Sandwiching C_0–semigroups

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Abstract - Let $\mathbf{T} = \{T(t)\}_{t\geqslant 0}$ be a C_0–semigroup on a Banach space X. We prove the following results:

(i) If X is separable, there exist separable Hilbert spaces X_0 and X_1, continuous dense embeddings $j_0 : X_0 \to X$ and $j_1 : X \to X_1$, and C_0–semigroups $\mathbf{T}_0$ and $\mathbf{T}_1$ on X_0 and X_1 respectively, such that $j_0 \circ T_0(t) = T(t) \circ j_0$ and $T_1(t) \circ j_1 = j_1 \circ T(t)$ for all $t \geqslant 0$.

(ii) If $\mathbf{T}$ is $\odot$–reflexive, there exist reflexive Banach spaces X_0 and X_1, continuous dense embeddings $j : D(A^2) \to X_0$, $j_0 : X_0 \to X$, $j_1 : X \to X_1$, and C_0–semigroups $\mathbf{T}_0$ and $\mathbf{T}_1$ on X_0 and X_1 respectively, such that $T_0(t) \circ j = j \circ T(t)$. $j_0 \circ T_0(t) = T(t) \circ j_0$ and $T_1(t) \circ j_1 = j_1 \circ T(t)$ for all $t \geqslant 0$, and such that $\sigma(A_0) = \sigma(A) = \sigma(A_1)$, where A_k is the generator of $\mathbf{T}_k$, $k = 0, \varnothing, 1$.

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0. Introduction

In this paper we investigate the following problem: given a strongly continuous semigroup of bounded linear operators (briefly, a C_0-semigroup) $\mathbf{T} = \{T(t)\}_{t\geqslant 0}$ on a Banach space X, is it possible to find spaces X_0 and X_1, continuous dense embeddings $j_0 : X_0 \to X$ and $j_1 : X \to X_1$, and C_0-semigroups $\mathbf{T}_0$ and $\mathbf{T}_1$ on X_0 and X_1, respectively, such that $j_0 \circ T_0(t) = T(t) \circ j_0$ and $T_1(t) \circ j_1 = j_1 \circ T(t)$ for all $t \geqslant 0$?

Naturally, this question is only meaningful if we require the spaces X_0 and X_1 and/or the semigroups $\mathbf{T}_0$ and $\mathbf{T}_1$ in some sense to be 'better' than X and $\mathbf{T}$. Below we provide two affirmative answers:

(i) If X is separable, then X_0 and X_1 may be chosen to be separable *Hilbert* spaces;
(ii) If $\mathbf{T}$ is $\odot-$reflexive, then X_0 and X_1 may be chosen to be *reflexive* Banach spaces, we may choose X_0 to be intermediate between $D(A^2)$ and X, and we may arrange that $\sigma(A_0) = \sigma(A) = \sigma(A_1)$, where A_k is the generator of $\mathbf{T}_k$, $k = 0, \varnothing, 1$.

These results are proved in Sections 1 and 2, respectively.

1. Sandwiching between Hilbert space semigroups

Throughout this section, we fix a Banach space X, a C_0-semigroup $\mathbf{T}$ on X. If H is a Banach space which is continuously embedded into X, for each $t > 0$ we define the linear subspace H_t of X by

$$H_t = \left\{ \int_0^t T(s)ih(s)\,ds : \ h \in L^2([0,t];H) \right\};$$

here $i : H \subset X$ denotes the inclusion mapping.

Theorem 1.1. *If H is a reflexive Banach space which is continuously embedded into X, then there exists another reflexive Banach space X_0, continuously embedded in X, such that:*

(i) $\mathbf{T}$ restricts to a C_0-semigroup $\mathbf{T}_0$ on X_0;
(ii) $H_t \subset X_0$ for all $t > 0$.

If X is separable, then H is separable as well. If H is a Hilbert space, then X_0 may be chosen to be a Hilbert space as well.

