Exponential stability of operators and operator semigroups

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Extending earlier results of Datko, Pazy and Littman on C_0-semigroups, and of Przyluski and Weiss on operators, we prove the following. Let T be a bounded linear operator on a Banach space X and let $r(T)$ denote its spectral radius. Let E be a Banach function space over $\mathbb{N}$ with the property that $\lim_{n\to\infty} \|\chi_{\{0,\dots,n-1\}}\|_E = \infty$. If for each $x \in X$ and $x^* \in X^*$ the map $n \mapsto \langle x^*, T^n x\rangle$ belongs to E, then $r(T) < 1$.

By applying this to Orlicz spaces E, the following result is obtained. Let T be a bounded linear operator on a Banach space X and let $\phi : \mathbb{R}_+ \to \mathbb{R}_+$ be a non-decreasing function with $\phi(t) > 0$ for all $t > 0$. If $\sum_{n=0}^{\infty} \phi\big(|\langle x^*, T^n x\rangle|\big) < \infty$ for all $\|x\|, \|x^*\| \leqslant 1$, then $r(T) < 1$. Assuming a Δ_2-condition on ϕ, a further improvement is obtained.

For locally bounded semigroups $\mathbf{T} = \{T(t)\}_{t\geqslant 0}$, we obtain similar results in terms of the maps $t \mapsto \|T(t)x\|$.

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0. Introduction

Let $\mathbf{T} = \{T(t)\}_{t\geqslant 0}$ be a C_0-semigroup on a Banach space X. Let $\omega(\mathbf{T})$ be its growth bound. It is a well-known theorem of Datko [Da] that $\omega(\mathbf{T}) < 0$ if

$$\int_0^\infty \|T(t)x\|^2 \, dt < \infty, \qquad \forall x \in X.$$

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This result was generalized by Pazy [P], who showed that the exponent $p = 2$ may be replaced by any $1 \leqslant p < \infty$. Recently, Littman [Li] showed that $\omega(\mathbf{T}) < 0$ if there exists a continuous, increasing function $\phi : \mathbb{R}_+ \to \mathbb{R}_+$ which satisfies $\phi(0) = 0$ and $\phi(t) > 0$ for all $t > 0$, such that

$$\int_0^\infty \phi(\|T(t)x\|)\, dt < \infty, \qquad \forall x \in X.$$

Pazy's result is recovered from the function $\phi(t) = t^p$.

Noting that for the spectral radius we have the formula $r(T(1)) = e^{\omega(\mathbf{T})}$ and that $T(n) = (T(1))^n$, it is natural to ask whether analogues of the above results hold for the powers of an arbitrary bounded linear operator T on X. Indeed, Zabczyk [Za] showed that $r(T) < 1$ if there is a $1 \leqslant p < \infty$ such that

$$\sum_{n=0}^\infty \|T^n x\|^p < \infty, \qquad \forall x \in X.$$

Recently, Weiss [We] proved that it is even sufficient that

$$\sum_{n=0}^\infty |\langle x^*, T^n x\rangle|^p < \infty, \qquad \forall x \in X, x^* \in X^*$$

for some $1 \leqslant p < \infty$. Earlier, Przyluski [Pr] had obtained the case $p = 1$ for weakly sequentially complete Banach spaces.

Comparing the results on semigroups and bounded operators, the natural question arises whether an analogue of Littman's theorem is valid for bounded operators. In this paper we show that this is indeed the case. In fact, we have the following more general result.

Theorem 0.1. *Let T be a bounded linear operator on a Banach space X and let $\phi : \mathbb{R}_+ \to \mathbb{R}_+$ be a non-decreasing function with $\phi(t) > 0$ for all $t > 0$.*

(i) If

$$\sum_{n=0}^\infty \phi\big(|\langle x^*, T^n x\rangle|\big) < \infty, \qquad \forall \|x\|, \|x^*\| \leqslant 1,$$

then $r(T) < 1$.

(ii) If ϕ satisfies a Δ_2-condition at 0, and if

$$\sum_{n=0}^\infty \phi\big(\alpha_n |\langle x^*, T^n x\rangle|\big) < \infty, \qquad \forall \|x\|, \|x^*\| \leqslant 1$$

for some non-negative sequence (α_n) with $\sum_n \alpha_n \phi(\alpha_n) = \infty$, then $r(T) < 1$.

The idea behind the proof is as follows. First we show that if E is a Banach function space over $\mathbb{N}$ with the property that $\lim_{n\to\infty} \|\chi_{\{0,\dots,n-1\}}\|_E = \infty$, and if for each $x \in X$ and $x^* \in X^*$ the function

$$f_{x,x^*}(n) := \langle x^*, T^n x\rangle$$

belongs to E, then $r(T) < 1$. This is proved in Section 2. The condition on E is necessary in order to exclude the Banach function spaces $E = c_0$ and l^∞; the example of the shift on l^2 shows that these spaces indeed have to be excluded.

In Section 3, we apply this to certain Orlicz sequence spaces E, in order to obtain a proof of Theorem 0.1.

In Section 4 we discuss the semigroup versions of our results. We do not assume $\mathbf{T}$ to be strongly continuous. In fact, in the theorems of Datko-Pazy and Littman, which we will derive as a consequence of the results for the discrete case, it suffices to have $\mathbf{T}$ locally bounded.

The main results in this paper are valid both for real and complex Banach spaces X. We will deal with complex scalars only. The real case can be derived from this by complexification as follows. First, for an operator T on a real Banach space X, we define $r(T) := r(T_{\mathbb{C}})$, where $T_{\mathbb{C}}$ is the complexification of T. Now if, for instance, $n \mapsto \langle x^*, T^n x\rangle$ belongs to some real Banach function space E for all $x \in X$ and $x^* \in X^*$, then $n \mapsto \langle x^*_{\mathbb{C}}, T^n_{\mathbb{C}} x_{\mathbb{C}}\rangle$ belongs to $E_{\mathbb{C}}$ for all $x_{\mathbb{C}} \in X_{\mathbb{C}}$ and $x^*_{\mathbb{C}} \in X^*_{\mathbb{C}}$.

