and the first follows from the countable additivity of the indefinite Pettis integral, cf. [DU, Ch. 2].

Theorem 1.4. *Let $\mathbf{T}$ be a positive C_0-semigroup on a Banach lattice E. Then the following assertions are equivalent:*
(i) $s(A) < 0$;
(ii) $\omega_1(\mathbf{T}) < 0$;
(iii) $\mathbf{T}$ *is weakly L^1-stable;*
(iv) $\mathbf{T}$ *is Pettis integrable;*
(v) $\omega_R(A) < 0$.

Proof: The equivalence of (i), (ii) and (v) was already mentioned in the introduction.

Assume (i). By [Na, Thm. A.IV.1.4 and Thm. C.IV.1.3], for each $x \in X$ there exists a $R(x) \in X$ such that

$$\lim_{\tau\to\infty} \int_0^\tau T(t)x\ dt = R(x).$$

In particular, the integrals on the left hand are uniformly bounded with respect to τ, say by a constant $M(x)$. Then for any $\tau > 0$, $0 \leqslant x \in X$ and $0 \leqslant x^* \in X^*$ we have

$$\int_0^\tau |\langle x^*, T(t)x\rangle|\ dt = \left|\int_0^\tau \langle x^*, T(t)x\rangle\ dt\right| \leqslant \|x^*\|\ \left\|\int_0^\tau T(t)x\ dt\right\| \leqslant \|x^*\| \cdot M(x).$$

Since τ is arbitrary, it follows that $\mathbf{T}$ is weakly L^1-stable.

The implication (iii)$\Rightarrow$(v) follows from Theorem 1.2.

If $\mathbf{T}$ is Pettis integrable, by definition $\mathbf{T}$ is weakly L^1-stable. This gives (iv)$\Rightarrow$(iii).

Finally, suppose $\mathbf{T}$ is weakly L^1-stable. Because of the equivalence (i)$\Leftrightarrow$(iii), also the semigroup defined by $U(t) := e^{\varepsilon t}T(t)$ is weakly L^1-stable for some small $\varepsilon > 0$. Therefore, $t \mapsto T(t)x = e^{-\varepsilon t}U(t)x$ is Pettis integrable by the observation preceding this theorem. ////

If $\mathbf{T}$ is bounded and one of the equivalent hypotheses in the theorem is fulfilled, then obviously $\mathbf{T}$ is weakly L^p-stable for *all* $1 \leqslant p < \infty$.

The following example shows that a weakly L^1-stable semigroup can have strictly positive growth bound $\omega_0(\mathbf{T})$. In particular, $\mathbf{T}$ need not be bounded. This partially answers a question raised in [We]; note the contrast with the Hilbert space case. Also, the example shows that a bounded weakly L^p-stable semigroup can have growth bound zero.

Example 1.5. Let $1 < p < \infty$ and let $X = L^p(\mathbb{R}_+) \cap L^1(\mathbb{R}_+, e^t dt)$. With the norm $\|f\| := \|f\|_{L^p(\mathbb{R}_+)} + \|f\|_{L^1(\mathbb{R}_+, e^t dt)}$, X is a Banach lattice. Define $T(t)f(s) := f(t+s)$. Then $\mathbf{T}$ is a positive C_0-semigroup with $\|T(t)\| = 1$ for all t and $s(A) = -1$; this is proved in [GVW]. Put $S(t) := e^{\frac{1}{2}t}T(t)$. Then $\omega(\mathbf{T}) = 0$, $\omega(\mathbf{S}) = \frac{1}{2}$, and by Theorem 1.4 and the remark after it, $\mathbf{T}$ is weakly L^p-stable for all $1 \leqslant p < \infty$ and $\mathbf{S}$ is weakly L^1-stable.

As in [GVW], this example can be modified to give a counterexample on a reflexive Banach lattice.

2. Fractional powers of closed operators

In this section we state some basic facts and definitions which we will use in the sequel. The two lemmas below are probably not new, but in view of the many different definitions of fractional powers in the literature and the difficulty to find references for the lemmas, we include their proofs.

A closed, densely defined linear operator A on a Banach space X is called *sectorial* if $(0,\infty) \subset \varrho(A)$ and there is a constant K such that

$$||R(\lambda, A)|| \leqslant \frac{K}{(\lambda+1)} \qquad \text{for all } \lambda > 0. \tag{2.1}$$

For a sectorial operator A, fractional powers of $-A$ are defined. If $\beta = 1$ we define $(-A)^{-\beta}$ as usual and for $0 < \beta < 1$ we use the (real) representation of the bounded operator $(-A)^{-\beta}$ given by

$$(-A)^{-\beta}x := \frac{\sin\beta\pi}{\pi} \int_0^\infty r^{-\beta} R(r, A)x\, dr \qquad (x \in X).$$

For any $n \in \mathbb{Z}$, the fractional power $(-A)^{n-\beta}$ is then defined by

$$(-A)^{n-\beta}x \;=\; (-A)^n(-A)^{-\beta}x \quad \text{for} \quad x \in D((-A)^{n-\beta}) = \{x \in X : (-A)^{-\beta}x \in D(A^n)\}.$$

The operators $(-A)^\beta$ ($\beta \in \mathbb{R}$) are closed, injective, and satisfy the semigroup property

$$(-A)^\alpha(-A)^\beta \;=\; (-A)^{\alpha+\beta} \qquad \text{if } \alpha \geqslant 0 \geqslant \beta \text{ or } \alpha, \beta \geqslant 0 \text{ or } \alpha, \beta \leqslant 0.$$

In particular, we have that $(-A)^{-\beta}(-A)^\beta x = x$ for every $\beta \in \mathbb{R}$ and $x \in D((-A)^\beta)$; in other words, $(-A)^{-\beta}$ is the inverse of the operator $(-A)^\beta$. We have inclusions

$$D((-A)^\beta) \subset D((-A)^\alpha) \quad \text{if } \beta \geqslant \alpha.$$

For details, we refer to [Ko].

If the resolvent satisfies (2.1), then there exist constants $d \geqslant 0$, $C > 0$ and $0 < \varphi < \pi$ such that $\|R(\lambda, A)\| \leqslant C(1 + |\lambda|)^{-1}$ for all λ in the sector $\{\lambda \in \mathbb{C} : |\arg \lambda| \leqslant \varphi\} \cup \{\lambda \in \mathbb{C} : |\lambda| \leqslant d\}$; this follows from an easy resolvent expansion argument. Hence, for every $\beta > 0$ and $x \in X$, the integral

$$\frac{1}{2\pi i}\int_\Gamma (-\lambda)^{-\beta} R(\lambda, A)x \, d\lambda, \tag{2.2}$$

where $\mu^{-\beta}$ is defined in terms of the principal branch of the logarithm, exists as a Bochner integral. Here, $\Gamma = \Gamma(\varphi, d) = \Gamma^{(1)}(\varphi, d) \cup \Gamma^{(2)}(\varphi, d) \cup \Gamma^{(3)}(\varphi, d)$ denotes the upwards oriented curve defined by

$$\begin{aligned}
\Gamma^{(1)}(\varphi, d) &= \{\lambda \in \mathbb{C} : |\lambda| \geqslant d,\ \arg\lambda = -\varphi\};\\
\Gamma^{(2)}(\varphi, d) &= \{\lambda \in \mathbb{C} : |\lambda| = d,\ |\arg\lambda| > \varphi\};\\
\Gamma^{(3)}(\varphi, d) &= \{\lambda \in \mathbb{C} : |\lambda| \geqslant d,\ \arg\lambda = \varphi\}.
\end{aligned}$$

Note that we use the argument function with values in $(-\pi, \pi]$. By Cauchy's theorem, the integral (2.2) is equal to the curve integral over $\Gamma(\tilde\varphi, \tilde d)$ for any $0 < \tilde\varphi \leqslant \varphi$ and $0 < \tilde d \leqslant d$. Letting the curve Γ collaps into the real line, we obtain the following.