Proof: The space $\mathcal{H} := L^2([0,\infty);H)$ is reflexive; if H is a Hilbert space, then $\mathcal{H}$ is a Hilbert space as well. Fix $M > 0$ and $\omega \in \mathbb{R}$ such that $\|T(t)\| \leqslant Me^{\omega t}$ for all $t \geqslant 0$, and fix $\alpha > \omega$. Define $S : \mathcal{H} \to X$ by

$$Sh := \int_0^\infty e^{-\alpha t}T(t)ih(t)\,dt.$$

We check that this integral exists as a Bochner integral in X and that S is a bounded operator from $\mathcal{H}$ into X. The integrand is strongly measurable, and

$$\begin{aligned}\int_0^\infty e^{-\alpha t}\|T(t)ih(t)\|_X\,dt &\leqslant \int_0^\infty e^{-\alpha t}\|T(t)\|_{\mathcal{L}(X)}\|i\|_{\mathcal{L}(H,X)}\|h(t)\|_H\,dt \\ &\leqslant M\|i\|_{\mathcal{L}(H,X)}\int_0^\infty e^{-(\alpha-\omega)t}\|h(t)\|_H\,dt \\ &\leqslant M\|i\|_{\mathcal{L}(H,X)}\Big(\int_0^\infty e^{-2(\alpha-\omega)t}\,dt\Big)^{1/2}\|h\|_{\mathcal{H}} =: C\|h\|_{\mathcal{H}}.\end{aligned}$$

On $X_0 := \operatorname{range} S$ we define a norm $\|\cdot\|_{X_0}$ by $\|Sh\|_{X_0} := \|\pi h\|_{\mathcal{H}/\ker S}$, where $\pi : \mathcal{H} \to \mathcal{H}/\ker S$ is the quotient map. The resulting space X_0 is isometrically isomorphic to $\mathcal{H}/\ker S$ and therefore reflexive; if H is a Hilbert space then X_0 is a Hilbert space as well. If X is separable, then also H is separable.

The quotient operator $\tilde{S} : \mathcal{H}/\ker S \to X$ defined by $\tilde{S}(\pi h) := Sh$, has norm $\leqslant C$. Consequently,

$$\|Sh\|_X = \|\tilde{S}(\pi h)\|_{\mathcal{H}/\ker S} \leqslant C\|\pi h\|_{\mathcal{H}/\ker S} = C\|Sh\|_{X_0}.$$

It follows that the inclusion $X_0 \subset X$ is continuous.

For $s \geqslant 0$ and $h \in \mathcal{H}$, define $h_s \in \mathcal{H}$ by

$$h_s(t) := \begin{cases} h(t-s), & t \geqslant s; \\ 0, & \text{otherwise.}\end{cases}$$

Then

$$T(s)(Sh) = \int_0^\infty e^{-\alpha t}T(t+s)ih(t)\,dt = e^{\alpha s}\int_s^\infty e^{-\alpha t}T(t)ih(t-s)\,dt = e^{\alpha s}Sh_s.$$

Hence, $T(s)(Sh) \in X_0$. If $g \in \mathcal{H}$ is such that $Sg = Sh$, then the above identity shows that $Sg_s = Sh_s$, which implies that $\|\pi h_s\|_{\mathcal{H}/\ker S} \leqslant \|\pi h\|_{\mathcal{H}/\ker S}$. Consequently,

$$\|T(s)(Sh)\|_{X_0} = e^{\alpha s}\|\pi h_s\|_{\mathcal{H}/\ker S} \leqslant e^{\alpha s}\|\pi h\|_{\mathcal{H}/\ker S} = e^{\alpha s}\|Sh\|_{X_0}.$$

It follows that $T(s)$ restricts to a bounded operator $T_0(s)$ on X_0 of norm $\leqslant e^{\alpha s}$. We check that the resulting semigroup $\mathbf{T}_0$ is strongly continuous on X_0. Let $i_0 : X_0 \subset X$ denote the inclusion mapping. By dominated convergence, for all $x^* \in X^*$ and $h \in \mathcal{H}$ we have

$$\begin{aligned}\lim_{s\downarrow 0}\langle T_0(s)Sh, i_0^*x^*\rangle &= \lim_{s\downarrow 0}\int_0^\infty e^{-\alpha t}\langle T(t+s)ih(t), x^*\rangle\,dt \\ &= \int_0^\infty e^{-\alpha t}\langle T(t)ih(t), x^*\rangle\,dt = \langle Sh, i_0x^*\rangle.\end{aligned}$$

