STOCHASTIC INTEGRATION IN UMD SPACES

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We report on a joint work with Mark Veraar and Lutz Weis [6].

Building upon previous work by Rosiński and Suchanecki [8] and Brzeźniak and the author [1], a systematic theory of stochastic integration for Banach space-valued functions with respect to Brownian motions has been constructed in [7] using a recent idea of Kalton and Weis to study vector-valued functions through certain operator-theoretic properties of the associated integral operators [4]. In the work presented here, the results of [7] are extended to a theory of stochastic integration for stochastic processes taking values in a UMD space.

Let (γ_n) be a sequence of independent standard Gaussian random variables on some probability space $(\Omega, \mathbb{P})$. A bounded operator $T : H \to E$ acting from a separable real Hilbert space H with orthonormal basis (h_n) into a real Banach space E is said to be *γ-radonifying* if the Gaussian sum $\sum_n \gamma_n Th_n$ converges in $L^2(\Omega; E)$. This definition is independent of the choice of (γ_n) and (h_n), and the vector space $\gamma(H,E)$ of all γ-radonifying operators from H to E is a Banach space with respect to the norm $\|\cdot\|_{\gamma(H,E)}$ defined by

$$\|T\|^2_{\gamma(H,E)} := \mathbb{E}\Big\| \sum_n \gamma_n Th_n \Big\|^2.$$

Let $W = (W(t))_{t\geqslant 0}$ be a Brownian motion on $(\Omega, \mathbb{P})$. The main result of [7] can be formulated as follows.

Theorem 1 ([7]). *For a function $\psi : [0,T] \to E$ such that $\langle \psi, x^*\rangle \in L^2(0,T)$ for all $x^* \in E^*$, the following assertions are equivalent:*

(1) *For every measurable set $A \subseteq [0,T]$ there exists an E-valued random variable η_A such that for all $x^* \in E^*$ we have*
$$\langle \eta_A, x^*\rangle = \int_A \langle \phi(t), x^*\rangle \, dW(t) \quad \textit{almost surely};$$

(2) *There exists an operator $S_\psi \in \gamma(L^2(0,T), E)$ such that for all $f \in L^2(0,T)$ and $x^* \in E^*$ we have*
$$\langle S_\psi f, x^*\rangle = \int_0^T f(t)\langle \psi(t), x^*\rangle \, dt.$$

Writing $\eta_A = \int_A \psi(t)\, dW(t)$, for all $1 \leqslant p < \infty$ we have $\mathbb{E}\Big\| \int_0^T \psi(t)\, dW(t)\Big\|^p \eqsim_p \|S_\psi\|^p_{\gamma(L^2(0,T),E))}$, with equality for $p = 2$.

If the equivalent conditions of the theorem are satisfied, then ψ is said to be *stochastically integrable* with respect to W.

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Denote by $\mathscr{F}^W = (\mathscr{F}_t^W)_{t\geqslant 0}$ the augmented filtration generated by W. A stochastic process $\phi : [0,T]\times\Omega \to E$ is said to be $\mathscr{F}^W$*-weakly progressive* if for all $x^*\in E^*$ the real-valued process $\langle \psi, x^*\rangle$ is progressively measurable with respect to $\mathscr{F}^W$. Such a process is said to be *elementary progressive* if it is of the form $\phi = \sum_{n=1}^N 1_{(t_n,t_{n+1}]}\otimes \xi_n$, where ξ_n is an $\mathscr{F}_{t_n}^W$-measurable simple random variable with values in E. Assuming that E is a UMD space, Garling [3] proved the following two-sided decoupling inequality for elementary progressive processes, valid for $1<p<\infty$:

$$\mathbb{E}_\Omega\Big\|\int_0^T \phi(t)\,dW(t)\Big\|^p \eqsim_{p,E} \mathbb{E}_{\Omega\times\tilde\Omega}\Big\|\int_0^T \phi(t)\,d\tilde W(t)\Big\|^p.$$

Here $\tilde W$ is a Brownian motion on a probability space $(\tilde\Omega,\tilde{\mathbb{P}})$ and the integral on the right hand side is defined pathwise with respect to Ω. By Fubini's theorem, the Kahane-Khinchine inequalities and Theorem 1, the right hand side is proportional to

$$\mathbb{E}_{\Omega\times\tilde\Omega}\Big\|\int_0^T \phi(t)\,d\tilde W(t)\Big\|^p \eqsim_p \mathbb{E}_\Omega\Big(\mathbb{E}_{\tilde\Omega}\Big\|\int_0^T \phi(t)\,d\tilde W(t)\Big\|^2\Big)^{p/2} = \mathbb{E}_\Omega\|S_\phi\|^p_{\gamma(L^2(0,T),E)},$$

