LOWER SEMICONTINUITY AND THE THEOREM OF DATKO AND PAZY

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Abstract. Let $\mathbf{T}$ be a C_0-semigroup on a real or complex Banach space X and let $J : C^+[0,\infty) \to [0,\infty]$ be a lower semicontinuous and nondecreasing functional on $C^+[0,\infty)$, the positive cone of $C[0,\infty)$, satisfying $J(c\mathbf{1}) = \infty$ for all $c > 0$. We prove the following result: if $\mathbf{T}$ is not uniformly exponentially stable, then the set

$$\{x \in X : \ J(\|T(\cdot)x\|) = \infty\}$$

is residual in X.

A C_0-semigroup $\mathbf{T} = \{T(t)\}_{t\geqslant 0}$ on a (real or complex) Banach space X is said to be *uniformly exponentially stable* if there exist constants $M \geqslant 1$ and $\omega > 0$ such that

$$\|T(t)\| \leqslant Me^{-\omega t}, \qquad t \geqslant 0.$$

A well-known result of Datko and Pazy [6] states that $\mathbf{T}$ is uniformly exponentially stable if there exists $p \in [1,\infty)$ such that

$$\int_0^\infty \|T(t)x\|^p\,dt < \infty, \qquad x \in X.$$

This result was generalized by Zabczyk [8], who showed that a C_0-semigroup on X is uniformly exponentially stable if there exists a convex nondecreasing function $\phi : [0,\infty) \to [0,\infty)$ with $\phi(0) = 0$ and $\phi(t) > 0$ for all $t > 0$ such that

$$\int_0^\infty \phi(\|T(t)x\|)\,dt < \infty, \qquad x \in X.$$

Zabczyk's result was improved and generalized to evolution families by Rolewicz [7, Theorem 1]. In the semigroup case Rolewicz's result reads as follows: if a C_0-semigroup $\mathbf{T}$ on X fails to be uniformly exponentially stable, then for every nondecreasing continuous function $\phi : [0,\infty) \to [0,\infty)$ with $\phi(0) = 0$ and $\phi(t) > 0$ for all $t > 0$ there exists a dense subset $D \subseteq X$ such that

$$\int_0^\infty \phi(\|T(t)x\|)\,dt = \infty, \qquad x \in D;$$

it is implicit in the proof of [7, Theorem 2] that D is in fact residual.

In [5] it is shown that $\mathbf{T}$ is uniformly exponentially stable if there exists a Banach function space E over $[0,\infty)$ with the property that

$$\lim_{t\to\infty} \|\mathbf{1}_{[0,t]}\|_E = \infty, \tag{1.1}$$

such that

$$\|T(\cdot)x\| \in E, \qquad x \in X.$$

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The Datko-Pazy theorem follows from this by taking $E = L^p[0,\infty)$. As is shown in [5], Rolewicz's version of the Datko-Pazy theorem can be derived as well by taking for E a suitable Orlicz space over $[0,\infty)$. This is a somewhat artificial construction, however. In this note we propose a more natural generalization of these results.

The proof of our main result, Theorem 4 below, is based upon results by Müller about the orbits of a single operator T. For the convenience of the reader, we recall these results first.

Proposition 1 ([3, Lemma 1])**.** *Let E be a finite-dimensional subspace of a Banach space X and let $\varepsilon > 0$. Then there exists a closed subspace $F \subseteq X$ of finite codimension such that*

$$\|e+f\| \geqslant (1-\varepsilon)\max\left\{\|e\|, \tfrac{1}{2}\|f\|\right\}, \qquad e \in E,\ f \in F.$$

Proposition 2 ([4, Lemma 2.2])**.** *Let T be a bounded linear operator on a Banach space X with $r(T) = r_{\rm ess}(T) = 1$. Then there is a constant $c > 0$ with the following property: for every $n \in \mathbb{N}$ and every subspace $Y \subseteq X$ of finite codimesion there exists $y \in Y$ with $\|y\| = 1$ such that*

$$\|T^j y\| \geqslant c, \qquad j = 0,\dots,n.$$

Here $r_{\rm ess}(T)$ denotes the essential spectral radius of T. In [4] this result is stated real spaces only, but the proof also works for complex spaces.