By the reflexivity of X_0, the restriction mapping $i_0^* : X^* \to X_0^*$ has dense range. Since $\mathbf{T}_0$ is locally bounded on X_0, it follows that $\mathbf{T}_0$ is weakly continuous on X_0. Hence by a standard result from semigroup theory [Pz, Theorem 2.1.4], $\mathbf{T}_0$ is strongly continuous on X_0. This proves (i).

To prove (ii) fix $t > 0$ and $h \in L^2([0,t], H)$. Define $\tilde{h} \in \mathcal{H}$ by

$$\tilde{h}(s) := \begin{cases} e^{\alpha s} h(s), & s \in [0,t]; \\ 0, & \text{otherwise.} \end{cases}$$

Then

$$\int_0^t T(s) i h(s)\, ds = S\tilde{h} \in X_0,$$

and therefore $H_t \subset X_0$ by definition of H_t. ∎

If $\mathbf{T}$ is uniformly exponentially stable, then X_0 may be chosen in such a way that $\mathbf{T}_0$ is uniformly exponentially stable on X_0: one takes $\omega < \alpha < 0$ in the above proof. This observation also applies to the results below, but we have no particular application for it. We do not know whether, in case $\mathbf{T}$ is uniformly bounded, it is possible to choose X_0 in such a way that $\mathbf{T}_0$ is uniformly bounded as well. However, using a weighted L^2 space to define $\mathcal{H}$, the proof of Theorem 1.1 can be modified to obtain $\mathbf{T}_0$ with at most linear growth.

The following theorem is in some sense 'dual' to Theorem 1.1. It depends on the following simple observation:

Lemma 1.2. *Suppose X is a separable Banach space. There exists a separable Hilbert space H which is densely and continuously embedded in X.*

Proof: Let (x_n) be a sequence of norm one vectors in X with dense linear span. Define a bounded operator $j : l^2 \to X$ by $j : (\alpha_n) \mapsto \sum_n n^{-1}\alpha_n x_n$. The restriction of j to $H := (\ker j)^\perp$ is an embedding. ∎

Theorem 1.3. *If X is separable, there exists a separable Hilbert space X_1 and a continuous dense embedding $j : X \to X_1$ such that*

$$T_1(t) jx := jT(t)x \qquad (x \in X)$$

defines a C_0−semigroup $\mathbf{T}_1$ on X_1.

Proof: Define

$$X^{\odot} := \{x^* \in X^* : \lim_{t \downarrow 0} \|T^*(t)x^* - x^*\| = 0\}.$$

Thus, $X^{\odot}$ is the largest subspace of X^* on which the adjoint semigroup $\mathbf{T}^*$ acts in a strongly continuous way. The space $X^{\odot}$ is a norm-closed, weak*−dense, $\mathbf{T}^*$−invariant subspace of X^* which induces an equivalent norm in X in the sense that there exists a constant $M \geqslant 1$ such that

$$M^{-1}\|x\| \leqslant \sup\left\{|\langle x, x^{\odot}\rangle| : \ x^{\odot} \in X^{\odot},\ \|x^{\odot}\| \leqslant 1\right\} \leqslant \|x\|$$

for all $x \in X$ [Ne, Chapter 1]. The restriction of $\mathbf{T}^*$ to $X^\odot$ will be denoted by $\mathbf{T}^\odot$.

By the separability of X we may choose a separable closed $\mathbf{T}^\odot-$invariant subspace Y of $X^\odot$ which still induces an equivalent norm in X. Let H_Y be any separable Hilbert space which is densely embedded in Y; such a Hilbert space exists by Lemma 1.2. By Theorem 1.1 (i) there exists a continuously embedded, $\mathbf{T}^\odot-$invariant, separable Hilbert space Y_0 in Y such that $\mathbf{T}^\odot$ restricts to a C_0-semigroup on Y_0. Put $\mathbf{T}_0^\odot := \mathbf{T}^\odot|_{Y_0}$.