Note that, both in the real and in the complex case, $r(T) < 1$ if and only if $\|T^n\| \leqslant Me^{-\omega n}$ for some M and $\omega > 0$, if and only if there exists an n such that $\|T^n\| < 1$.

1. Banach function spaces

In this section we briefly recall some facts about Banach function spaces. For more details, we refer to the books [Z1], [MN], [BS].

Let (Ω, Σ, μ) be a σ-finite positive measure space. Let $M(\mu)$ be the vector space of μ-measurable functions $\Omega \to \mathbb{C}$, identifying functions which are equal μ-a.e. A *Banach function norm* is a function $\rho : M(\mu) \to [0, \infty]$ with the following properties:

(N1) $\rho(f) = 0$ if and only $f = 0$;
(N2) if $|f| \leqslant |g|$ μ-a.e., then $\rho(f) \leqslant \rho(g)$;
(N3) $\rho(af) = |a|\rho(f)$ for all scalars $a \in \mathbb{C}$ and all $\rho(f) < \infty$;
(N4) $\rho(f + g) \leqslant \rho(f) + \rho(g)$ for all $f, g \in M(\mu)$.

Let $E = E_\rho$ be the set $\{f \in M(\mu) : \|f\|_E := \rho(f) < \infty\}$. Then E is easily seen to be a normed linear space. If E is complete, then it is called a *Banach function space* over (Ω, Σ, μ). Note that E is an ideal in $M(\mu)$: if

$|f| \leqslant |g|$ μ-a.e. with $g \in E$, then also $f \in E$ (and $\|f\|_E \leqslant \|g\|_E$). This property of Banach function spaces, which we will call the *ideal property*, will be used repeatedly in what follows. Every Banach function space E is a Dedekind complete Banach lattice. We say that E is *carried* by a subset Ω' of Ω if the following is true: whenever $H \subset \Omega'$ is a measurable set of positive measure, then there exists a function $f \in E$ that is not zero μ-a.e. on H. In order to exclude the pathological situation that Ω is larger than the 'joint support' of the functions in E, we will always assume that E is carried by Ω. This is no loss of generality, for one can prove that there always exists a maximal subset Ω' of Ω such that E is carried by Ω' [Z1, Thm. 67.2].

The above definition of a Banach function space can be found in the books [Z1] and [MN]. Some authors, e.g. [BS] include into the definition the further hypothesis that the characteristic function sets of finite measure be in E. In the present setting, this need not be the case.

If $f_n \to f$ in norm in a Banach function space E, then there is a subsequence (f_{n_k}) converging to f pointwise μ-a.e; use [MN, Prop. 2.6.3] or [Z1, Ex. 64.1].

We will be interested in Banach function spaces over $\mathbb{N}$ and $\mathbb{R}_+ = [0, \infty)$ (with the counting measure and the Lebesgue measure, respectively).

Let E be a Banach function space over $\mathbb{N}$. Since by assumption E is carried by $\mathbb{N}$, the characteristic function of each $n \in \mathbb{N}$ belongs to E. Indeed, there is a function $f \in E$ with $|f(n)| > 0$. But then $\chi_{\{n\}} \leqslant |f(n)|^{-1}|f| \in E$, and consequently $\chi_{\{n\}} \in E$. By taking finite sums, it follows that the characteristic function of each finite subset of $\mathbb{N}$ belongs to E. We define

$$\Psi_E(n) := \|\chi_{\{0,\dots,n-1\}}\|_E$$

and

$$\Psi_E(\infty) := \lim_{n\to\infty} \Psi_E(n).$$

For $E = l^p$, $1 \leqslant p < \infty$ we have $\Psi_E(n) = n^{1/p}$ and $\Psi_E(\infty) = \infty$, and for $E = c_0$ and l^∞ we have $\Psi_E(n) = \Psi_E(\infty) = 1$.

If E is a Banach function space over $\mathbb{R}_+$, then we set $\Psi_E(n) = \chi_{[0,t]}$, provided the characteristic function $\chi_{[0,t]}$ belongs to E. In all situations to be discussed later, this is indeed the case. Finally, we set $\Psi_E(\infty) := \lim_{t\to\infty} \Psi_E(t)$.

2. An abstract sufficient condition for power stability

Let T be a bounded operator on a complex Banach space X. T is said to be *power stable* if the spectral radius of T satisfies $r(T) < 1$. By Gelfand's formula for the spectral radius, T is power stable if and only if $\|T^n\| < 1$ for some $n \in \mathbb{N}$, if and only if there exist numbers $M \geqslant 1$ and $\omega > 0$ such that $\|T^n\| \leqslant Me^{-n\omega}$ for all $n \in \mathbb{N}$. In this section, we will derive an abstract sufficient condition for an operator to be power stable.

We start with the following easy lemma.

Lemma 2.1. *Suppose $r(T) \geqslant 1$ and $0 < \varepsilon < 1$. Then for each $N \in \mathbb{N}$, there exist norm one vectors $x_N \in X$ and $x_N^* \in X^*$ such that*

$$|\langle x_N^*, T^n x_N \rangle| \geqslant \varepsilon, \quad n = 0, 1, ..., N-1.$$

Proof: Fix $N \in \mathbb{N}$ and let $\lambda \in \partial\sigma(T)$ be any point on the boundary of the spectrum of T. Then $|\lambda| \geqslant 1$ and λ is an approximate eigenvalue. Let (y_n) be an approximate eigenvector corresponding to λ. Since for each $k \in \mathbb{N}$,

$$\lim_{n\to\infty} \|T^k y_n - \lambda^k y_n\| = 0,$$

we may choose n_0 so large that $\|T^k y_{n_0} - \lambda^k y_{n_0}\| \leqslant \frac{1}{2}(1-\varepsilon)$, $k = 0, 1, ..., N-1$. Put $x_N = y_{n_0}$ and let $x_N^* \in X^*$ be any norm one vector such that $|\langle x_N^*, x_N\rangle| > \frac{1}{2}(1+\varepsilon)$. Then for $k = 0, 1, ..., N-1$ we have

$$|\langle x_N^*, T^k x_N\rangle| \geqslant |\lambda|^k |\langle x_N^*, x_N\rangle| - \frac{1}{2}(1-\varepsilon) \geqslant \varepsilon.$$

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For real Banach spaces, this lemma is wrong. A counterexample is provided by rotation over α in $\mathbb{R}^2$, with $\alpha/(2\pi)$ irrational. For this operator, Lemma 2.1 fails for every choice of ε. We leave the easy proof to the reader.