Lemma 2.1. *For every $\beta > 0$ and $x \in X$, the integral (2.2) equals $(-A)^{-\beta}x$.*

Proof: If $\beta \in \mathbb{N}$, the statement follows by Cauchy's integral formula applied to the right half plane. In the case that $0 < \beta < 1$, consider for $0 < |\theta| \leqslant \varphi$ the curve $\Gamma(\theta) = \{re^{i\theta} : r > 0\}$. Using that

$$\lim_{d \downarrow 0} \int_{\Gamma^{(2)}(\theta, d)} (-\lambda)^{-\beta} R(\lambda, A)x \, d\lambda = 0,$$

it follows from Cauchy's theorem that

$$\begin{aligned}
\frac{1}{2\pi i}\int_\Gamma (-\lambda)^{-\beta} R(\lambda, A)x \, d\lambda &= \frac{1}{2\pi i}\int_{\Gamma(\theta)} (-\lambda)^{-\beta} R(\lambda, A)x \, d\lambda \\
&\qquad - \frac{1}{2\pi i}\int_{\Gamma(-\theta)} (-\lambda)^{-\beta} R(\lambda, A)x \, d\lambda \\
&= \frac{1}{2\pi i}\int_0^\infty (re^{i(\theta-\pi)})^{-\beta} R(re^{i\theta}, A)x \, e^{i\theta} dr \\
&\qquad - \frac{1}{2\pi i}\int_0^\infty (re^{i(-\theta+\pi)})^{-\beta} R(re^{-i\theta}, A)x \, e^{-i\theta} dr \\
&\to \frac{1}{2\pi i}\int_0^\infty r^{-\beta}(e^{i\pi\beta} - e^{-i\pi\beta}) R(r, A)x \, dr \qquad \text{as } \theta \to 0 \\
&= \frac{\sin\beta\pi}{\pi}\int_0^\infty r^{-\beta} R(r, A)x \, dr = (-A)^{-\beta}x.
\end{aligned} \tag{2.3}$$

The convergence is a consequence of Lebesgue's convergence theorem, the integrands in the integrals being dominated by the function $r \mapsto Cr^{-\beta}(1+r)^{-1}$. For $\beta > 1$, $\beta \notin \mathbb{N}$, we use that

$$\frac{R(\lambda, A)x}{(-\lambda)^{[\beta]}} \;=\; R(\lambda, A)(-A)^{-[\beta]}x \;+\; \sum_{k=0}^{[\beta]-1} \frac{(-A)^{-k-1}x}{(-\lambda)^{[\beta]-k}},$$

where $[\beta]$ denotes the unique integer such that $[\beta] \leqslant \beta < [\beta]+1$. The integrals in

$$\frac{1}{2\pi i}\int_\Gamma (-\lambda)^{-\beta} R(\lambda, A)x\, d\lambda$$
$$= \frac{1}{2\pi i}\int_\Gamma (-\lambda)^{-\beta+[\beta]} R(\lambda, A)(-A)^{-[\beta]}x\, d\lambda \;+\; \sum_{k=0}^{[\beta]-1} \frac{1}{2\pi i}\int_\Gamma (-\lambda)^{-\beta+k}\, d\lambda\, (-A)^{-k-1}x$$

exist as Bochner integrals. Cauchy's theorem applied to the left half plane yields that

$$\int_\Gamma (-\lambda)^{-\beta+k}\, d\lambda \;=\; 0 \qquad \text{for every } 0 \leqslant k \leqslant [\beta]-1.$$

Since $0 < \beta - [\beta] < 1$, it follows by (2.3) that

$$\frac{1}{2\pi i}\int_\Gamma (-\lambda)^{-\beta} R(\lambda, A)x\, d\lambda \;=\; (-A)^{-[\beta]}(-A)^{-\beta+[\beta]}x \;=\; (-A)^{-\beta}x.$$

////

Under certain conditions, the complex representation (2.2) of $(-A)^{-\beta}x$ can be changed to a curve integral over a line.

Lemma 2.2. *Let $a < 0$ and let A be a sectorial operator. Assume that the resolvent of A is uniformly bounded in $\{\lambda \in \mathbb{C} : Re\lambda \geqslant a\}$. Then for all $\beta > 0$ and $x \in D(A)$ we have*

$$(-A)^{-\beta}x \;=\; \frac{1}{2\pi i}\int_{\mathrm{Re}\lambda = a} (-\lambda)^{-\beta} R(\lambda, A)x\, d\lambda.$$

Proof: For all $\lambda \in \{\mathrm{Re}\lambda \geqslant a\}$, $\lambda \neq 0$, we have

$$\|R(\lambda, A)x\| = \left\|\frac{1}{\lambda}[R(\lambda, A)(\lambda - A)x + R(\lambda, A)Ax]\right\| \leqslant \frac{1}{|\lambda|}\big(||x|| + M||Ax||\big), \tag{2.4}$$

where $M = \sup\{\|R(\lambda, A)\| : \mathrm{Re}\lambda \geqslant a\}$. Hence the lemma follows from Lemma 2.1 and Cauchy's theorem. ////

3. The abscissae $\omega_R^p(A)$ and Paley-Wiener type lemmas for semigroups

Let A be a closed linear operator on a Banach space X, let $1 \leqslant p \leqslant \infty$ and let $w \in \mathbb{R}$. We say that the resolvent of A *belongs to $H^p(w)$* if $\{\mathrm{Re}\lambda > w\} \subset \varrho(A)$ and for each $x \in X$ we have

$$\sup\Big\{\int_{-\infty}^{\infty} \|R(t+is, A)x\|^p\; ds :\; t > w\Big\} < \infty;$$

in case $p = \infty$, as usual, we take the sup-norm along the lines $t + i\mathbb{R}$. It is not hard to see, cf. the argument at the beginning of Section 1, that for each $t > w$ there exists a constant M_t such that

$$\Big(\int_{-\infty}^{\infty} \|R(t+is, A)x\|^p \ ds\Big)^{\frac{1}{p}} \leqslant M_t\|x\|, \quad \text{for all } x \in X. \tag{3.1}$$

Moreover, for all $t > w$,

$$B_t := \sup\{\|R(\lambda, A)\| : \ \mathrm{Re}\lambda \geqslant t\} < \infty. \tag{3.2}$$

In fact, for functions $f \in H^p(w, X)$ and $t > w$ we have the estimate

$$\sup_{s\in\mathbb{R}} \|f(t+is)\| \leqslant \frac{1}{(\pi(t-w'))^{\frac{1}{p}}}\|f\|_{H^p(w',X)}, \qquad w < w' < t \tag{3.3}$$

cf. [HP, Thm. 6.4.2]. Applying this to $f(\lambda) = R(\lambda, A)x$, by the uniform boundedness theorem we obtain (3.2). We define $\omega_R^p(A)$ to be the abscissa

$$\omega_R^p(A) = \inf\{w \in \mathbb{R} : \ A \text{ has } H^p(w)\text{-resolvent}\}.$$

If the resolvent fails to be in $H^p(w)$ for all $w \in \mathbb{R}$, we put $\omega_R^p(A) = \infty$. As before, we write $\omega_R(A)$ instead of $\omega_R^\infty(A)$ for the abscissa of uniform boundedness of the resolvent of A. By (3.2), $\omega_R(A) \leqslant \omega_R^p(A)$ for all $1 \leqslant p \leqslant \infty$.