Lemma 3. *Let T be a bounded linear operator on a Banach space X, and assume that its spectral radius satisfies $r(T) \geqslant 1$. Then for all $x \in X$ and $\delta > 0$ there exists a constant $C > 0$ with the following property: for all $n \in \mathbb{N}$ there exists $y \in X$ such that $\|x-y\| < \delta$ and $\|T^j y\| \geqslant C$ for all $j = 0,\dots,n$.*

Proof. Without loss of generality we may assume that $r(T) = 1$.

If $r_{\rm ess}(T) < 1$, then T has an eigenvalue λ with $|\lambda| = 1$, and we may proceed as in part A of the proof of [4, Theorem 2.3].

Suppose next that $r_{\rm ess}(T) = 1$. Let c be the constant from Proposition 2. Fix $n \in \mathbb{N}$ and let E denote the finite–dimensional linear subspace of X spanned by the set $\{T^j x:\ j = 0,\dots,n\}$. By Proposition 1, there exists a closed subspace F of X of finite codimension such that

$$\|e+f\| \geqslant \tfrac{1}{2}\max\left\{\|e\|, \tfrac{1}{2}\|f\|\right\}, \qquad e \in E,\ f \in F.$$

Let $F' = \{f \in F : T^j f \in F,\ j = 0,\dots n\}$. The assumption $r_{\rm ess}(T) = 1$ implies that X is infinite-dimensional, and therefore F' is a nontrivial closed subspace of X of finite codimension. By Proposition 2 there exists a vector $f \in F'$ with $\|f\| = 1$ and $\|T^j f\| \geqslant c$, $j = 0,\dots,n$. Let $y := x + \frac{1}{2}\delta f$. Then $\|x-y\| < \delta$ and

$$\|T^j y\| = \|T^j x + \tfrac{1}{2}\delta T^j f\| \geqslant \tfrac{1}{8}\delta\|T^j f\| \geqslant \tfrac{1}{8}c\,\delta, \qquad j = 0,\dots,n. \qquad \blacksquare$$

If T is a Hilbert space operator and if there is a $\lambda \in \sigma(T)$ with $|\lambda| = r(T) \geqslant 1$ which is not an eigenvalue, in the lemma we may take any constant $0 < C < \delta$; this result is due to Beauzamy [1, Theorem 2.A.1].

We denote by $C[0,\infty)$ the space of all continuous functions on $[0,\infty)$. With the topology of uniform convergence on compact sets, this is a separable Fréchet space. By $C^+[0,\infty)$ we denote the positive cone of $C[0,\infty)$.

Recall that a subset of a topological space is called *residual* if its complement is of the first category.

Theorem 4. *Let $J : C^+[0,\infty) \to [0,\infty]$ be a map with the following properties:*

(1) *J is lower semicontinuous;*
(2) *J is nondecreasing, i.e. $0 \leqslant f \leqslant g$ implies $J(f) \leqslant J(g)$;*
(3) *$J(c\mathbf{1}) = \infty$ for all $c > 0$.*

Let $\mathbf{T}$ be a C_0−semigroup on a Banach space X which is not uniformly exponentially stable. Then the set

$$\big\{x \in X : \; J\big(\|T(\cdot)x\|\big) = \infty\big\}$$

is residual.

Proof. For $k = 1, 2, \dots$ let

$$X_k = \big\{x \in X : \; J\big(\|T(\cdot)x\|\big) > k\big\}.$$

The lower semicontinuity of J implies that each X_k is open. It suffices to prove that each X_k is dense.