Since H_Y is dense in Y, (ii) of Theorem 1.1 shows that the inclusion $j : Y_0 \subset Y$ is dense. Therefore the adjoint map $j^* : Y^* \to Y_0^*$ is injective with dense range; here Y_0^* denotes the Banach space dual of the (Hilbert) space Y_0.

Since Y induces an equivalent norm in X, X is canonically isomorphic to a norm closed, weak$^*-$dense, $(\mathbf{T}^\odot|_Y)^*-$invariant subspace of Y^*. Under this identification, j^* restricts to an injective map from X into Y_0. We claim that this restriction still has dense range. Indeed, j^* is weak$^*-$to$-$weakly continuous as a map from Y^* to Y_0^*, being an adjoint operator taking values in a reflexive space. The claim now follows from the fact that X is weak$^*-$dense in Y^*.

We have obtained dense embedding $j_1 := j^*|_X$ from X into $X_1 := Y_0^*$. The adjoint semigroup $\mathbf{T}_1 := (\mathbf{T}_0^\odot)^*$ is a C_0-semigroup on X_1, being the adjoint of a strongly continuous semigroup on a reflexive space. For all $x \in X$ and $x_1 \in X_1$ we have

$$\begin{aligned}\langle x_1, T_1(t) j_1 x\rangle &= \langle x, jT_0^\odot(t) x_1\rangle \\ &= \langle x, T^\odot(t) j x_0\rangle \\ &= \langle T(t)x, jx_1\rangle \\ &= \langle x_1, j_1 T(t) x\rangle.\end{aligned}$$

This shows that $T_1(t) \circ j_1 = j_1 \circ T(t)$. Finally we observe that $X_1 = Y_0^*$ can be given the structure of a Hilbert space in a natural way by providing it with the inner product of Y_0. ∎

As a corollary we find that every semigroup on a separable Banach space is sandwiched between two Hilbert space semigroups:

Corollary 1.4. *If $\mathbf{T}$ is a C_0-semigroup on a separable Banach space X, then there exist separable Hilbert spaces X_0 and X_1, continuous dense embeddings $j_0 : X_0 \to X$ and $j_1 : X \to X_1$, and C_0-semigroups $\mathbf{T}_0$ and $\mathbf{T}_1$ on X_0 and X_1 respectively, such that $j_0 \circ T_0(t) = T(t) \circ j_0$ and $T_1(t) \circ j_1 = j_1 \circ T(t)$ for all $t \geqslant 0$.*

Proof: Apply Theorem 1.1 and Lemma 1.2 to obtain X_0 and $\mathbf{T}_0$, and apply Theorem 1.3 to obtain X_1 and $\mathbf{T}_1$. ∎

The constructions in Theorems 1.1 and 1.3 have their origin in the theory of stochastic differential equations (SDE's) on Hilbert spaces.

If $\mathbf{T}$ is C_0-semigroup on a Hilbert space X and $Q \in \mathcal{L}(X)$ is a positive self-adjoint operator such that

$$\sup_{t>0} \operatorname{Trace} Q_t < \infty,$$

where the positive self-adjoint operators $Q_t \in \mathcal{L}(X)$ are defined by

$$Q_t x := \int_0^t T(s)QT^*(s)x\,ds \qquad (x \in X),$$

then $Q_\infty x := \lim_{t\to\infty} Q_t x$ defines a positive self-adjoint operator Q_∞ which is the covariance operator of a unique centered Gaussian measure μ on X. It was shown in [CG] that the reproducing kernel Hilbert space H_∞ associated with this measure is $\mathbf{T}-$invariant; this space H_∞ is continuously embedded in X. This situation is covered by Theorem 1.1 in the following way. If we take H to be the reproducing kernel Hilbert space associated with Q (this space is continuously embedded in X) and let $\alpha = 0$, then the space X_0 constructed in Theorem 1.1 coincides with H_∞.