We say that the resolvent of A *belongs to* $H^p(w)$ *weakly* if for all $x \in X$ and $x^* \in X^*$ the function $\lambda \mapsto \langle x^*, R(\lambda, A)x\rangle$ belongs to $H^p(w)$. We denote by weak-$\omega_R^p(A)$ the corresponding abscissa, i.e. the infimum of all w such that $R(\lambda, A)$ belongs to $H^p(w)$ weakly.

If $R(\lambda, A)$ belongs to $H^p(w)$ weakly and $t > w$, there is a constant M_t such that

$$\Big(\int_{-\infty}^{\infty} |\langle x^*, R(t+is, A)x\rangle|^p \ ds\Big)^{\frac{1}{p}} \leqslant M_t\|x\| \ \|x^*\|, \quad \text{for all } x \in X, x^* \in X^*. \tag{3.4}$$

This follows from the observations at the beginning of Section 1. Moreover, for all $t > w$ we have

$$B_t := \sup\{\|R(\lambda, A)\| : \ \mathrm{Re}\lambda \geqslant t\} < \infty.$$

In particular, $\omega_R(A) = \text{weak-}\omega_R(A) \leqslant \text{weak-}\omega_R^p(A)$ for all $1 \leqslant p \leqslant \infty$.

Our next aim is to prove that the functions $\omega_R^p(A)$ and weak-$\omega_R^p(A)$ are non-increasing in p. Let $L_0^\infty(\mathbb{R}, X)$ denote the closure in $L^\infty(\mathbb{R}, X)$ of the set of all X-valued step functions of the form $\sum_{n=1}^N x_n\chi_{F_n}$, where $x_n \in X$ and $F_n \subset \mathbb{R}$ is measurable and has *finite* Lebesgue measure. Clearly, $C_0(\mathbb{R}, X) \subset L_0^\infty(\mathbb{R}, X)$. Also, it is well-known [HP] that the restriction of an $f \in H^p(w, X)$ to each line $t + i\mathbb{R}$ belongs to $L_0^\infty(\mathbb{R}, X)$ $(1 \leqslant p < \infty$, $t > w)$.

The basic fact we will use is that for all $1 \leqslant p < \infty$ and $0 < \theta < 1$, the complex interpolation method gives

$$\big(L^p(\mathbb{R}, X), L_0^\infty(\mathbb{R}, X)\big)_{[\theta]} = L^q(\mathbb{R}, X), \tag{3.5}$$

where $1/q = (1-\theta)/p$ [BL, Thm. 5.1.2].

Proposition 3.1. *The functions $\omega_R^p(A)$ and weak-$\omega_R^p(A)$ are non-increasing in p. Moreover, for all $1 \leqslant p \leqslant \infty$ we have either $\omega_R^p(A) = \infty$ or $\omega_R^p(A) = \omega_R(A)$.*

Proof: Fix $1 \leqslant p \leqslant q \leqslant \infty$. We claim that $H^p(w,X) \subset H^q(w+\varepsilon, X)$ for all $\varepsilon > 0$.

By (3.2), $H^p(w,X) \subset H^\infty(w+\varepsilon,X)$ for all $\varepsilon > 0$, so we may assume that $p < q < \infty$. Let $\delta \geqslant \varepsilon > 0$ be arbitrary and let $0 < \theta < 1$ be such that $1/q = (1-\theta)/p$. Define the linear operators $T_\delta^{(p)} : H^p(w,X) \to L^p(\mathbb{R}, X)$ and $T_\delta^{(\infty)} : H^p(w,X) \to L_0^\infty(\mathbb{R}, X)$ by $f \mapsto f|_{w+\delta+i\mathbb{R}}$. Then $\|T_\delta^{(p)}\| \leqslant 1$ and $\|T_\delta^{(\infty)}\| \leqslant (\pi\delta)^{-\frac{1}{p}}$ by (3.3). By complex interpolation, it follows that

$$\|f|_{w+\delta+i\mathbb{R}}\|_{L^q(\mathbb{R},X)} \leqslant (\pi\delta)^{-\frac{\theta}{p}} \|f\|_{H^p(\mathbb{R},X)} \leqslant (\pi\varepsilon)^{-\frac{q-p}{pq}} \|f\|_{H^p(\mathbb{R},X)}.$$

This proves that $H^p(w,X) \subset H^q(w+\varepsilon; X)$.

Applying this to the functions $\lambda \mapsto R(\lambda, A)x$ shows that $\omega_R^q(A) \leqslant \omega_R^p(A)$.

So far, we have proved that $p \to \omega_R^p(A)$ is non-increasing. Now suppose that for some fixed p we have $\omega_R^p(A) < \infty$. By (3.2), we see that $\omega_R(A) \leqslant \omega_R^p(A)$; we will prove that the converse inequality also holds. Let $b > \omega_R^p(A)$, let $\omega_R(A) \leqslant t_0 \leqslant b$ and let $t_0 \leqslant t \leqslant b$ be arbitrary and fixed. Let B_{t_0} be defined by (3.2). By the resolvent equation, for all $s \in \mathbb{R}$ we have

$$\|R(t+is,A)\| = \|[I + (b-t)R(t+is,A)]R(b+is,A)\| \leqslant (1 + B_{t_0}(b-t))\|R(b+is,A)\|.$$

Consequently,

$$\left(\int_{-\infty}^{\infty} ||R(t+is,A)x||^p \, ds\right)^{\frac{1}{p}} \leqslant (1 + B_{t_0}(b-t)) \left(\int_{-\infty}^{\infty} ||R(b+is,A)x||^p \, ds\right)^{\frac{1}{p}}.$$

Since $t_0 \leqslant t \leqslant b$ is arbitrary and the resolvent is in $H^p(b)$, this shows that the resolvent is in $H^p(t_0)$.

Finally, we have to prove the inequality weak-$\omega_R^q(A) \leqslant$ weak-$\omega_R^p(A)$. This is done similarly as in the strong case, using the spaces $H^p(w)$ and the functions $\lambda \mapsto \langle x^*, R(\lambda, A)x\rangle$. ////

Recall that a Banach space X has *Fourier type* p for some $1 \leqslant p \leqslant 2$, if the Fourier transform $\mathcal{F}$ extends to a bounded operator

$$\mathcal{F} : L^p(\mathbb{R}, X) \to L^{p'}(\mathbb{R}, X),$$

$\frac{1}{p} + \frac{1}{p'} = 1$, between the Lebesgue-Bochner spaces $L^p(\mathbb{R}, X)$ and $L^{p'}(\mathbb{R}, X)$. In other words, it is assumed that the vector-valued Hausdorff-Young theorem holds for the exponent p.

Every Banach space has Fourier type 1 but only a Hilbert space has Fourier type 2 [Kw]. The Banach space $L^r(\Omega, \mu)$ has Fourier type $\min\{r, r'\}$ [Pe]. Every B-convex (in particular every uniformly convex) space has a Fourier type for some p with $1 < p \leqslant 2$ [Bo]. Recall that a complex Banach space is *B-convex* if it does not contain the spaces l_n^1 uniformly, or equivalently, if it is of type p for some $1 < p \leqslant 2$. For more information we refer to [Pi].

From (3.5) and the Riemann-Lebesgue Lemma one sees that if a Banach space has Fourier type p for some $1 \leqslant p \leqslant 2$, then it has Fourier type q for all $1 \leqslant q \leqslant p$.