Fix $k \geqslant 1$ and let $B(x, \delta)$ be an open ball with centre $x \in X$ and radius $\delta > 0$. We will show that $X_k \cap B(x, \delta) \neq \varnothing$.

Since $\mathbf{T}$ is not uniformly exponentially stable we have $r(T(1)) \geqslant 1$. By Lemma 3 there exists a constant $C > 0$ with the following property: for each $n = 0, 1, \dots$ there exists an $y_n \in X$ with $\|x - y_n\| < \delta$ and $\|T(j)y_n\| \geqslant C$ for all $j = 0, \dots, n$. Then,

$$\|T(t)y_n\| \geqslant \frac{C}{M}, \qquad t \in [0, n],$$

where $M := \sup_{0 \leqslant s \leqslant 1} \|T(s)\|$.

Let $(f_n)_{n \geqslant 0} \subseteq C^+[0,\infty)$ be a sequence with

$$0 \leqslant f_n \leqslant \frac{C}{M}\, \mathbf{1}_{[0,n]}, \qquad n = 0, 1, \dots$$

and

$$\lim_{n\to\infty} f_n = \frac{C}{M}\,\mathbf{1} \quad \text{uniformly on compact sets.}$$

Then,

$$\|T(t)y_n\| \geqslant \frac{C}{M}\, \mathbf{1}_{[0,n]}(t) \geqslant f_n(t), \qquad t \in [0,\infty), \; n = 0, 1, \dots.$$

By the monotonicity and lower semicontinuity of J we obtain

$$\liminf_{n\to\infty} J(\|T(\cdot)y_n\|) \geqslant \liminf_{n\to\infty} J(f_n) \geqslant J\big(\frac{C}{M}\,\mathbf{1}\big) = \infty.$$

In particular, there exists an index n_0 such that $J(\|T(\cdot)y_{n_0}\|) > k$. Therefore, $y_{n_0} \in X_k \cap B(x, \delta)$, showing that the intersection is nonempty. ∎

The semigroup case of Rolewicz's theorem follows from Theorem 4 by taking

$$J(f) = \int_0^\infty \phi(f(t))\, dt, \qquad f \in C^+[0,\infty).$$

This functional satisfies the three assumptions of Theorem 4; lower semicontinuity follows from Fatou's lemma. In fact, if $\mathbf{T}$ is not uniformly exponentially stable, we obtain the somewhat stronger result that the set

$$\left\{x \in X : \; \int_0^\infty \phi(\|T(t)x\|)\, dt = \infty\right\}$$

is residual.

The result from [5] mentioned above involving Banach function spaces satisfying (1.1) also follows from Theorem 4: take

$$J(f) := \lim_{t\to\infty} \|\mathbf{1}_{[0,t]}f\|_E = \sup_{t\geqslant 0} \|\mathbf{1}_{[0,t]}f\|_E.$$

To see that J is lower semicontinuous we argue as follows. For each $t \geqslant 0$, the map $J_t(f) := \|\mathbf{1}_{[0,t]}f\|_E$ is continuous. Indeed, if $f_n \to f$ uniformly on compact sets, then $\mathbf{1}_{[0,t]}f_n \to \mathbf{1}_{[0,t]}f$ uniformly. Therefore, given $\varepsilon > 0$, for n large enough we have

$$|\mathbf{1}_{[0,t]}f_n - \mathbf{1}_{[0,t]}f| \leqslant \varepsilon \mathbf{1}_{[0,t]}$$

in E, and therefore by the triangle inequality,

$$|J(f_n) - J(f)| \leqslant \|\mathbf{1}_{[0,t]}f_n - \mathbf{1}_{[0,t]}f\|_E \leqslant \varepsilon \|\mathbf{1}_{[0,t]}\|_E.$$