In the next lemma, $f_{x,x^*} : \mathbb{N} \to \mathbb{C}$ denotes the map $f_{x,x^*}(n) = \langle x^*, T^n x\rangle$.

Lemma 2.2. *Let T be a bounded operator on X and E a Banach function space over $\mathbb{N}$ such that $f_{x,x^*} \in E$ for all $x \in X$ and $x^* \in X^*$. Then there is a constant M such that $\|f_{x,x^*}\|_E \leqslant M\|x\|\,\|x^*\|$.*

Proof: For each $x \in X$ define $T_x : X^* \to E$ by $T_x(x^*) = f_{x,x^*}$. We claim that each T_x is closed. Indeed, suppose $x_n^* \to x^*$ in X^* and $T_x(x_n^*) \to y^*$ in E. Since norm convergent sequences in a Banach function space admit a pointwise a.e. convergent subsequence, we must have $y^* = T_x(x^*)$.

By the closed graph theorem, each T_x is bounded. Similarly, for each $x^* \in X^*$ the map $T_{x^*} : X \to E$, $T_{x^*}(x) = f_{x,x^*}$ is bounded. Now the result follows easily form the uniform boundedness theorem. ////

Theorem 2.3. *Let T be a bounded operator on a Banach space X. Let E be a Banach function space over $\mathbb{N}$ with $\Psi_E(\infty) = \infty$. If, for each $x \in X$ and $x^* \in X^*$, the function*

$$f_{x,x^*}(n) := \langle x^*, T^n x\rangle$$

belongs to E, then $r(T) < 1$.

Proof: Let M be as in Lemma 2.2 and suppose for contradiction that $r(T) \geqslant 1$. Fix $0 < \varepsilon < 1$ and let $N \in \mathbb{N}$ be arbitrary. Set $\Omega_N = \{0, ..., N-1\}$ and let x_N and x_N^* be as in Lemma 2.1. Then $\|f_{x_N,x_N^*}\|_E \leqslant M$. From $|\chi_{\Omega_N}| \leqslant \varepsilon^{-1}|f_{x_N,x_N^*}|$ and the ideal property of Banach function spaces, we see that $\chi_{\Omega_N} \in E$ and $\Psi_E(n) = \|\chi_{\Omega_N}\|_E \leqslant \varepsilon^{-1}M$. Since this holds for all N, we have $\Psi_E(\infty) \leqslant \varepsilon^{-1}M$, a contradiction. ////

Theorem 2.3 can be applied to various 'weighted' Banach function spaces. We start with the abstract idea. For two functions α, β on $\mathbb{N}$ we let $\alpha\beta$ denote the pointwise product.

Corollary 2.4. *Let T be a bounded operator on X. Let E be a Banach function space over $\mathbb{N}$ and let $\alpha = (\alpha_n)_{n=0}^{\infty}$ be a non-negative sequence of scalars such that*

$$\lim_{n\to\infty} \|\alpha\chi_{\{0,\dots,n-1\}}\|_E = \infty. \tag{2.1}$$

If the map $n \mapsto \alpha_n\langle x^*, T^n x\rangle$ belongs to E for all $x \in X$ and $x^* \in X^*$, then $r(T) < 1$.

Proof: We may assume that $T \neq 0$. Let $\mu_n := \|\chi_{\{n\}}\|_E$, $n = 0, 1, \dots$ Then $\mu_n > 0$ and we can define a sequence $(\tilde{\alpha}_n)$ by

$$\tilde{\alpha}_k = \begin{cases} \alpha_k, & \alpha_k \neq 0; \\ 2^{-k}\mu_k^{-1}\|T\|^{-k}, & \alpha_k = 0. \end{cases}$$

The set $E_{\tilde{\alpha}}$ of all sequences $y = (y_n)$ for which $\tilde{\alpha}y \in E$, with norm $\|y\|_{E_{\tilde{\alpha}}} := \|\tilde{\alpha}y\|_E$, is a Banach function space (note that (N1) holds because $(\tilde{\alpha}_n)$ is strictly positive). By (2.1) and the fact that $(\tilde{\alpha}_n) \geqslant (\alpha_n)$, we have $\Psi_{E_{\tilde{\alpha}}}(\infty) = \infty$.

Since the functions $n \mapsto \tilde{\alpha}_n\langle x^*, T^n x\rangle$ belong to E for all $x \in X$ and $x^* \in X^*$, the functions $n \mapsto \langle x^*, T^n x\rangle$ belong to $E_{\tilde{\alpha}}$. Therefore we can apply Theorem 2.3. ////

Applying Corollary 2.4 to $E = l^p$ and $\alpha_k := \beta_k^{1/p}$ gives us the following result, which will be generalized at the end of the next section.

Corollary 2.5. *Let (β_n) a non-negative sequence with $\sum_{n=0}^{\infty}\beta_n = \infty$. Let T be a bounded operator on X. If, for some $1 \leqslant p < \infty$,*

$$\sum_{k=0}^{\infty} \beta_k |\langle x^*, T^k x\rangle|^p < \infty, \quad \forall x \in X, x^* \in X^*,$$

then $r(T) < 1$.

In [Ne], this corollary has been applied in order to obtain the following result on the weak orbits of an operator with spectral radius one:

Corollary 2.6 [Ne]. *Let T be a bounded operator on a Banach space X with $r(T) = 1$. Let $(\alpha_n) \in c_0$ be of norm one. Then each sequence (n_k) has a subsequence (n_{k_j}) with the property that there exist norm one vectors $x \in X$, $x^* \in X^*$ such that*

$$|\langle x^*, T^{n_{k_j}} x\rangle| \geqslant |\alpha_{k_j}|, \quad j = 0, 1, \dots$$

This result exhibits a connection between our theory and the theory of orbits of Beauzamy [B]. In Section 4 we will show, conversely, that results from the theory of orbits can be used for the study of stability.