Now let A be a sectorial operator on X with $\omega_R(A) < 0$, and fix $\omega_R(A) < a < 0$ and $x \in D(A)$. By (2.4), for all $\beta > 0$ the integral

$$(Sx)(t) := \frac{1}{2\pi i} \int_{\mathrm{Re}\lambda = a} e^{\lambda t}(-\lambda)^{-\beta} R(\lambda, A)x \, d\lambda, \tag{3.6}$$

where $\mu^{-\beta}$ is defined in terms of the principle branch of the logarithm, exists as a Bochner integral. By Cauchy's theorem, the integral does not depend on the value of a. Moreover, (2.4) shows that there is a constant C_a, which only depends on A and a, such that

$$\|(Sx)(t)\| \leqslant C_a e^{at} \|x\|_{D(A)}. \tag{3.7}$$

We also define

$$(S_a x)(t) := e^{-at}(Sx)(t). \tag{3.8}$$

Lemma 3.2. *Assume X has Fourier type p for some $1 \leqslant p \leqslant 2$ and assume A is a sectorial operator with $\omega_R^q(A) < 0$ for some $p \leqslant q \leqslant \infty$. Then $\omega_R(A) < 0$, and for all $\beta > \frac{1}{p} - \frac{1}{q}$ and $a > \omega_R^q(A)$, formula (3.8) extends to a bounded linear operator $S_a : X \to L^{p'}(\mathbb{R}_+, X)$.*

Proof: Fix $x \in D(A)$. Since $\omega_R(A) \leqslant \omega_R^q(A) < 0$, the definition of $(Sx)(t)$ makes sense. Without loss of generality, we may assume $\omega_R^q(A) < a < 0$. Then

$$(Sx)(t) = \frac{e^{at}}{2\pi} \int_{-\infty}^{\infty} e^{-ist} g(s)\, ds = \frac{e^{at}}{2\pi} \mathcal{F}(g)(t)$$

where $g(s) = (-a+is)^{-\beta} R(a-is, A)x$. Since $\frac{1}{r} := \frac{1}{p} - \frac{1}{q} < \beta$, we have by (3.1),

$$\Big(\int_{-\infty}^{\infty} ||g(s)||^p ds\Big)^{\frac{1}{p}} \leqslant \Big(\int_{-\infty}^{\infty} |a-is|^{-\beta r} ds\Big)^{\frac{1}{r}} \Big(\int_{-\infty}^{\infty} ||R(a-is, A)x||^q ds\Big)^{\frac{1}{q}} \leqslant C||x||.$$

Since X has Fourier type p, for all $x \in D(A)$ we get

$$||e^{-a(\cdot)}(Sx)(\cdot)||_{L^{p'}(\mathbb{R}_+, X)} \leqslant C_1 ||x||.$$

Therefore, S_a extends to a bounded linear operator $S_a : X \to L^{p'}(\mathbb{R}_+, X)$. ////

It is easy to see that a generator whose resolvent is uniformly bounded in the open right half plane (or equivalently, $\omega_R(A) < 0$; cf. Lemma 4.1) is sectorial. In that case, the integral (3.6) can be expressed in terms of $\mathbf{T}$:

Lemma 3.3. *Let A be the generator of a C_0-semigroup $\mathbf{T}$ with $\omega_R(A) < 0$. Then for all $\beta > 0$, $\omega_R(A) < a < 0$ and $x \in D(A)$ we have $(Sx)(t) = T(t)(-A)^{-\beta}x$ for all $t \geqslant 0$.*

Proof: First let $x \in D(A^2)$. By Cauchy's theorem,

$$\frac{1}{2\pi i} \int_{\mathrm{Re}\lambda = a} e^{\lambda t} (-\lambda)^{-\beta-1}\, d\lambda = 0. \tag{3.9}$$

Using this and the equation $\frac{1}{\lambda} AR(\lambda, A)x = R(\lambda, A)x - \frac{x}{\lambda}$, we see that

$$(Sx)(t) = \frac{1}{2\pi i} \int_{\mathrm{Re}\lambda = a} e^{\lambda t} (-\lambda)^{-\beta-1} R(\lambda, A)(-Ax)\; d\lambda.$$

Since $-Ax \in D(A)$, we may differentiate under the integral sign and obtain

$$\begin{aligned} \frac{d}{dt}(Sx)(t) &= \frac{1}{2\pi i} \int_{\mathrm{Re}\lambda = a} \lambda e^{\lambda t} (-\lambda)^{-\beta-1} R(\lambda, A)(-Ax)\, d\lambda \\ &= A\Big(\frac{1}{2\pi i} \int_{\mathrm{Re}\lambda = a} e^{\lambda t} (-\lambda)^{-\beta-1} R(\lambda, A)(-Ax)\, d\lambda\Big) = A((Sx)(t)). \end{aligned}$$

To obtain the second identity, we used $\lambda = (\lambda - A) + A$, (3.9), and the closedness of A. Also, $(Sx)(0) = (-A)^{-\beta}x$ by Lemma 2.2. We have shown that $(Sx)(\cdot)$ is the solution to the abstract Cauchy problem

$$\begin{aligned} \frac{d}{dt}u(t) &= Au(t) \\ u(0) &= (-A)^{-\beta}x. \end{aligned}$$

By the uniqueness of classical solutions, we must have $(Sx)(t) = T(t)(-A)^{-\beta}x$.

Next, let $x \in D(A)$ and choose a sequence $(x_n) \subset D(A^2)$ such that $x_n \to x$ in the graph norm of $D(A)$. Let $t \geqslant 0$ be fixed. By (3.7), $(Sx_n)(t) \to (Sx)(t)$ in X. The boundedness of $(-A)^{-\beta}$ implies that also $(Sx_n)(t) = T(t)(-A)^{-\beta}x_n \to T(t)(-A)^{-\beta}x$, and the lemma follows. ////

The above two lemmas will now be used to derive two Paley-Wiener type results, giving L^p-stability of certain orbits of a semigroup in terms of H^r-abscissae of its Laplace transform, i.e. of the resolvent.

Lemma 3.4. *Suppose X has Fourier type p for some $1 \leqslant p \leqslant 2$ and let $p \leqslant q \leqslant \infty$. Furthermore, assume A is the generator of a C_0-semigroup $\mathbf{T}$ satisfying $\omega_R^q(A) < 0$. Then for all $\beta > \frac{1}{p} - \frac{1}{q}$, $a > \omega_R^q(A)$, and $x \in D((-A)^\beta)$, the map $t \mapsto e^{-at}T(t)x$ belongs to $L^{p'}(\mathbb{R}_+, X)$.*

Proof: Without loss of generality, assume $\omega_R^q(A) < a < 0$.

Let $y \in D((-A)^\beta)$ be arbitrary and put $x := (-A)^\beta y$. Choose a sequence $x_n \to x$ with $x_n \in D(A)$ for each n. By Lemma 3.2, $S_a x_n \to S_a x$ in $L^{p'}(\mathbb{R}_+, X)$. By taking a subsequence if necessary, we may assume that $(S_a x_n)(t) \to (S_a x)(t)$ for almost all $t \geqslant 0$. For any such t, by Lemma 3.3 we have

$$e^{-at}T(t)(-A)^{-\beta}x = \lim_{n\to\infty} e^{-at}T(t)(-A)^{-\beta}x_n = \lim_{n\to\infty}(S_a x_n)(t) = (S_a x)(t).$$

Therefore, $e^{-at}T(\cdot)y$ is equal a.e. to the function $(S_a x)(\cdot) \in L^{p'}(\mathbb{R}_+, X)$. ////

We will now give the 'weak' analogue of Lemma 3.4.

Lemma 3.5. *Let $1 \leqslant p \leqslant 2$ and $p \leqslant q \leqslant \infty$. Furthermore, let A be the generator of a C_0-semigroup $\mathbf{T}$ satisfying weak-$\omega_R^q(A) < 0$. Then for all $\beta > \frac{1}{p} - \frac{1}{q}$, $a >$ weak-$\omega_R^q(A)$, $x \in D((-A)^\beta)$ and $x^* \in X^*$, the map $t \mapsto e^{-at}\langle x^*, T(t)x\rangle$ belongs to $L^{p'}(\mathbb{R}_+)$.*

Proof: Without loss of generality, assume weak-$\omega_R^q(A) < a < 0$.