Being the supremum of a family of continuous maps, J is lower semicontinuous. Thus, if $\mathbf{T}$ is not uniformly exponentially stable, then

$$\{x \in X : \ \|T(\cdot)x\| \notin E\} \ \text{is residual.} \tag{1.2}$$

The norm of a Banach function space E is said to have the *Fatou property* if the following holds: if f is a measurable function and $(f_n)_{n\geqslant 0}$ is a sequence in E such that $\sup_{n\geqslant 0} \|f_n\|_E < \infty$ and $0 \leqslant f_n \uparrow f$, then $f \in E$ and $\lim_{n\to\infty} \|f_n\|_E = \|f\|_E$. For Banach function spaces E whose norm has the Fatou property, in particular for $E = L^p[0,\infty)$ with $1 \leqslant p < \infty$, the result contained in (1.2) can be proved in a more elementary way as follows.

Suppose E is a Banach function space satisfying (1.1) whose norm has the Fatou property, and assume that the set of all $x \in X$ with $\|T(\cdot)x\| \in E$ is of the second category. We will show that $\mathbf{T}$ is uniformly exponentially stable.

For $k = 1, 2, \ldots$ define

$$X_k = \big\{x \in X : \ \big\| \|T(\cdot)x\| \big\|_E \leqslant k\big\}.$$

In order to prove that X_k is closed, suppose that $x_n \to x$ in X with $x_n \in X_k$ for all $n \geqslant 0$. Defining $f_n := \|T(\cdot)x_n\| \in E$ and $f := \|T(\cdot)x\|$, we have $\mathbf{1}_{[0,j]}f \in E$ and $\mathbf{1}_{[0,j]}f_n \to \mathbf{1}_{[0,j]}f$ uniformly, and hence in E, as $n \to \infty$. It follows that

$$\|\mathbf{1}_{[0,j]}f\|_E = \lim_{n\to\infty} \|\mathbf{1}_{[0,j]}f_n\|_E \leqslant \limsup_{n\to\infty} \|f_n\|_E \leqslant k.$$

By the Fatou property, it follows that $f \in E$ and

$$\|f\|_E = \lim_{j\to\infty} \|\mathbf{1}_{[0,j]}f\|_E \leqslant k.$$

Therefore $x \in X_k$ and X_k is closed.

Since by assumption $\bigcup_{k\geqslant 1} X_k$ is of the second category, at least one X_{k_0} has nonempty interior. Let $B(x_0, \delta_0)$ be an open ball with centre x_0 and radius δ_0 contained in X_{k_0}. Then by the triangle inequality in E, the open ball $B(0, \delta_0)$ is contained in X_{2k_0}. But then for all nonzero $x \in X$ and $0 < \delta < \delta_0$,

$$\big\| \|T(\cdot)x\| \big\|_E = \frac{\|x\|}{\delta} \cdot \big\| \|T(\cdot)(\delta x/\|x\|)\| \big\|_E \leqslant \frac{\|x\|}{\delta} \cdot 2k_0 < \infty.$$

This shows that $\|T(\cdot)x\| \in E$ for all $x \in E$, and we may apply the result from [5] (or the Datko-Pazy theorem if $E = L^p[0,\infty)$) to conclude that E is uniformly exponentially stable.

Remark 5. L. Weis has kindly pointed out that for $E = L^p[0,\infty)$ this, and related residuality results, have been obtained by V. Wrobel (preprint).

A C_0-semigroup $\mathbf{T}$ on X is said to be *strongly stable* if

$$\lim_{t\to\infty} \|T(t)x\| = 0, \qquad x \in X.$$

Every uniformly exponentially stable semigroup is strongly stable, but the converse is not true: a simple counterexample is the semigroup of left translations on $C_0[0,\infty)$.

As a consequence, a function J satisfying the three assumptions of Theorem 4 cannot be finitely valued on the subset $C_0^+[0,\infty)$ of all positive functions vanishing at infinity. Indeed, the existence of such J would imply that every strongly stable C_0-semigroup is uniformly exponentially stable. In fact we have the following simple observation. First note that conditions 2 and 3 of Theorem 4 imply that $J(f) = \infty$ whenever $f \geqslant c\mathbf{1}$ for some $c > 0$.