3. Orlicz spaces

A certain class of Banach function spaces, the Orlicz spaces, is of special interest. To these we turn now. For more details, we refer to [Z2].

Let $\phi : \mathbb{R}_+ \to [0,\infty]$ be a function with $\phi(0) = 0$ which is non-decreasing, left-continuous, and not identically 0 or ∞ on $(0,\infty)$. Define

$$\Phi(t) := \int_0^t \phi(s)\ ds.$$

A function Φ of this form is called a *Young function*.

Let (Ω, Σ, μ) be a positive σ-finite measure space, and let $f : \Omega \to \mathbb{C}$ be a measurable function. Let Φ be a Young function. We define

$$M^\Phi(f) := \int_\Omega \Phi(|f(\omega)|)\ d\omega.$$

The set L^Φ of all f for which there exists a $k > 0$ such that $M^\Phi(kf) < \infty$ is easily checked to be a linear space. In case (Ω, Σ, μ) is either non-atomic or purely atomic, with the norm

$$\rho^\Phi(f) := \inf\{k : M^\Phi(\frac{1}{k}f) \leqslant 1\}$$

the space (L^Φ, ρ^Φ) becomes a Banach function space over Ω. Spaces of this type are called *Orlicz spaces.* Note that every Orlicz space contains the characteristic functions of sets of finite measure; in particular it is carried by Ω.

Trivial examples of Orlicz spaces are l^p and $L^p(\mathbb{R}_+)$ (over $\mathbb{N}$ and $\mathbb{R}_+$, respectively), $1 \leqslant p < \infty$. They are obtained from $\phi(t) = pt^{p-1}$, $t > 0$, (so $\Phi(t) = t^p$). Similarly, l^∞ and $L^\infty(\mathbb{R}_+)$ are obtained from $\phi = \Phi$ given by

$$\phi(t) = \begin{cases} 0, & 0 \leqslant t \leqslant 1, \\ \infty, & t > 1. \end{cases}$$

The following lemma describes when the characteristic function of an Orlicz space L^Φ satisfies $\Psi_{L^\Phi}(\infty) = \infty$.

Lemma 3.1. *Suppose Ω is either $\mathbb{N}$ or $\mathbb{R}_+$. Then $\Psi_{L^\Phi}(\infty) = \infty$ if and only if $\phi(t) > 0$ for all $t > 0$.*

Proof: Put $\Omega_n = \{0, ..., n-1\}$ if $\Omega = \mathbb{N}$ and $\Omega_n = [0, n]$ if $\Omega = \mathbb{R}_+$. Note that $\chi_{\Omega_n} \in L^\Phi$. We must prove that $\lim_n \|\chi_{\Omega_n}\|_{L^\Phi} = \infty$.

Since $\phi(t) > 0$ for all $t > 0$, the same holds for Φ. Since ϕ is not identically ∞ on $(0,\infty)$ there is a $t_0 > 0$ such that $0 < \Phi(t) < \infty$ for all $t \in (0, t_0)$. Also note that Φ is continuous on $[0, t_0)$ and strictly increasing. From these facts it follows that for all n large enough there is a unique point $t_n \in (0, t_0)$ such that $\Phi(t_n) = n^{-1}$; moreover $t_n \to 0$ as $n \to \infty$. Calculating, we find

$$\begin{aligned}
\|\chi_{\Omega_n}\|_{L^\Phi} &= \inf\{k : M^\Phi(\frac{1}{k}\chi_{\Omega_n}) \leqslant 1\} \\
&= \inf\{k : \int_\Omega \Phi(\frac{1}{k}\chi_{\Omega_n}(\omega))\ d\omega \leqslant 1\} \\
&= \inf\{k : \int_{\Omega_n} \Phi(\frac{1}{k})\ d\omega \leqslant 1\} \\
&= \inf\{k : \Phi(\frac{1}{k}) \leqslant \frac{1}{n}\} = \frac{1}{t_n}.
\end{aligned}$$

But $t_n^{-1} \to \infty$, which proves the 'if' part of the lemma.

Conversely, suppose that $\phi = 0$ on the interval $[0, t_0]$ for some $t_0 > 0$. A calculation as above then shows that

$$\lim_{n\to\infty} \|\chi_{\Omega_n}\|_{L^\Phi} = \frac{1}{t_0} < \infty.$$

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Lemma 3.2. *Let $\phi : \mathbb{R}_+ \to \mathbb{R}_+$ be left-continuous and non-decreasing with $\phi(0) = 0$ and $\phi(t) > 0$ for all $t > 0$. Suppose Ω is either $\mathbb{N}$ or $\mathbb{R}_+$. Then there exists a Young function Φ such that $f \in L^\Phi$ for all $f \in L^\infty(\Omega)$ which satisfy*

$$\int_\Omega \phi\Big(|f(\omega)|\Big)\ d\mu(\omega) < \infty. \tag{3.1}$$

If moreover $\mu(\Omega) = \infty$, then Φ can be chosen such that $\Psi_{L^\Phi}(\infty) = \infty$.

Proof: By replacing ϕ by some multiple of ϕ, we may assume that $\phi(1) = 1$. Define

$$\tilde\phi(t) := \begin{cases} \phi(t), & 0 \leqslant t \leqslant 1, \\ 1, & t > 1. \end{cases}$$

Let Φ be the Young function

$$\Phi(t) = \int_0^t \tilde\phi(s)\ ds.$$

Now fix $f \in L^\infty(\Omega)$ satisfying (3.1). Let $\Omega_f := \{\omega \in \Omega : |f(\omega)| \geqslant 1\}$. From

$$\int_\Omega \phi\Big(|f(\omega)|\Big)\ d\mu(\omega) \geqslant \int_{\Omega_f} \phi\Big(|f(\omega)|\Big)\ d\mu(\omega) \geqslant \int_{\Omega_f} \phi(1)\ d\mu(\omega) = \mu(\Omega_f)$$

it follows that $\mu(\Omega_f) < \infty$. Hence,

$$\int_{\Omega_f} \Phi(|f|)\ d\mu \leqslant \int_{\Omega_f} \Phi(\|f\|_\infty)\ d\mu = \Phi(\|f\|_\infty)\mu(\Omega_f) < \infty.$$

Also, noting that $\Phi(t) \leqslant \tilde{\phi}(t) = \phi(t)$ for all $t \in [0,1]$, we have

$$\int_{\Omega\setminus\Omega_f} \Phi(|f|)\, d\mu \leqslant \int_{\Omega\setminus\Omega_f} \phi(|f|)\, d\mu < \infty.$$

It follows that

$$M^{\Phi}(f) = \int_{\Omega} \Phi(|f|)\, d\mu < \infty.$$

But then trivially $f \in L^{\Phi}$.