Using (3.4) and the Hausdorff-Young theorem, the argument of Lemma 3.2 can be modified to show that

$$\|e^{-a(\cdot)}\langle x^*, (Sx)(\cdot)\rangle\|_{L^{p'}(\mathbb{R}_+)} \leqslant C\|x\|\,\|x^*\|, \quad \forall x \in D(A), x^* \in X^*. \tag{3.10}$$

Now fix $x^* \in X^*$. Define $S_{a,x^*} : D(A) \to L^{p'}(\mathbb{R}_+)$ by $(S_{a,x^*}x)(t) := \langle x^*, (S_a x)(t)\rangle$. It follows from (3.10) that S_{a,x^*} extends to a bounded operator $S_{a,x^*} : X \to L^{p'}(\mathbb{R}_+)$.

Let $y \in D((-A)^\beta)$ and $x^* \in X^*$ be arbitrary and put $x := (-A)^\beta y$. Choose a sequence $x_n \to x$ with $x_n \in D(A)$ for each n. Then $S_{a,x^*}x_n \to S_{a,x^*}x$ in $L^{p'}(\mathbb{R}_+)$. By taking a subsequence if necessary, we may assume that $(S_{a,x^*}x_n)(t) \to (S_{a,x^*}x)(t)$ for almost all $t \geqslant 0$. For any such t, we have

$$\begin{aligned} e^{-at}\langle x^*, T(t)(-A)^{-\beta}x\rangle &= \lim_{n\to\infty} e^{-at}\langle x^*, T(t)(-A)^{-\beta}x_n\rangle = \lim_{n\to\infty}\langle x^*, (S_a x_n)(t)\rangle \\ &= \lim_{n\to\infty}(S_{a,x^*}x_n)(t) = (S_{a,x^*}x)(t). \end{aligned}$$

Therefore, the function $e^{-a(\cdot)}\langle x^*, T(\cdot)y\rangle$ is equal a.e. to the function $S_{a,x^*}x \in L^{p'}(\mathbb{R}_+)$. ////

4. The abscissae $\omega_R^p(A)$ and stability

In this section, we will apply Lemma 3.4 to obtain stability results for C_0-semigroups. The first lemma, along with its proof, is taken from [Hu1].

Lemma 4.1. *Let A be a closed operator on a Banach space X. If the resolvent of A is uniformly bounded in the open right half plane, then $\omega_R(A) < 0$.*

Proof: Let $\|R(\lambda, A)\| \leqslant M$ for all $\mathrm{Re}\lambda > 0$. Put $\sigma := (2M)^{-1}$. Then for all $\nu \in \mathbb{R}$ and $\mu \leqslant 0 < \lambda$ such that $\lambda - \mu < \sigma$, the series

$$R_{\mu+i\nu} := \sum_{n=0}^{\infty} (\lambda - \mu)^n R(\lambda + i\nu, A)^{n+1} \tag{4.1}$$

converges absolutely. Therefore the function $z \mapsto R_z$ is a analytic operator valued function for $\mathrm{Re}z > -\sigma$, and it coincides with $R(z, A)$ whenever $z \in \varrho(A)$. We claim that actually $R_z = R(z, A)$ for all $\mathrm{Re}z > -\sigma$. Suppose the contrary. Then there is a $z \in \partial\sigma(A) \cap \{\mathrm{Re}z > -\sigma\}$. But if we let $z_n \to z$ inside $\varrho(A) \cap \{\mathrm{Re}z > -\sigma\}$, we see that $\|R(z_n, A)\|$, hence also $\|R_{z_n}\|$, tends to ∞, a contradicition to the analyticity of $z \mapsto R_z$. Finally, the uniform boundedness of the resolvent on the half plane $\mathrm{Re}\lambda > -\sigma$ follows immediately from (4.1). ////

The next lemma gives a sufficient condition in order that $\omega_R^q(A) < 0$ for certain values of q.

Lemma 4.2. *Suppose X has Fourier type p for some $1 \leqslant p \leqslant 2$, and suppose A is the generator of a C_0-semigroup $\mathbf{T}$ on X. If the resolvent of A is uniformly bounded in the open right half plane, then $\omega_R^{p'}(A) = \omega_R(A) < 0$ ($\frac{1}{p} + \frac{1}{p'} = 1$).*

Proof: First we note that by Lemma 4.1, $\omega_R(A) < 0$. Choose $b \geqslant 0$ so that $||T(t)|| \leqslant Ce^{\frac{b}{2}t}$ for some $C < \infty$ and all $t > 0$. Fix $t \geqslant b$. Since X has Fourier type p and

$$R(t + is, A)x \;=\; \int_0^\infty e^{-is\xi} \left(e^{-t\xi}T(\xi)x\right) d\xi,$$

we get

$$\begin{aligned}\left(\int_{-\infty}^{\infty} ||R(t+is, A)x||^{p'} \, ds\right)^{\frac{1}{p'}} &\leqslant ||\mathcal{F}||_{L^p(\mathbb{R},X)\to L^{p'}(\mathbb{R},X)} \left(\int_0^\infty ||e^{-t\xi}T(\xi)x||^p \, d\xi\right)^{\frac{1}{p}} \\ &\leqslant ||\mathcal{F}||_{L^p(\mathbb{R},X)\to L^{p'}(\mathbb{R},X)} \left(\int_0^\infty e^{-\frac{1}{2}bp\xi} \, d\xi\right)^{\frac{1}{p}} \|x\|.\end{aligned} \tag{4.2}$$

This shows that $\omega_R^{p'}(A) \leqslant b$, and therefore $\omega_R^{p'}(A) = \omega_R(A)$ by Proposition 3.1. ////

Before proceeding with the main result of this section, let us observe the following simple consequence of this lemma and Proposition 3.1.

Corollary 4.3. *If A generates a C_0-semigroup on a Banach space with Fourier type p for some $1 \leqslant p \leqslant 2$, then for all $p' \leqslant q \leqslant \infty$ one has $\omega_R^{p'}(A) =$ weak$-\omega_R^{p'}(A) = \omega_R^q(A) =$ weak$-\omega_R^q(A) = \omega_R(A)$.*

If $\mathbf{T}$ is a C_0-semigroup with generator A on a Banach space X whose resolvent $R(\lambda, A)$ is uniformly bounded in the open right half plane, then by the Hille-Yosida theorem A is a densely defined sectorial operator. Consequently, the fractional powers of $-A$ are defined. Recall from the introduction the notation $\omega_\beta(\mathbf{T}) = \sup\{\omega(x) : x \in D((-A)^\beta)\}$.

Theorem 4.4. *Assume that X has Fourier type p for some $1 \leqslant p \leqslant 2$, and that A generates a C_0-semigroup $\mathbf{T}$ whose resolvent is uniformly bounded in the right half plane. Then for all $\beta > \frac{1}{p} - \frac{1}{p'}$ one has $\omega_\beta(\mathbf{T}) < 0$.*

Proof: By Lemma 4.1, $\omega_R(A) < 0$. Lemma 4.2 yields that $\omega_R^{p'}(A) = \omega_R(A) < 0$. Now fix an arbitrary $x \in D((-A)^\beta)$. Applying Lemma 3.4 with $q = p'$, for $\beta > \frac{1}{p} - \frac{1}{p'}$ and $a > \omega_R(A)$ we get that $e^{-at}T(\cdot)x$ is equal a.e. to $(S_a((-A)^\beta x))(\cdot) \in L^{p'}(\mathbb{R}_+, X)$. By a standard argument (cf. [Pa, Thm. 4.4.1]), this implies that $e^{-a(\cdot)}T(\cdot)x$ is bounded. Hence, $\omega(x) \leqslant a$. ////

Corollary 4.5. *Let A be the generator of a C_0-semigroup on a Banach space X. If the resolvent of A is uniformly bounded in the right half plane, then $\omega_{1+\varepsilon}(\mathbf{T}) < 0$ for every $\varepsilon > 0$.*

Indeed, this follows from Lemma 4.1, Theorem 4.4 and the fact that every Banach space has Fourier type 1. In fact, the corollary is an immediate consequence of (3.7) and Lemmas 3.3 and 4.1. This result can be generalized considerably, cf. [NS].