As to Theorem 1.3, a duality construction which in some sense resembles the one presented here was used in [BRS] to show that to every C_0-semigroup $\mathbf{T}$ on a Hilbert space X, another Hilbert space X_1 and a Hilbert-Schmidt embedding $j_1 : X \to X_1$ can be associated in such a way that $\mathbf{T}$ extends to a C_0-semigroup $\mathbf{T}_1$ on X_1. This result is applied to the study of SDE's with cylindrical noise.

2. The $\odot-$reflexive case

In this section we will prove versions of the above results for $\odot$-reflexive semigroups. It turns out that for this class of semigroups it is possible to control the spectra of the generators of $\mathbf{T}_0$ and $\mathbf{T}_1$, the price to pay being that X_0 and X_1 are obtained only as reflexive Banach spaces.

We start with a lemma about equality of spectra, which may be compared to [Ar, Proposition 1.1].

Lemma 2.1. *Let A be a closed operator with domain $D(A)$ on a Banach space X. Suppose X_0 is a Banach space such that $D(A^2) \subset X_0 \subset X$ with continuous inclusions. Denote the part of A in X_0 by A_0. If $\varrho(A) \cap \varrho(A_0) \neq \emptyset$, then $\sigma(A_0) = \sigma(A)$.*

Proof: First we prove the inclusion $\varrho(A) \subset \varrho(A_0)$. Pick $\lambda \in \varrho(A)$ and fix an arbitrary $\mu \in \varrho(A) \cap \varrho(A_0)$. If $x_0 \in X_0$, then $R(\lambda, A)R(\mu, A)x_0 \in D(A^2) \subset X_0$. Therefore,

$$R(\lambda, A)x_0 = R(\mu, A_0)x_0 + (\mu - \lambda)R(\lambda, A)R(\mu, A)x_0 \in X_0.$$

Hence $R(\lambda, A)X_0 \subset X_0$, and the restriction $R(\lambda, A)|_{X_0}$ defines a bounded operator on X_0 by the closed graph theorem. Clearly $R(\lambda, A)|_{X_0}$ is a two-sided inverse for $\lambda - A_0$, so $\lambda \in \varrho(A_0)$.

To prove the inclusion $\varrho(A_0) \subset \varrho(A)$, pick $\lambda \in \varrho(A_0)$ and fix an arbitrary $\mu \in \varrho(A)$. Define a bounded operator R_λ on X by

$$R_\lambda x := R(\mu, A)x + (\mu - \lambda)R(\mu, A)^2 x + (\mu - \lambda)^2 R(\lambda, A_0)R(\mu, A)^2 x \qquad (x \in X).$$

Then it is easily verified that R_λ is a two-sided inverse of $\lambda - A$, so $\lambda \in \varrho(A)$. ∎

We now return to the setting of Section 1 and assume that A is the generator of a C_0-semigroup $\mathbf{T}$ on X.

Lemma 2.2. *If H is a Banach space such that $D(A) \subset H \subset X$ with continuous inclusions, then $D(A^2) \subset H_t$ for all $t > 0$.*

Proof: Fix $t > 0$ and choose $\omega \in \varrho(A) \cap \mathbb{R}$ so large that $\|e^{-\omega t}T(t)|_{D(A)}\|_{\mathcal{L}(D(A))} < 1$. Then the restriction to $D(A)$ of $I - e^{-\omega t}T(t)$ is invertible in $D(A)$, and for $x \in D(A)$ we have

$$\begin{aligned}(\omega - A)^{-1}x &= (I - e^{-\omega t}T(t))(\omega - A)^{-1}(I|_{D(A)} - e^{-\omega t}T(t)|_{D(A)})^{-1}x \\ &= -\int_0^t e^{-\omega s}T(s)(I|_{D(A)} - e^{-\omega t}T(t)|_{D(A)})^{-1}x\,ds.\end{aligned}$$