where $S_\phi : \Omega\to\gamma(L^2(0,T),E)$ satisfies $\langle S_\phi(\omega)f, x^*\rangle = \int_0^T f(t)\langle\phi(t,\omega),x^*\rangle\,dt$ for all $f\in L^2(0,T)$ and $x^*\in E^*$ almost surely. As a consequence, the mapping $S_\phi\mapsto \int_0^T\phi(t)\,dW(t)$ extends to an isomorphism from the closure in $L^p(\Omega;\gamma(L^2(0,T),E))$ of the elementary progressive processes onto a certain closed subspace of $L^p(\Omega;E)$. Using a version of the Pettis measurability theorem for $\mathscr{F}^W$-measurable processes in combination with Itô's martingale representation theorem and approximation arguments, the range of this isomorphism can be identified as the subspace of all mean zero $\mathscr{F}_T^W$-measurable elements of $L^p(\Omega;E)$. The result is an extension of Itô's martingale representation theorem to UMD-valued processes, which is the main ingredient in the proof of the following theorem:

Theorem 2. *Let E be a UMD space and let $p\in(1,\infty)$. For a weakly progressive process $\phi:[0,T]\times\Omega\to E$ such that $\langle\phi,x^*\rangle\in L^p(\Omega;L^2(0,T))$ for all $x^*\in E^*$, the following assertions are equivalent:*

1. *For every measurable set $A\subseteq[0,T]$ there exists a random variable $\eta_A\in L^p(\Omega;E)$ such that for all $x^*\in E^*$ we have*
$$\langle\eta_A,x^*\rangle = \int_A\langle\phi(t),x^*\rangle\,dW(t) \quad \text{in } L^p(\Omega);$$
2. *There exists a random variable $S_\phi\in L^p(\Omega;\gamma(L^2(0,T),E))$ such that for all $f\in L^2(0,T)$ and $x^*\in E^*$ we have*
$$\langle S_\phi(\omega)f,x^*\rangle = \int_0^T f(t)\langle\phi(t,\omega),x^*\rangle\,dt \quad \text{for almost all } \omega\in\Omega.$$

Writing $\eta_A = \int_A\phi(t)\,dW(t)$ we have $\mathbb{E}\big\|\int_0^T\phi(t)\,dW(t)\big\|^p \eqsim_{p,E} \mathbb{E}\|S_\phi\|^p_{\gamma(L^2(0,T),E)}$.

If the equivalent conditions of the theorem are satisfied, then ϕ is said to be L^p-*stochastically integrable* with respect to W. Note that the scalar stochastic integral on the right hand side in (1) is well defined in $L^p(\Omega)$ by the Burkholder-Davis-Gundy inequalities. When combined with our generalized Itô representation theorem, the equivalence of norms in the last line of the theorem leads to Burkholder-Davis-Gundy inequalities for UMD-valued $\mathscr{F}^W$-martingales.

Theorem 2 can be applied to show that every continuous L^p-martingale $(M_t)_{t\geqslant 0}$ with respect to the filtration $\mathscr{F}^W$, with values in a UMD space E and satisfying $M_0 = 0$, is L^p-stochastically integrable with respect to W on every interval $[0,T]$ and satisfies

$$\mathbb{E}\Big\|\int_0^T M_t\,dW(t)\Big\|^p \lesssim_{p,E} T^{\frac{p}{2}}\,\mathbb{E}\|M_T\|^p.$$

In particular this applies to the continuous L^p-martingale $M_t := \int_0^t \phi(s)\,dW(s)$, where ϕ is an L^p-stochastically integrable process with values in E.

The idea to use decoupling inequalities to construct a theory of stochastic integration in UMD spaces is due to McConnell [5] who used convergence in probability rather than L^p-convergence. McConnell first generalized Garling's inequalities to obtain decoupling inequalities for tangent sequences with values in UMD spaces and used these to prove that a progressive process with values in a UMD space E is stochastically integrable if and only if its trajectories are stochastically integrable almost surely as E-valued functions. His arguments depend heavily on the equivalence of the UMD property and the geometric notion of ζ-convexity. By using stopping time arguments, our Theorem 2 can be localized to recover McConnell's result under somewhat weaker measurability assumptions. An advantage of this approach is that it uses the UMD property in a direct and elementary way through Garling's inequality. An Itô formula is obtained as well.

Our results can be extended to processes with values in $\mathscr{L}(H,E)$, where H is a separable real Hilbert space and E is a real UMD space; the integrator is then an H-cylindrical Brownian motion. In a subsequent paper we shall apply the results to the study of existence, uniqueness, and regularity of certain classes of nonlinear stochastic evolution equations in E, thereby extending parts of the theory of stochastic evolution equations in Hilbert spaces developed by Da Prato and Zabczyk [2] and many others, to the setting of UMD spaces.

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