As we observed after Lemma 1.1, a weakly L^p-stable semigroup has uniformly bounded resolvent in a right half plane $\{\mathrm{Re}\,\lambda > -\varepsilon\}$. Combining this with Corollary 4.5 it follows that $\omega_{1+\varepsilon}(\mathbf{T}) < 0$ for all $\varepsilon > 0$ if $\mathbf{T}$ is a weakly L^p-stable C_0-semigroup for some $1 \leqslant p < \infty$.

Theorem 4.4 implies a refinement of a stability result of Wrobel [Wr]. He proved that $\omega_1(\mathbf{T}) < 0$ for C_0-semigroups on B-convex Banach spaces whose generators A have uniformly bounded resolvents in the right half plane.

Theorem 4.6. *Let A be the generator of a C_0-semigroups on a B-convex Banach space X. If the resolvent of A is uniformly bounded in the right half plane, then there exists an $\varepsilon > 0$ such that $\omega_{1-\varepsilon}(\mathbf{T}) < 0$.*

This follows from the fact that X has Fourier type p for some $1 < p \leqslant 2$ and Theorem 4.4.

Theorem 4.7. *Let A be the generator of a positive C_0-semigroup on a space L^p, $1 \leqslant p < \infty$. Then $s(A) = \omega_R^q(A)$ for all $\max\{p, p'\} \leqslant q \leqslant \infty$. If $s(A) < 0$, then for all $\beta > |\frac{1}{p} - \frac{1}{p'}|$ we have $\omega_\beta(\mathbf{T}) < 0$.*

This follows from the facts that L^p is of Fourier type $\min\{p, p'\}$ and that for positive semigroups one knows that $s(A) = \omega_R(A)$. Of course, Theorem 4.7 is true for non-positive semigroups as well, provided we replace $s(A)$ by $\omega_R(A)$.

5. Stability of weakly L^p-stable semigroups

In this section we will apply Lemma 3.5 to obtain stability results for C_0-semigroups.

Theorem 5.1. *Let $\mathbf{T}$ be a C_0-semigroup on a Banach space X.*

(i) If weak-$\omega_R^q(A) < 0$ for some $1 \leqslant q \leqslant \infty$, then for all $\beta > 1 - \frac{1}{q}$ we have $\omega_\beta(\mathbf{T}) \leqslant$ weak-$\omega_R^q(A) < 0$. In particular, $\omega_1(\mathbf{T}) \leqslant$ weak-$\omega_R^q(A)$ for all $1 \leqslant q < \infty$.

(ii) If the resolvent is uniformly bounded in the open right half plane and if $2 \leqslant q < \infty$, then for all $\beta > 1 - \frac{1}{q}$, $x \in D((-A)^\beta)$ and $x^* \in X^*$, the map $\langle x^*, T(\cdot)x\rangle$ belongs to $L^q(\mathbb{R}_+)$. In particular, if $x \in D(A)$, then for all $x^* \in X^*$ we have $\langle x^*, T(\cdot)x\rangle \in L^q(\mathbb{R}_+)$ for all $2 \leqslant q < \infty$.

Proof: (i) For all weak-$\omega_R^q(A) < a < 0$, Lemma 3.5 (with $p = 1$) shows that the function $\langle x^*, e^{-a(\cdot)}T(\cdot)x\rangle$ is bounded for all $x \in D((-A)^\beta)$ and $x^* \in X^*$. Therefore, $\omega_\beta(A) \leqslant a$ by the uniform boundedness theorem.

(ii) Apply Lemma 3.5 (with $q = \infty$). ////

We will now prepare some lemmas for a somewhat less trivial application of Lemma 3.5.

Lemma 5.2. *Let $1 \leqslant p \leqslant 2$ and let $\mathbf{T}$ be a weakly L^p-stable C_0-semigroup on X. Then the resolvent belongs to $H^{p'}$ weakly.*

Proof: Let $t > 0$ and $s \in \mathbb{R}$ be arbitrary. Let $x \in X$ and $x^* \in X^*$ be arbitrary. By Lemma 1.1 (iii) we have

$$\langle x^*, R(t+is, A)x\rangle = \int_0^\infty e^{-is\xi}\Big(e^{-t\xi}\langle x^*, T(\xi)x\rangle\Big)\, d\xi = (\mathcal{F}g)(s), \tag{5.1}$$

where $\mathcal{F}$ is the Fourier transform and

$$g(\xi) = \chi_{\mathbb{R}_+}(\xi)e^{-t\xi}\langle x^*, T(\xi)x\rangle.$$

Note that a priori we obtain the integral in (5.1) as an improper integral, but the conditions on $\mathbf{T}$ and t ensure that the integral is actually a Lebesgue integral.

Let $K = K_{x,x^*} := \|\langle x^*, T(\cdot)x\rangle\|_{L^p(\mathbb{R}_+)}$. By the Hausdorff-Young theorem, there is a constant C_p such that

$$\|\langle x^*, R(t+i(\cdot), A)x\rangle\|_{L^{p'}(\mathbb{R})} \leqslant C_p\|g(\cdot)\|_{L^{p'}(\mathbb{R})} \leqslant C_pK.$$

Since $t > 0$ is arbitrary, the lemma follows. ////

Lemma 5.3. [Ko, Thm. 6.4] *Let A be a sectorial operator. For all $\omega \geqslant 0$ and $\beta \in \mathbb{R}$ we have $D((-A)^\beta) = D((-(A-\omega))^\beta)$.*

For our purposes, it is in fact sufficient to know that for all $\varepsilon > 0$, $D((-A)^{\beta+\varepsilon}) \subset D((-(A-\omega))^\beta)$.

Lemma 5.4. *Let $1 \leqslant p \leqslant 2$, and let A be the generator of a weakly L^q-stable C_0-semigroup $\mathbf{T}$ for some $1 \leqslant q \leqslant 2$. Then for all $\beta > \frac{1}{p} - \frac{1}{q'}$, $(\frac{1}{q} + \frac{1}{q'} = 1)$, $x \in D((-A)^\beta)$, $x^* \in X^*$, and $\varepsilon > 0$, the map $t \mapsto e^{-\varepsilon t}\langle x^*, T(t)x\rangle$ belongs to $L^{p'}(\mathbb{R}_+)$.*

Proof: Let $x \in D((-A)^\beta)$. By Lemma 5.2, the resolvent $R(\lambda, A)$ belongs to $H^{q'}$ weakly, so $R(\lambda, A-2\varepsilon)$ belongs to $H^{q'}(-2\varepsilon)$ weakly. By Lemma 5.3, $x \in D((-(A-2\varepsilon))^\beta$. Therefore we can apply Lemma 3.5 to the semigroup $T_{2\varepsilon}(t) := e^{-2\varepsilon t}T(t)$ and $a = -\varepsilon$ and find that $t \mapsto e^{-\varepsilon t}\langle x^*, T(t)x\rangle$ belongs to $L^{p'}(\mathbb{R}_+)$. ////

Now we are in a position to prove the main result of this section.