But

$$(I|_{D(A)} - e^{-\omega t}T(t)|_{D(A)})^{-1}x = \sum_{n=0}^{\infty} e^{-n\omega t}T(nt)x \in D(A) \subset H,$$

the sum being absolutely convergent in $D(A)$. Therefore the function h defined by

$$h(s) := -e^{-\omega s}(I|_{D(A)} - e^{-\omega t}T(t)|_{D(A)})^{-1}x \qquad (s \in [0,t])$$

belongs to $L^2([0,t]; H)$. From

$$(\omega - A)^{-1}x = \int_0^t T(s)h(s)\,ds,$$

we conclude that $(\omega - A)^{-1}x \in H_t$. ∎

Denote $X^{\odot *} := (X^{\odot})^*$ and $X^{\odot\odot} := (X^{\odot})^{\odot}$, the $\odot-$dual of $X^{\odot}$ with respect to the C_0-semigroup $\mathbf{T}^{\odot}$. Define a map $k : X \to X^{\odot *}$ by

$$\langle x^{\odot}, kx\rangle := \langle x, x^{\odot}\rangle \qquad (x^{\odot} \in X^{\odot}).$$

Since $X^{\odot}$ induces an equivalent norm in X, the map k is an isomorphic embedding, and it is easy to see that $kX \subset X^{\odot\odot}$. If $kX = X^{\odot\odot}$, then $\mathbf{T}$ is said to be $\odot-$*reflexive*. By a theorem of de Pagter [Pa] (cf. also [Ne, Theorem 2.5.2]), this happens if and only if there exists $\mu \in \varrho(A)$ such that $R(\mu, A)$ is a weakly compact operator; in this case $R(\mu, A)$ is weakly compact for all $\mu \in \varrho(A)$.

Lemma 2.3. *If $\mathbf{T}$ is $\odot-$reflexive, then there exists a reflexive Banach space H such that $D(A) \subset H \subset X$ with continuous inclusions.*

Proof: Fix any $\lambda \in \varrho(A)$. Then $R(\lambda, A)$ is weakly compact. If $(x_n) \subset D(A)$ is a sequence which is bounded with respect to the graph norm of $D(A)$, then $((\lambda - A)x_n)$ is a bounded sequence in X and therefore the identity $x_n = R(\lambda, A)((\lambda - A)x_n)$ shows that the sequence (x_n) is relatively weakly compact in X. This shows that the inclusion $D(A) \subset X$ is weakly compact. Hence by the factorization theorem of Davis-Figiel-Johnson-Pelczynski [DFJP], it factors through a reflexive Banach space. Accordingly there exists a reflexive Banach space H_0 and bounded operators $T_0 : D(A) \to H_0$ and $T_1 : H_0 \to X$ such that $T_1 T_0 x = x$ for all $x \in D(A)$. Let $H := \operatorname{range} T_1$; H is a reflexive Banach space with respect to the norm $\|T_1 h_0\| := \|\pi h_0\|_{H_0/\ker T_1}$, where $\pi : H_0 \to H_0/\ker T_1$ is the quotient mapping. We now have $D(A) \subset H \subset X$ with continuous inclusions, the first one being given by $T_1 \circ T_0$. ∎

Theorem 2.4. *Suppose* $\mathbf{T}$ *is a* $\odot-$*reflexive semigroup on a Banach space* X*. Then there exist reflexive Banach spaces* X_0 *and* X_1 *and a continuous dense embedding* $j : X \to X_1$ *such that:*

(i) $D(A^2) \subset X_0 \subset X$ *with continuous and dense inclusions;*
(ii) $\mathbf{T}$ *restricts to a* C_0-*semigroup* $\mathbf{T}_0$ *on* X_0*;*
(iii) $T_1(t)jx := jT(t)x$ $(x \in X)$ *defines a* C_0-*semigroup on* X_1*;*
(iv) $\sigma(A_0) = \sigma(A) = \sigma(A_1)$*, where* A_k *is the generator of* $\mathbf{T}_k$*,* $k = 0, 1$*.*