Proposition 6. *Let $J : C^+[0,\infty) \to [0,\infty]$ be lower semicontinuous, and assume that $J(f) = \infty$ for all $f \in C^+[0,\infty)$ for which there exists a constant $c > 0$ such that $f \geqslant c\mathbf{1}$. Then the set*

$$\big\{f \in C_0^+[0,\infty) : \ J(f) = \infty\big\}$$

is residual in $C_0^+[0,\infty)$, endowed with the topology of uniform convergence.

Proof. Suppose, for a contradiction, that the set $F := \big\{f \in C_0^+[0,\infty) : \ J(f) < \infty\big\}$ is of the second category in $C_0^+[0,\infty)$. For $k = 1, 2, \dots$ let

$$F_k := \big\{f \in C_0^+[0,\infty) : \ J(f) \leqslant k\big\}.$$

As a subset of $C_0^+[0,\infty)$, each F_k is closed. Indeed, if $f_n \to f$ uniformly with $f_n \in F_k$ for all n, then $f_n \to f$ uniformly on compact sets, and the lower semicontinuity of J gives $J(f) \leqslant \liminf_{n\to\infty} J(f_n) \leqslant k$. Since by assumption $\bigcup_{k\geqslant 1} F_k$ is of the second category in $C_0^+[0,\infty)$, there is an F_{k_0} with nonempty interior relative to $C_0^+[0,\infty)$.

Let $B(f_0,\delta_0)$ be an open ball in $C_0^+[0,\infty)$ contained in F_{k_0}, and fix $0 < \delta < \delta_0$ arbitrary. Choose a sequence $(g_n)_{n\geqslant 0}$ in $C_0^+[0,\infty)$ such that $0 \leqslant g_n \leqslant \delta\mathbf{1}$ and $g_n \to \delta\mathbf{1}$ uniformly on compact sets. We have $f_0 + g_n \in B(f_0,\delta_0)$ for each n, and $\lim_{n\to\infty}(f_0+g_n) = f_0+\delta\mathbf{1}$ uniformly on compact sets. By the lower semicontinuity of J,

$$J\big(f_0 + \delta\mathbf{1}\big) \leqslant \liminf_{n\to\infty} J(f_0 + g_n) \leqslant k_0,$$

a contradiction. ∎

We do not know whether Theorem 4 remains true if the conditions 2 and 3 are replaced by the condition

2′. $J(f) = \infty$ for all $f \in C^+[0,\infty)$ with $f \geqslant c\mathbf{1}$ for some $c > 0$.

We are going to check next that none of the three conditions in Theorem 4 can be omitted.

Example 7. Define

$$J(f) := \begin{cases} 0, & f \in C_0^+[0,\infty), \\ \infty, & \text{otherwise}, \end{cases} \qquad f \in C^+[0,\infty).$$

Then J is nondecreasing, $J(c\mathbf{1}) = \infty$ for all $c > 0$, but J is not lower semicontinuous. If $\mathbf{T}$ is a C_0-semigroup which is strongly stable, then $J(\|T(\cdot)x\|) = 0$ for all $x \in X$, but $\mathbf{T}$ need not be uniformly exponentially stable.

In order to give an example showing that the second condition of Theorem 4 cannot be omitted we need some preparation.

Let us call a subset K of $C^+[0,\infty)$ *solid* if from $0 \leqslant f \leqslant g$ and $g \in K$ it follows that $f \in K$.