Theorem 5.5. *Let $\mathbf{T}$ be a C_0-semigroup which is weakly L^q-stable for some $1 \leqslant q \leqslant 2$. Then for all $\beta > \frac{1}{q}$ we have $\omega_\beta(\mathbf{T}) \leqslant 0$.*

Proof: An application of Lemma 5.4, with $p = 1$, shows that $t \mapsto e^{-\varepsilon t}\langle x^*, T(t)x\rangle$ is bounded for all $x \in D((-A)^\beta)$, $x^* \in X^*$, and $\varepsilon > 0$. The theorem follows from the uniform boundedness theorem. ////

6. Existence of L^p-solutions

In this section we apply the ideas of Section 3 to the abstract Cauchy problem. Let A be a sectorial operator on a Banach space X of Fourier type p for some $1 \leqslant p \leqslant 2$. Assume that there is a q with $p \leqslant q \leqslant \infty$ such that $\omega_R^q(A) < 0$. Let $\beta > \frac{1}{p} - \frac{1}{q}$ and $\omega_R^q(A) < a < 0$, and consider the map S_a of Lemma 3.2.

Lemma 6.1. *For all $x \in X$*

$$\int_0^t e^{as}(S_a x)(s)\,ds \;\in\; D(A) \qquad \text{for almost all } t > 0, \tag{6.1}$$

and

$$e^{at}(S_a x)(t) \;=\; A\left(\int_0^t e^{as}(S_a x)(s)\,ds\right) \;+\; (-A)^{-\beta}x \qquad \text{for almost all } t > 0. \tag{6.2}$$

Proof: Fix $x \in D(A)$. We have

$$y_t := \int_0^t e^{as}(S_a x)(s)\,ds = \int_0^t (Sx)(s)\,ds = \frac{1}{2\pi i}\int_{\mathrm{Re}\lambda=a} \frac{1}{\lambda}(e^{\lambda t}-1)\,(-\lambda)^{-\beta}R(\lambda,A)x\,d\lambda.$$

Since $\frac{1}{\lambda}AR(\lambda,A)x = R(\lambda,A)x - \frac{x}{\lambda}$, the integrals in

$$\begin{aligned} z_t &:= \frac{1}{2\pi i}\int_{\mathrm{Re}\lambda=a} \frac{1}{\lambda}(e^{\lambda t}-1)\,(-\lambda)^{-\beta}AR(\lambda,A)x\,d\lambda \\ &= \frac{1}{2\pi i}\int_{\mathrm{Re}\lambda=a} e^{\lambda t}(-\lambda)^{-\beta}R(\lambda,A)x\,d\lambda \;-\; \frac{1}{2\pi i}\int_{\mathrm{Re}\lambda=a} (-\lambda)^{-\beta}R(\lambda,A)x\,d\lambda \\ &\qquad\qquad +\; \frac{1}{2\pi i}\int_{\mathrm{Re}\lambda=a} (e^{\lambda t}-1)(-\lambda)^{-\beta-1}x\,d\lambda \end{aligned} \tag{6.3}$$

exist as Bochner integrals. The closedness of A implies that $y_t \in D(A)$ and $Ay_t = z_t$.
The last integral in (6.3) is zero by Cauchy's theorem. Then (6.3) and Lemma 2.2 yield that

$$\begin{aligned} (Sx)(t) &= A\left(\int_0^t (Sx)(s)\,ds\right) + \frac{1}{2\pi i}\int_{\mathrm{Re}\lambda=a} (-\lambda)^{-\beta}R(\lambda,A)x\,d\lambda \\ &= A\left(\int_0^t e^{as}(S_a x)(s)\,ds\right) + (-A)^{-\beta}x. \end{aligned}$$

This proves the lemma in case $x \in D(A)$.

For an arbitrary $x \in X$, choose $x_n \in D(A)$ with $x_n \to x$ in X. Using Lemma 3.2, we see that that $(S_a x_n) \to (S_a x)$ in the norm of $L^{p'}(\mathbb{R}_+, X)$ and therefore, for all $t > 0$,

$$\int_0^t e^{as}(S_a x_n)(s)\,ds \to \int_0^t e^{as}(S_a x)(s)\,ds.$$

By taking a subsequence if necessary, we may assume that $(S_a x_n)(t) \to (S_a x)(t)$ almost everywhere. Hence, for almost all t,

$$e^{at}(S_a x_n)(t) - (-A)^{-\beta}x_n \to e^{at}(S_a x)(t) - (-A)^{-\beta}x.$$

Since (6.1) and (6.2) hold for all x_n and A is closed, we see that (6.1) and (6.2) hold for x. ////

Consider the abstract Cauchy problem

$$(ACP) \qquad u'(t) = Au(t) \quad (t \geqslant 0), \qquad u(0) = x.$$

(a) A function $u : \mathbb{R}_+ \to X$ is called a L^p_w*-solution* of (ACP) if $e^{-w(\cdot)}u(\cdot) \in L^p(\mathbb{R}_+, X)$ and

$$u(t) = A\left(\int_0^t u(s)\,ds\right) + x \qquad \text{for almost all } t > 0 \text{ and } u(0) = x.$$

We say that u is a L^p*-solution* if it is a L^p_w-solution for some $w \in \mathbb{R}$.

(b) The function $u : \mathbb{R}_+ \to X$ is called a $W^{1,p}_w$*-solution* of (ACP) if $e^{-w(\cdot)}u(\cdot) \in W^{1,p}(\mathbb{R}_+, X)$ and

$$u'(t) = Au(t) \quad \text{for almost all } t > 0 \text{ and } u(0) = x.$$

By a $W^{1,p}$*-solution*, we mean a $W^{1,p}_w$-solution for some $w \in \mathbb{R}$.

Definition (a) says essentially that a L^p-solution is a mild solution, but a L^p-solution is not necessarily continuous. By part (b), it follows in particular that a $W^{1,p}$-solution is a strong solution in the sense of Pazy, see [Pa, p. 109]. The name L^p-solution is motivated by the following notion of Beals [Be].

(c) $u : \mathbb{R}_+ \to X$ is a L^2*-solution in the sense of Beals* if $\|u(\cdot)\|$ is locally L^2, and for each $y^* \in D(A^*)$ the function $\langle y^*, u(\cdot)\rangle$ is equal a.e. to a locally absolutely continuous function $u_{y^*}(t)$ with $u_{y^*}(0) = \langle y^*, x\rangle$, $\frac{d}{dt}u_{y^*}(t) = \langle A^*y^*, u_{y^*}(t)\rangle$.

Proposition 6.2. *For each $q \geqslant 2$, a L^q-solution u of (ACP) is a L^2-solution in the sense of Beals.*

Proof: Let u be a L^q-solution of (ACP). Clearly, $\|u(\cdot)\|$ is locally L^2. Let $y^* \in D(A^*)$ and define

$$u_{y^*}(t) := \langle A^*y^*, \int_0^t u(s)\,ds\rangle + \langle y^*, x\rangle.$$

Then $u_{y^*}(0) = \langle y^*, x\rangle$, $u_{y^*}(\cdot)$ is locally absolutely continuous with $\frac{d}{dt}u_{y^*}(t) = \langle A^*y^*, u(t)\rangle$, and $u_{y^*}(t) = \langle y^*, u(t)\rangle$ a.e. ////

Theorem 6.3. *Let A be a sectorial operator on a Banach space X of Fourier type p for some $1 \leqslant p \leqslant 2$. Let $p \leqslant q \leqslant \infty$ and assume that $\omega^q_R(A) < \infty$. Then for all $w > \omega^q_R(A)$ and $\beta > \frac{1}{p} - \frac{1}{q}$ the following assertions hold.*
(a) (ACP) has a $L^{p'}_w$-solution for every $x \in D((-A)^\beta)$.
(b) (ACP) has a $W^{1,p'}_w$-solution for every $x \in D((-A)^{\beta+1})$.