Proof: Let X_0 be the space of Theorem 1.1. By Lemmas 2.3 and 2.2 we have $D(A^2) \subset H_t$ for all $t > 0$, and therefore $D(A^2) \subset X_0$ by Theorem 1.1 (ii). By the closed graph theorem this inclusion is continuous. Since $D(A^2)$ is dense in X, the inclusion $X_0 \subset X$ is dense. From $D(A_0^2) \subset D(A^2) \subset X_0$ we see that the inclusion $D(A^2) \subset X_0$ is dense as well. This proves (i) and (ii). Equality of the spectra $\sigma(A_0) = \sigma(A)$ follows from Lemma 2.1 and the easy observation that A_0 is indeed the part of A in X_0.

The space X_1 is constructed as in the proof of Theorem 1.3, except for the following modifications. We now take $Y := X^\odot$, notice that the strongly continuous adjoint semigroup $\mathbf{T}^\odot$ is $\odot-$reflexive (see e.g. [Ne, Corollary 2.5.8]), and apply Lemma 2.3 to see that there exists a reflexive space Y_0 such that:

(i) $D((A^\odot)^2) \subset Y_0 \subset Y = X^\odot$ with dense inclusions;
(ii) $\mathbf{T}^\odot$ restricts to a C_0-semigroup $\mathbf{T}_0^\odot$ on Y_0;
(iii) $\sigma(A_0^\odot) = \sigma(A^\odot)$.

The adjoint $j^* : X^{\odot *} \to X_1 := Y_0^*$ of the inclusion $j : Y_0 \subset X^\odot$ is injective with dense range, and its restriction $j_1 = j^*|_X$ to X has dense range in X_1 again. Recalling the spectra of a generator, its adjoint, and its $\odot-$adjoint always agree, it follows that for the adjoint semigroup $\mathbf{T}_1 := (\mathbf{T}_0^\odot)^*$ on X_1 we have $\sigma(A_1) = \sigma(A_0^\odot) = \sigma(A^\odot) = \sigma(A)$; cf. [Ne, Section 1.4]. ∎

Remark 2.5.

(i) It would be interesting to know whether $D(A^2)$ can be replaced by $D(A)$ in the above result.

(ii) Assertions (i) and (ii) of Theorem 2.4 actually characterize $\odot-$reflexive semigroups. In fact, if the inclusion mapping $D(A^2) \subset X$ factors through a reflexive Banach space Y, then by factoring $R(\mu, A)^2$ through $D(A^2)$ it follows that $R(\mu, A)^2$ factors through Y as well, and therefore $R(\mu, A)^2$ is weakly compact. It is easy to prove [Pa] that then also $R(\mu, A)$ is weakly compact, and hence $\mathbf{T}$ is $\odot-$reflexive.

(iii) The 'metamathematical' interpretation of Theorem 2.4 is as follows. Suppose $\mathcal{C}$ is a set of conditions which implies a certain property $\mathcal{P}$ for C_0-semigroups on reflexive Banach spaces. Then $\mathcal{C}$ also implies property $\mathcal{P}$ for $\odot-$reflexive semigroups, provided the conditions $\mathcal{C}$ are stable under similarity transformations and both $\mathcal{C}$ and $\mathcal{P}$ are stable under continuous dense inclusions. Indeed, if $\mathbf{T}$ is $\odot-$reflexive and verifies the conditions $\mathcal{C}$, then its restriction $\mathbf{T}_{D(A^2)}$ to $D(A^2)$ also verifies $\mathcal{C}$ (use the similarity transformation $T_{D(A^2)}(t) = R(\lambda, A)^2 T(t)(\lambda - A)^2$).

Hence $\mathbf{T}_0$ verifies $\mathcal{C}$ as well (inject $D(A^2)$ into X_0). By reflexivity it follows that $\mathbf{T}_0$ has property $\mathcal{P}$, and therefore (by injecting into X) $\mathbf{T}$ has property $\mathcal{P}$.

This provides a canonical way of extending certain results for C_0-semigroups on reflexive spaces to arbitrary $\odot-$reflexive semigroups. ∎

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