Proposition 8. *Let K be a closed, convex, solid subset of $C^+[0,\infty)$ not containing any nonzero constant function. If $\mathbf{T}$ is not uniformly exponentially stable, then the set of all $x \in X$ with the property $c\,\|T(\cdot)x\| \notin K$ for all $c > 0$ is residual.*

Proof. Since K is closed and convex, its Minkowski functional

$$J_K(f) := \inf\{\lambda > 0 : \ f \in \lambda K\}, \qquad f \in C^+[0,\infty),$$

is lower semicontinuous. Since K is solid, $0 \leqslant f \leqslant g$ implies $J_K(f) \leqslant J_K(g)$. Since K does not contain any nonzero constant function we have $J_K(c\mathbf{1}) = \infty$ for all $c > 0$. By Theorem 4, the set of all $x \in X$ with $J_K(\|T(\cdot)x\|) = \infty$ is residual. Noting that $J_K(\|T(\cdot)x\|) < \infty$ if and only if $c\,\|T(\cdot)x\| \in K$ for some $c > 0$, this gives the result. ∎

The solidity of K was needed only to verify condition 2 of Theorem 4. Thus if Theorem 4 were true for every functional J satisfying only conditions 1 and 3, then Proposition 8 would be true for every closed convex subset K of $C^+[0,\infty)$ containing 0. The following example shows that this is not true, however.

Example 9. Let $\mathbf{T}$ be a C_0-semigroup on a Banach space X which is strongly stable but not uniformly exponentially stable.

Let $0 < \varepsilon < 1$ and $n \geqslant 0$ be fixed and define

$$K_{\varepsilon,n} = \big\{f \in C^+[0,\infty) : \ \varepsilon f(n) \geqslant f(n+1)\big\}.$$

This set is closed and convex, it contains no nonzero constant function, but it is not solid. In order to obtain a contradiction let us assume that Proposition 8 may be applied to the set $K_{\varepsilon,n}$. We then find that the set

$$X_{\varepsilon,n} = \{x \in X : \ \varepsilon\|T(n)x\| < \|T(n+1)x\|\}$$

is residual. Let $(\varepsilon_k)_{k\geqslant 0}$ be a sequence with $0 < \varepsilon_k < 1$ for all $k \geqslant 0$ and $\varepsilon_k \uparrow 1$ as $k \to \infty$. Then

$$\|T(n)x\| \leqslant \|T(n+1)x\|$$

if and only if $x \in \bigcap_{k\geqslant 0} X_{\varepsilon_k,n} =: X_n$, and this set is residual. Next,

$$\|T(n)x\| \leqslant \|T(n+1)x\|, \qquad n = 0, 1, \ldots$$

if and only if $x \in \bigcap_{n\geqslant 0} X_n$, and this set is again residual. But since we assumed that $\mathbf{T}$ is strongly stable, $\bigcap_{n\geqslant 0} X_n = \{0\}$, a contradiction.

The next example shows that condition 3 in Theorem 4 cannot be relaxed too much.

Example 10. Let $J(f) := |\{t \in [0,\infty) : \ f(t) > \varepsilon\}|$, where $|\cdot|$ denotes the Lebesgue measure and $\varepsilon > 0$ is fixed. Then J is lower semicontinuous and nondecreasing, and $J(c\mathbf{1}) = \infty$ if and only if $c > \varepsilon$. If $\mathbf{T}$ is a strongly stable semigroup on X, then $J(\|T(\cdot)x\|) < \infty$ for all $x \in X$, but $\mathbf{T}$ need not be uniformly exponentially stable.

For $p \in [1, \infty)$ the functional

$$J_p(f) = \int_0^\infty (f(t))^p \, dt, \qquad f \in C^+[0, \infty), \tag{1.3}$$

ocuuring in the Datko-Pazy theorem is not only nondecreasing but also convex. It is not possible, however, to replace 'nondecreasing' by 'convex' in Theorem 4, as is shown by the following example.