Finally, the function $\tilde{\phi}$ also satisfies $\tilde{\phi}(t) > 0$ for all $t > 0$. Therefore, if $\mu(\Omega) = \infty$, then $\Psi_{L^{\Phi}}(\infty) = \infty$ by Lemma 3.1. ////

We remark that Lemmas 3.1 and 3.2 can be proved for arbitrary positive σ-finite measure spaces (Ω, Σ, μ) which are either non-atomic or purely atomic.

Now we are in a position to prove the following theorem.

Theorem 3.3. *Let T be a bounded linear operator on a Banach space X and let $\phi : \mathbb{R}_+ \to \mathbb{R}_+$ be non-decreasing with $\phi(t) > 0$ for all $t > 0$. If*

$$\sum_{n=0}^{\infty} \phi\Big(|\langle x^*, T^n x\rangle|\Big) < \infty, \qquad \forall \|x\|, \|x^*\| \leqslant 1,$$

then $r(T) < 1$.

Proof: Define $\tilde{\phi}(0) = 0$ and $\tilde{\phi}(t) := \lim_{s\uparrow t} \phi(s)$ for $t > 0$. Then $\tilde{\phi}$ is left-continuous and satisfies $\tilde{\phi}(0) = 0$ and $0 < \tilde{\phi}(t) \leqslant \phi(t)$ for all $t > 0$. Therefore, if we replace ϕ by $\tilde{\phi}$ is necessary, we may assume that ϕ is left-continuous and satisfies $\phi(0) = 0$.

For $x \in X$ and $x^* \in X^*$ both of norm $\leqslant 1$, define $f_{x,x^*}(n) = \langle x^*, T^n x\rangle$. From the finiteness of the above sum, it follows that $\lim_n f_{x,x^*}(n) = 0$. In particular, $f_{x,x^*} \in l^{\infty} = L^{\infty}(\mathbb{N})$. Therefore, by Lemma 3.2 there is a Young function Φ such that $f_{x,x^*} \in L^{\Phi}$ for all $x \in X$ and $x^* \in X^*$ of norm $\leqslant 1$. Moreover, $\Psi_{L^{\Phi}}(\infty) = \infty$ by Lemma 3.1. Since L^{Φ} is a linear space, in fact $f_{x,x^*} \in L^{\Phi}$ for *all* $x \in X, x^* \in X^*$. Now we can apply Theorem 2.3. ////

We will now work out Corollary 2.4 for Orlicz spaces, assuming that the function ϕ satisfies a so-called Δ_2-*condition at* 0, i.e. we assume that there is an $\varepsilon > 0$ and a constant K such that for all $t \in [0, \varepsilon]$ we have $\phi(t) \leqslant K\phi(\frac{1}{2}t)$.

Theorem 3.4. *Let T be a bounded operator on X. Let $\phi : \mathbb{R}_+ \to \mathbb{R}_+$ be a non-decreasing function satisfying a Δ_2-condition at 0 with $\phi(0) = 0$ and $\phi(t) > 0$ for $t > 0$, and let $\alpha = (\alpha_n)$ be a non-negative scalar sequence such that*

$$\sum_{n=0}^{\infty} \alpha_n \phi(\alpha_n) = \infty. \tag{3.2}$$

If, for all $x \in X$ and $x^* \in X^*$ of norm $\leqslant 1$, we have

$$\sum_{n=0}^{\infty} \phi(\alpha_n |\langle x^*, T^n x\rangle|) < \infty,$$

then $r(T) < 1$.

Proof: Let $\phi(t) \leqslant K\phi(\frac{1}{2}t)$ for all $t \in [0,\varepsilon]$. Observe that without loss of generality we may assume $0 < \varepsilon \leqslant 1$ and for all n replace α_n by $\inf(\alpha_n, \varepsilon)$. To see this, consider the cases where $\{n \in \mathbb{N} : \alpha_n \geqslant \varepsilon\}$ is finite and infinite separately.

Also, without loss of generality, we may assume that ϕ is left-continuous. To see this, replace ϕ by the function $\tilde{\phi}$ of Theorem 3.3 and observe that for each $0 \leqslant t \leqslant \varepsilon$, $\phi(t) \leqslant K\phi(\frac{1}{2}t) \leqslant K\tilde{\phi}(t)$. In view of $\alpha_n \leqslant \varepsilon$ and $\tilde{\phi} \leqslant \phi$, it follows that (3.1) and (3.2) hold for $\tilde{\phi}$ as well. Also, by taking limits it follows that $\tilde{\phi}$ satisfies the Δ_2-condition at 0, with the same constants ε and K.

Finally we may assume that $\phi(1) = 1$.

Let $E = L^\Phi$ be the Orlicz space defined as in Lemma 3.2. Then we have $\lim_{n\to\infty} \|\alpha\chi_{\{0,\dots,n-1\}}\|_E = \infty$. Indeed, suppose the contrary and let $m > \sup_N \|\alpha\chi_{\{0,\dots,N-1\}}\|_E$. Then by definition of the norm of E, for all $N \in \mathbb{N}$ we have

$$\sum_{n=0}^{N-1} \Phi\big(\frac{\alpha_n}{m}\big) \leqslant 1.$$

Hence, $\sum_{n=0}^{\infty} \Phi(\alpha_n/m) \leqslant 1$. Subclaim: $\sum_{n=0}^{\infty}(\alpha_n/m)\phi(\alpha_n/m) < \infty$. Indeed, for all but finitely many n we must have $\alpha_n/m \leqslant \varepsilon$, for otherwise the sum involving Φ would be infinite. For these n, the Δ_2 condition implies that $\tilde{\phi}(s) = \phi(s) \geqslant K^{-1}(ms/\alpha_n)^k\phi(\alpha_n/m)$ for all $0 < s \leqslant \alpha_n/m$, where $k = (\ln K)/(\ln 2)$, and hence $\Phi(\alpha_n/m) \geqslant K^{-1}(k+1)^{-1}(\alpha_n/m)\phi(\alpha_n/m)$. This establishes the subclaim.