Proof: (a) Fix $x \in D((-A)^\beta)$. By scaling A, we may assume that $\omega^q_R(A) < w < 0$; after doing so, Lemma 5.3 guarantees that still $x \in D((-A)^\beta)$. Put $y = (-A)^\beta x$ and $u(t) = e^{wt}(S_w y)(t)$ where S_w is the map of Lemma 3.2. Then $e^{-w(\cdot)}u(\cdot) \in L^{p'}(\mathbb{R}_+, X)$ and

$$u(t) \;=\; e^{wt}(S_w y)(t) \;=\; A\left(\int_0^t e^{ws}(S_w y)(s)\,ds\right) + (-A)^{-\beta}y \;=\; A\left(\int_0^t u(s)\,ds\right) + x \qquad \text{a.e.}$$

(b) Fix $x \in D((-A)^{\beta+1})$. After rescaling A as in (i) we have $x \in D((-A)^{\beta+1})$. Put $y = -(-A)^{\beta+1}x$ and $u(t) = \int_0^t e^{ws}(S_w y)(s)\,ds + x$. Then $e^{-wt}\frac{d}{dt}u(t) = (S_w y)(t)$ belongs to $L^{p'}(\mathbb{R}_+, X)$. It follows easily from (3.6) that $A^{-1}S_w = S_w A^{-1}$. Also, since $x = A^{-1}(-A)^{-\beta}y$, it follows from (6.2) that $e^{-wt}u(t) = A^{-1}(S_w y)(t) = (S_w(A^{-1}y))(t)$ belongs to $L^{p'}(\mathbb{R}_+, X)$. It follows that also $\frac{d}{dt}\big(e^{-wt}u(t)\big)$ belongs to $L^{p'}(\mathbb{R}_+, X)$. Furthermore, $u(0) = x$ and, since $(-A)^{-\beta}y = Ax$,

$$\begin{aligned}\frac{d}{dt}u(t) \;&=\; e^{-wt}(S_w y)(t) \;=\; A\left(\int_0^t e^{-ws}(S_w y)(s)\,ds\right) + (-A)^{-\beta}y \\ &=\; A\left(\int_0^t e^{-ws}(S_w y)(s)\,ds + x\right) \;=\; Au(t) \qquad \text{a.e.}\end{aligned}$$

////

If X has Fourier type p, then $\omega_R^{p'}(A) \leqslant \omega_0(\mathbf{T})$, where $\frac{1}{p} + \frac{1}{p'} = 1$. Therefore, the condition $\omega_R^q(A) < \infty$ is automatically satisfied for all $p' \leqslant q \leqslant \infty$.

Beals [Be] proves the following. If A is a densely defined closed operator on a Hilbert space H, such that the resolvent is uniformly bounded in some right half plane, then for each $x \in D(A)$, the problem (ACP) has a unique L^2-solution in the sense of Beals. If in addition we assume that A is sectorial we can improve this as follows:

Corollary 6.4. *Let A be a sectorial operator on a B-convex Banach space X. Then there exists an $\varepsilon > 0$ such that for each $x \in D((-A)^{1-\varepsilon})$, the problem (ACP) has a unique L^2-solution in the sense of Beals.*

Proof: As we observed at the beginning of this section, X is of Fourier type p for some $1 < p \leqslant 2$. Therefore the existence assertions follow by combining the previous two results. Uniqueness of L^2-solutions is proved in [Be]. ////

Corollary 6.5. *Let A be a sectorial operator on a Hilbert space H. Then for each $\beta > \frac{1}{2}$ and $x \in D((-A)^{\beta})$, the problem (ACP) has a unique L^2-solution in the sense of Beals. Moreover, if $\omega_R^q < 0$ for some $q \geqslant 2$, then this solution is globally L^2.*

Proof: We only need to prove the globality assertion, which follows from Theorem 6.3. ////

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7. References

[Be] R. Beals, On the abstract Cauchy problem, *J. Funct. Anal.* 10 (1972), 281-299.

[BL] J. Bergh and J. Löfström, *Interpolation Spaces, An Introduction*, Grundlehren 223, Springer-Verlag, Berlin-Heidelberg-New York (1976).

[Bo] J. Bourgain, Vector-valued Hausdorff-inequalities and applications, *Springer Lect. Notes in Math.* 1317 (1986/87), 239-249.

[DU] J. Diestel and J.J. Uhl, *Vector Measures*, Math. Surveys nr. 15, Amer. Math. Soc., Providence, R.I. (1977).

[Ge] L. Gearhart, Spectral theory for contraction semigroups on Hilbert spaces, *Trans. Am. Math. Soc.* 236 (1978), 385-394.

[GVW] G. Greiner, J. Voigt, and M. Wolff, On the spectral bound of the generator of semigroups of positive operators, *J. Operator Th.* 5 (1981), 245-256.

[HP] E. Hille and R.S. Phillips, *Functional Analysis and Semi-groups*, Amer. Math. Soc. Colloq. Publ., vol. 31, Amer. Math. Soc., rev. ed., Providence, R.I. (1957).

[Ho] K. Hoffman, *Banach Spaces of Analytic Functions*, Englewood Cliffs, New York (1962).

[Hu1] F. Huang, Characteristic conditions for exponential stability of linear dynamical systems in Hilbert spaces, *Ann. Diff. Eq.* 1 (1985), 43-56.

[Hu2] F. Huang, Exponential stability of linear systems in Banach spaces, *Chin. Ann. of Math.* 10B (1989), 332-340.

[HK] F. Huang and L. Kangsheng, A problem of exponential stability for linear dynamical systems in Hilbert spaces, *Kexue Tongbao* 33 (1988), 460-462.

[Ko] H. Komatsu, Fractional powers of operators, *Pacific J. Math.* 19 (1966), 285-346.

[Kw] S. Kwapien, Isomorphic characterizations of inner product spaces by orthogonal series with vector valued coefficients, *Studia Math.* 44 (1972), 583-595.

[Na] R. Nagel (ed.), *One-parameter Semigroups of Positive Operators*, Springer Lect. Notes in Math. 1184 (1986).

[NS] J.M.A.M. van Neerven and B. Straub, On the existence and growth of classical solutions of the abstract Cauchy problem for a generator of an α-times integrated semigroup, submitted.

[Ne] F. Neubrander, Laplace transform and asymptotic behavior of strongly continuous semigroups, *Houston Math. J.* 12 (1986), 549-561.

[Pa] A. Pazy, *Semigroups of Linear Operators and Applications to Partial Differential Equations*, Springer-Verlag, Berlin-Heidelberg-New York-Tokyo (1983).

[Pe] J. Peetre, Sur la transformation de Fourier des fonctions à valeurs vectorielles, *Rend. Sem. Mat. Univ. Padova* 42 (1969), 15-26.

[Pi] G. Pisier, *Factorisation of Linear Operators and Geometry of Banach Spaces*, CBMS, Regional Conference Series, No. 60, AMS, Providence (1986).

[Sl] M. Slemrod, Asymptotic behavior of C_0-semigroups as determined by the spectrum of the generator, *Indiana Univ. Math. J.* 25 (1976), 783-792.

[W] L. Weis, Stability of positive semigroups on $L_p(\mu)$, to appear in: Proc. Am. Math. Soc.

[We] G. Weiss, Weak L^p-stability of a linear semigroup on a Hilbert space implies exponential stability, *J. Diff. Eq.* 76 (1988), 269-285.

[Wr] V. Wrobel, Asymptotic behavior of C_0-semigroups in B-convex spaces, *Indiana Univ. Math. J.* 38 (1989), 101-114.

[Za] A.C. Zaanen, *Integration*, 2nd ed., North Holland (1967).