Example 11. Let $K_{\varepsilon,n}$ be the closed convex set of Example 9. Clearly, $\lambda K_{\varepsilon,n} = K_{\varepsilon,n}$ for all $\lambda > 0$, and therefore its Minkowski functional $J_{\varepsilon,n}$ is given by

$$J_{\varepsilon,n}(f) = \begin{cases} 0, & f \in K_{\varepsilon,n}; \\ \infty, & \text{else.} \end{cases}$$

In particular, $J_{\varepsilon,n}$ is convex. As we have seen, $J_{\varepsilon,n}$ is also lower semicontinuous and $J_{\varepsilon,n}(c\mathbf{1}) = \infty$ for all $c > 0$. Now let us assume that the conclusion of Theorem 4 holds for the functionals $J_{\varepsilon,n}$. Then the conclusion of Proposition 8 holds for the sets $K_{\varepsilon,n}$, and it was shown in Example 9 that this leads to a contradiction.

For $p \in (0, 1)$, the functional J_p defined by (1.3) is concave. Our final result shows that Theorem 4 does remain true if we replace 'nondecreasing' by 'concave'.

Theorem 12. *Let $J : C^+[0, \infty) \to [0, \infty]$ be a map with the following properties:*

(1) *J is lower semicontinuous;*
(2) *J is concave;*
(3) *$J(c\mathbf{1}) = \infty$ for all $c > 0$.*

Let $\mathbf{T}$ be a C_0-semigroup on a Banach space X which is not uniformly exponentially stable. Then the set

$$\big\{x \in X : \; J\big(\|T(\cdot)x\|\big) = \infty\big\}$$

is residual.

Proof. For $k = 1, 2, \ldots$ let X_k be the open set

$$X_k := \big\{x \in X : \; J(\|T(\cdot)y\|) > k\big\}.$$

Let $B(x, \delta)$ be an arbitrary open ball in X; we will show that $X_k \cap B(x, 2\delta) \neq \varnothing$. Proceeding as in the proof of Theorem 4, we construct a sequence $(y_n)_{n \geqslant 0}$ contained in $B(x, \delta)$ and a sequence $(f_n)_{n \geqslant 0}$ in $C^+[0, \infty)$ with

$$\|T(\cdot)y_n\| \geqslant \frac{C}{M} \mathbf{1}_{[0,n]} \geqslant f_n, \qquad n = 0, 1, \ldots$$

and

$$\lim_{n \to \infty} J(f_n) = \infty.$$

Let $(\alpha_n)_{n \geqslant 0}$ be a sequence of real numbers satisfying $0 < \alpha_n \leqslant \frac{1}{2}$ for all $n \geqslant 0$ and

$$\lim_{n \to \infty} \alpha_n = 0, \qquad \lim_{n \to \infty} \alpha_n J(f_n) = \infty.$$

Noting that $\alpha_n / (1 - \alpha_n) \leqslant 1$ for all $n \geqslant 0$ and using the concavity of J, it follows that

$$\begin{aligned} &\liminf_{n \to \infty} J\big(\big\|T(\cdot)\big((1 - \alpha_n)y_n\big)\big\|\big) \\ &\qquad \geqslant \liminf_{n \to \infty} \left(\alpha_n J(f_n) + (1 - \alpha_n) J\left(\|T(\cdot)y_n\| - \frac{\alpha_n}{1 - \alpha_n} f_n\right)\right) \\ &\qquad \geqslant \lim_{n \to \infty} \alpha_n J(f_n) = \infty. \end{aligned}$$

Given a fixed integer $k \geqslant 1$, it follows that there exists an index n_0 such that

$$J\big(\big\|T(\cdot)\big((1-\alpha_n)y_n\big)\big\|\big) > k, \qquad n \geqslant n_0.$$

On the other hand, since $\lim_{n\to\infty} \alpha_n = 0$, there exists an index $n_1 \geqslant n_0$ such that $(1-\alpha_n)y_n \in B(x, 2\delta)$ for all $n \geqslant n_1$. For such n we have $(1-\alpha_n)y_n \in X_k \cap B(x, 2\delta)$, showing that the intersection is nonempty. ∎