Let $j \in \mathbb{N}$ be such that $m \leqslant 2^j$. From the Δ_2-condition it follows that

$$\sum_{n=0}^{\infty} \phi\alpha_n(\alpha_n) \leqslant mK^j \sum_{n=0}^{\infty} \frac{\alpha_n}{m}\phi\big(\frac{\alpha_n}{m}\big) < \infty.$$

Thus we have arrived at a contradiction and the claim is proved.

Let $\|x\| \leqslant 1$, $\|x^*\| \leqslant 1$. Since $\sum_{n=0}^{\infty} \phi(\alpha_n|\langle x^*, T^n x\rangle|) < \infty$, we must have $\lim_n \alpha_n\langle x^*, T^n x\rangle = 0$. Hence, the map $n \mapsto \alpha_n\langle x^*, T^n x\rangle$ is bounded and therefore belongs to E by Lemma 3.2. By linearity, the same is true for *all* $x \in X$ and $x^* \in X^*$. Now the result follows from Corollary 2.4. ////

Since $\phi(t) = t^{p-1}$ satisfies a Δ_2-condition at 0 for each $1 \leqslant p < \infty$, Corollary 2.5 is contained as a special case in Theorem 3.4.

By completely different methods, which will be developed in the next section in the semigroup setting, we can prove the following strong analogue of Theorem 3.4.

Theorem 3.5. *Let T be a bounded operator on X. Let $\phi : \mathbb{R}_+ \to \mathbb{R}_+$ be a non-decreasing function satisfying a Δ_2-condition at 0 with $\phi(0) = 0$ and $\phi(t) > 0$ for $t > 0$, and let $\alpha = (\alpha_n)$ be a non-negative scalar sequence such that $\sum_{n=0}^{\infty} \phi(\alpha_n) = \infty$. If, for all $x \in X$ of norm $\leqslant 1$, we have $\sum_{n=0}^{\infty} \phi(\alpha_n\|T^n x\|) < \infty$, then $r(T) < 1$.*

4. The semigroup case

In this section we will apply the above results to semigroups of operators. Recall that a family $\mathbf{T} = \{T(t)\}_{t \geqslant 0}$ of bounded linear operators on X is called a *semigroup* if

(S1) $T(0) = I$;
(S2) $T(t)T(s) = T(t+s), \quad \forall t, s \geqslant 0$;

A semigroup is *locally bounded* if $\sup_{0 \leqslant t \leqslant \varepsilon} \|T(t)\| < \infty$ for some $\varepsilon > 0$, and *strongly continuous* or a C_0-*semigroup* if it satisfies

(S3) $\lim_{t \to \infty} \|T(t)x - x\| = 0, \quad \forall x \in X.$

By the uniform boundedness theorem, a C_0-semigroup is locally bounded. Now let $\mathbf{T}$ be a locally bounded semigroup. By the semigroup property (S2), there are constants $M \geqslant 1$ and $\omega \in \mathbb{R}$ such that $\|T(t)\| \leqslant Me^{\omega t}$ for all $t \geqslant 0$. Therefore it makes sense to define

$$\omega(\mathbf{T}) = \inf\{\omega \in \mathbb{R} : \exists M \geqslant 1 \text{ such that } \|T(t)\| \leqslant Me^{\omega t} \text{ for all } t \geqslant 0\}.$$

Thus, $\omega(\mathbf{T}) < 0$ if and only $\mathbf{T}$ is *uniformly exponentially stable*, i.e., there exists $M \geqslant 1$ and $\omega > 0$ such that $\|T(t)\| \leqslant Me^{-\omega t}$.

For the spectral radius of the operators $T(t)$ we have the formula [Na, Prop. A-III.1.1]

$$r(T(t)) = e^{\omega(\mathbf{T})t}.$$

Hence $\omega(\mathbf{T}) < 0$ if and only $r(T(1)) < 1$, and this is the case if and only if there exists a $t > 0$ such that $\|T(t)\| < 1$.

We start with en example of a positive C_0-semigroup on a reflexive Banach space X with $\omega(\mathbf{T}) > 0$ such that each map $t \mapsto \langle x^*, T(t)x \rangle$ belongs to $L^1(\mathbb{R}_+)$.

Example 4.1. Let $\mathbf{S}$ be the C_0-semigroup of Greiner, Voigt and Wolff [GVW] and put $\mathbf{T} := e^{\frac{1}{2}t}\mathbf{S}$. Then $\omega(\mathbf{T}) = \frac{1}{2}$. The spectral bound of the generator of $\mathbf{T}$ being negative, from [Na, Thm. A-IV.1.4 and Thm. C-IV.1.3] and the positivity one easily deduces that $\langle x^*, T(\cdot)x \rangle \in L^1(\mathbb{R}_+)$ for all $x \in X$ and $x^* \in X^*$. It can be shown that these maps also belong to $L^2(\mathbb{R}_+)$; see [NSW].

This example shows that there is no hope of carrying over the results of Sections 2 and 3 to the maps $\langle x^*, T(\cdot)x \rangle$. However, it turns out that we can carry over the results to the maps $\|T(\cdot)x\|$.

Theorem 4.2. *Let $\mathbf{T}$ be a locally bounded semigroup on a Banach space X and let E be a Banach function space over $\mathbb{R}_+$ with $\Psi_E(\infty) = \infty$. If, for each $x \in X$, the map $t \mapsto \|T(t)x\|$ belongs to E, then $\omega(\mathbf{T}) < 0$.*

If $\mathbf{T}$ is strongly measurable, in particular if $\mathbf{T}$ is a C_0-semigroup, this can be proved by replacing the role of $L^p(\mathbb{R}_+)$ by E in the proof of Pazy [P, Thm IV.4.1]. Pazy's proof uses the fact that $L^p(\mathbb{R}_+;X)$ is a Banach space. In our case, due to the strong measurability of $\mathbf{T}$, the maps $f_x(t) = T(t)x$ define elements $f_x \in E(X)$, which is a Banach space by a modification of the argument in [Z1, Thm. 64.2].

Strong measurability need not be assumed, however. In fact, Theorem 4.2 can be deduced from Theorem 2.3 by applying the latter to the operator $T(1)$. We will outline the argument; the routine details are left to the reader.

Let E be a Banach function space over $\mathbb{R}_+$. Let $E_{\mathbb{N}}$ be the set of all functions f on $\mathbb{N}$ for which

$$g := \sum_{n=0}^{\infty} f(n)\chi_{[n,n+1]} \in E.$$

For $f \in E_{\mathbb{N}}$ we define

$$\|f\|_{E_{\mathbb{N}}} := \Big\|\sum_{n=0}^{\infty} f(n)\chi_{[n,n+1]}\Big\|_E.$$

One easily checks that the space $E_{\mathbb{N}}$ is a Banach function space over $\mathbb{N}$. Moreover, if $\Psi_E(\infty) = \infty$, then also $\Psi_{E_{\mathbb{N}}}(\infty) = \infty$.

Now let $\mathbf{T}$ and E be as in Theorem 4.2. We claim that if $f_x(t) := \|T(t)x\|$ belongs to E for all $x \in X$, then $g_x(n) := \|T(n+1)x\|$ belongs to $E_{\mathbb{N}}$ for all $x \in X$. Indeed, first one shows, as in [P], that $\mathbf{T}$ is bounded. After replacing the norm of X by the equivalent norm defined by $|\!|\!|x|\!|\!| := \sup_{t\geqslant 0}\|T(t)x\|$, we may assume that each map $t \mapsto |\!|\!|T(t)x|\!|\!|$ is non-increasing. Then the claim readily follows from $\sum_n g_x(n)\chi_{[n,n+1]}(\cdot) \leqslant |\!|\!|T(\cdot)x|\!|\!| \leqslant C\|T(\cdot)x\|$, the ideal property of Banach function spaces and the fact that $\|T(\cdot)x\| \in E$.

In particular, for all $x \in X$ and $x^* \in X^*$ the map $n \mapsto \langle x^*, T(n+1)x\rangle$ belongs to $E_{\mathbb{N}}$. Now we use the following variant of Theorem 2.3: if the maps $n \mapsto \langle x^*, T^{n+1}x\rangle$ belong to E for all $x \in X$ and $x^* \in X^*$, then $r(T) < 1$. This follows by a simple modification of the proof of Theorem 2.3. Therefore, $r(T(1)) < 1$ and Theorem 4.2 follows from the formula $r(T(t)) = e^{\omega(\mathbf{T})t}$. ////

Remark 4.3.

(i) The hypotheses of Theorem 4.2 imply that the maps $\|T(\cdot)x\|$ have to be measurable (which is much weaker than strong measurability of $T(\cdot)x$). In fact, Theorem 4.2 remains true if we only assume that for each $x \in X$ there exists a function $g_x \in E$ such that $\|T(t)x\| \leqslant g_x(t)$ a.e. Indeed, choose $\omega \in \mathbb{R}$ so large that $\|T(t)\| \leqslant Me^{\omega t}$. Then the semigroup $\mathbf{S}$ defined by $S(t) = e^{-\omega t}T(t)$ is bounded. We change to the equivalent norm $|\!|\!|\cdot|\!|\!|$ (which is now defined in terms of $\mathbf{S}$). The functions $|\!|\!|S(\cdot)x|\!|\!|$ are non-decreasing, hence measurable. Hence also $|\!|\!|T(\cdot)x|\!|\!|$ is measurable. Then from $|\!|\!|T(\cdot)x|\!|\!| \leqslant Cg_x(t)$ for some constant C, it follows that $|\!|\!|T(\cdot)x|\!|\!| \in E$.

The same observation applies to Theorems 4.4 and 4.7 below.

(ii) Also, in Theorem 4.2 it is implicitly assumed that $\chi_{[0,t]} \in E$ for all $t > 0$ (cf. the end of Section 1). If $\mathbf{T}$ is not nilpotent, this is automatically implied by the other hypotheses. Indeed, for each $t > 0$ there is an $x \in X$ such that $T(t)x \neq 0$. But then also $T(s)x \neq 0$ for all $s \in [0,t]$, and it follows that $\chi_{[0,t]} \leqslant C^{-1}\|T(\cdot)x\|$, where $C = \inf_{s\in[0,t]} \|T(s)x\|$. Since $\|T(\cdot)x\| \in E$, also $\chi_{[0,t]} \in E$. Note that $C > 0$; this follows easily from the local boundedness of $\mathbf{T}$.
On the other hand, if $\mathbf{T}$ is nilpotent, then $\omega(\mathbf{T}) = -\infty$ and there is nothing to prove.

As in Theorem 3.3 we can prove:

Theorem 4.4. *Let $\mathbf{T}$ be a locally bounded semigroup on X. Suppose $\phi : \mathbb{R}_+ \to \mathbb{R}_+$ is non-decreasing with $\phi(0) = 0$ and $\phi(t) > 0$ for all $t > 0$. If*

$$\int_0^\infty \phi\Big(\|T(t)x\|\Big)\,dt < \infty, \qquad \forall \|x\| \leqslant 1,$$

then $\omega(\mathbf{T}) < 0$.

Next we assume a Δ_2-contition at 0on ϕ and try to prove an analogue of Theorem 3.4. Arguing as in the proof of 3.4, from the finiteness of the integrals $\int_0^\infty \phi\big(\alpha(t)\|T(t)x\|\big)\,dt$, one would like to conclude that $\alpha(\cdot)\|T(\cdot)x\| \in L^\infty(\mathbb{R}_+)$ in order to apply Lemma 3.2. This seems problematic, however. There is another approach which does not refer to the theory of Orlicz spaces, but instead uses the following non-trivial result of Müller [Mü].