By a well-known result of Datko [2], a C_0-semigroup $\mathbf{T}$ on X is uniformly exponentially stable if and only if there exists $p \in [1,\infty)$ such that $\mathbf{T} * f \in L^p([0,\infty);X)$ for all $f \in L^p([0,\infty);X)$. Here, the convolution $\mathbf{T} * f$ is defined by

$$(\mathbf{T} * f)(t) = \int_0^t T(t-s)f(s)\,ds, \qquad t \geqslant 0.$$

In fact, let $x \in X$ be arbitrary and define $f_x \in L^p([0,\infty);X)$ by

$$f_x(t) = \begin{cases} T(t)x, & t \in [0,1), \\ 0, & t \geqslant 1. \end{cases}$$

Then,

$$(\mathbf{T} * f_x)(t) = \begin{cases} t\,T(t)x, & t \in [0,1), \\ T(t)x, & t \geqslant 1. \end{cases}$$

Since by assumption $\mathbf{T} * f_x \in L^p([0,\infty);X)$ for all $x \in X$, it follows that $T(\cdot)x \in L^p([0,\infty);X)$ for all $x \in X$. Therefore $\mathbf{T}$ is uniformly exponentially stable by the Datko-Pazy theorem. An easy modification of this argument shows that it is enough to have $\mathbf{T} * f \in L^p([0,\infty);X)$ for all $f \in C_c((0,\infty);X)$, the space of continuous $X-$valued functions with compact support in $(0,\infty)$; cf. the proof below.

The following result extends Datko's theorem to the setting of lower semicontinuous functionals:

Theorem 13. *Let $\mathbf{T}$ be a C_0-semigroup on X and let $J : C^+[0,\infty) \to [0,\infty]$ be a map with the following properties:*

(1) *J is lower semicontinuous;*
(2) *J is nondecreasing;*
(3) *$J(f) = \infty$ for all $f \in C^+[0,\infty)$ satisfying $\liminf_{t\to\infty} f(t) > 0$.*

If $J(\|\mathbf{T} * f\|) < \infty$ for all $f \in C_c((0,\infty);X)$, then $\mathbf{T}$ is uniformly exponentially stable.

Proof. Fix an arbitrary nonzero $0 \leqslant \phi \in C_c(0,\infty)$, with support in $[a,b]$ say, and define

$$\psi(t) = \int_0^t \phi(s)\,ds, \qquad t \geqslant 0.$$

Then for all $t \geqslant b$ we have $\psi(t) = \psi(b) > 0$. For $x \in X$ let $f_x \in C_c((0,\infty);X)$ be defined by

$$f_x(t) := \phi(t)T(t)x, \qquad t \geqslant 0.$$

Then $J(\|\mathbf{T} * f_x\|) < \infty$ by assumption. By the Baire category theorem, there is a ball $B(x_0,r)$ in X and an $N \in \mathbb{N}$ such that $J(\|\mathbf{T} * f_x\|) \leqslant N$ for all $x \in B(x_0,r)$. Now suppose $\mathbf{T}$ were not uniformly exponentially stable. Then $r(T) \geqslant 1$, and by Lemma 3 there exists a constant $c > 0$ such that for all $n \in \mathbb{N}$ we can find $y_n \in B(x_0,r)$ with

$$\|T(t)y_n\| \geqslant c, \qquad t \in [0,n].$$

Noting that

$$(\mathbf{T} * f_{y_n})(t) = \int_0^t \phi(s)T(t-s)T(s)y_n \, ds = \psi(t)T(t)y_n, \qquad t \geqslant 0,$$

we see that

$$J(c\psi) \leqslant \liminf_{n\to\infty} J(\psi \cdot c\mathbf{1}_{[0,n]}) \leqslant J(\mathbf{T} * f_{y_n}) \leqslant N.$$

On the other hand, from $\liminf_{t\to\infty} c\psi(t) = c\psi(b) > 0$ it follows that $J(c\psi) = \infty$, and we have arrived at a contradiction. ∎

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