Lemma 4.5. *Let T be a bounded operator on a Banach space X with spectral radius $r(T) \geqslant 1$, and let $0 < \varepsilon < 1$. Then for all $(\gamma_n) \in c_0$ of norm one there exists a norm one vector $x \in X$ such that*

$$\|T^k x\| \geqslant \varepsilon|\gamma_k|, \quad \forall k = 0, 1, 2, ...$$

If $\mathbf{T}$ is a locally bounded semigroup, we can apply the lemma to the operator $T(1)$ and obtain:

Lemma 4.6. *Let $\mathbf{T}$ be a locally bounded semigroup on X with $\omega(\mathbf{T}) \geqslant 0$. Then there is a constant $C > 0$ with the following property. For all $\gamma \in C_0(\mathbb{R}_+)$ of norm one there exists a norm one vector $x \in X$ such that*

$$\|T(t)x\| \geqslant C|\gamma(t)|, \quad \forall t \geqslant 0.$$

Proof: Put $M = \sup_{0\leqslant t\leqslant 1} \|T(t)\|$ and note that whenever $t \geqslant s \geqslant 0$, $t - s \leqslant 1$, we have $\|T(s)x\| \geqslant M^{-1}\|T(t)x\|$.

Define $\beta \in C_0(\mathbb{R}_+)$ of norm one as follows. First let $\alpha \in C_0(\mathbb{R}_+)$ be a norm-one function such that $\alpha(t) \downarrow 0$ and $\alpha \geqslant |\gamma|$. Then define $\beta \in C_0(\mathbb{R}_+)$ by

$$\beta(t) = \begin{cases} \alpha(0), & 0 \leqslant t < 1; \\ \alpha(t-1), & t \geqslant 1. \end{cases}$$

Let $T := T(1)$. By Lemma 4.5, we can choose $x \in X$ of norm one such that $\|T^k x\| \geqslant \frac{1}{2}\beta(k)$ for all k. For $t \in \mathbb{R}$ we let $E(t)$ be the integer part of t. Then for all $t \geqslant 0$ we have

$$\begin{aligned}\|T(t)x\| &\geqslant M^{-1}\|T(E(t)+1)x\| \geqslant \frac{1}{2}M^{-1}\beta(E(t)+1)\\ &\geqslant \frac{1}{2}M^{-1}\alpha(E(t)) \geqslant \frac{1}{2}M^{-1}\alpha(t) \geqslant \frac{1}{2}M^{-1}|\gamma(t)|.\end{aligned}$$

////

Before we proceed, we note that this lemma immediately leads to an alternative proof of Theorem 4.4: apply the lemma to any norm one function $\gamma \in C_0(\mathbb{R}_+)$ such that $\int_0^\infty \phi\big(C|\gamma(t)|\big)\,dt = \infty$.

Theorem 4.7. *Let* $\mathbf{T}$ *be a locally bounded semigroup on* X*. Let* $\phi : \mathbb{R}_+ \to \mathbb{R}_+$ *be a non-decreasing function satisfying a* Δ_2*-condition with* $\phi(0) = 0$ *and* $\phi(t) > 0$ *for* $t > 0$*, and let* α *be a non-negative measurable function on* $\mathbb{R}_+$ *such that* $\phi \circ \alpha \in L^1_{loc}(\mathbb{R}_+)$ *and*

$$\int_0^\infty \phi\big(\alpha(t)\big)dt = \infty.$$

If

$$\int_0^\infty \phi\Big(\alpha(t)\|T(t)x\|\Big)\,dt < \infty, \quad \forall \|x\| \leqslant 1,$$

then $\omega(\mathbf{T}) < 0$.

Proof: Put $t_0 := 0$ and let $t_1 > 0$ be so large that $\int_0^{t_1} \phi(\alpha(t))\,dt \geqslant 1$. Inductively, suppose $t_1 < ... < t_{n-1}$ have been chosen such that

$$\int_{t_{k-1}}^{t_k} \phi(2^{-k+1}\alpha(t))\,dt \geqslant 1, \quad k = 1, ..., n-1.$$

Since $\phi \circ \alpha \in L^1_{loc}(\mathbb{R}_+)$, we have $\int_{t_{n-1}}^\infty \phi(\alpha(t))\,dt = \infty$. Hence also

$$\int_{t_{n-1}}^\infty \phi(2^{-n+1}\alpha(t))\,dt = \infty$$

by the Δ_2-condition. Therefore, for $t_n > t_{n-1}$ large enough,

$$\int_{t_{n-1}}^{t_n} \phi(2^{-n+1}\alpha(t))\,dt \geqslant 1.$$

This completes the induction step.

Suppose, for a contradiction, that $\omega(\mathbf{T}) \geqslant 0$. Let $\gamma \in C_0(\mathbb{R}_+)$ be a norm one function such that $\gamma(t) \geqslant 2^{-n+1}$ for $t \in [t_{n-1}, t_n]$; $n = 1, 2, ...$ Fix $m \in \mathbb{N}$

such that $2^{-m} < C$, where C is the constant of Lemma 4.6. By that lemma, there is a norm one vector $x \in X$ such that $\|T(t)x\| \geqslant 2^{-m}\gamma(t)$ for all $t \geqslant 0$. But then, using the Δ_2-condition and the fact that $\alpha \leqslant \varepsilon$,

$$\int_0^\infty \phi(\alpha(t)\|T(t)x\|)\ dt \geqslant K^{-m}\int_0^\infty \phi(\alpha(t)\gamma(t))\ dt$$
$$\geqslant K^{-m}\sum_{n=1}^\infty \int_{t_{n-1}}^{t_n} \phi(2^{-n+1}\alpha(t))\ dt = \infty.$$

This contradiction concludes the proof. ////

Theorem 3.5 is proved in the same way.

An interesting special case of Theorem 4.7 is the following improvement of the Datko-Pazy theorem, which can also be obtained more directly along the lines of Corollary 2.5.

Corollary 4.8. *If* $\mathbf{T}$ *is a locally bounded semigroup with*

$$\int_0^\infty \beta(t)\|T(t)x\|^p\ dt < \infty \quad \forall x \in X,$$

where $0 \leqslant \beta \in L^1_{loc}(\mathbb{R}_+)$ *satisfies* $\int_0^\infty \beta(t)\ dt = \infty$, *then* $\omega(\mathbf{T}) < 0$.

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