The Norm of a Complex Banach Lattice

J.M.A.M. van Neerven [1]

Department of Mathematics
TU Delft
P.O. Box 5031, 2600 GA Delft
The Netherlands
J.vanNeerven@twi.tudelft.nl

In this note we study the problem how the complexification of a real Banach space can be normed in such a way that it becomes a complex Banach space from the point of view of the theory of cross-norms on tensor products on Banach spaces. In particular we show that the norm of a complex Banach lattice can be interpretated in terms of the l-tensor product of real Banach lattices.

1991 Mathematics Subject Classification: 46B42, 46M05
Keywords & phrases: Complexification of Banach space, cross-norm, Banach lattice, l-tensor product, cone absolutely summing operator, operator ideal.

0. Introduction

Let X be a real Banach space and let $X_{\mathbb{C}}$ be its complexification. There are various ways to introduce a norm on $X_{\mathbb{C}}$ which makes it into a complex Banach space. In this note we study this problem systematically by means of cross-norms. The main idea is the following. Regarding $X_{\mathbb{C}}$ as the tensor product $X \otimes \mathbb{C}$ and identifying the realification $(X_{\mathbb{C}})_{\mathbb{R}}$ of $X_{\mathbb{C}}$ with $X \otimes \mathbb{R}^2$ (both tensor products are with respect to $\mathbb{R}$), we show that every 'reasonable' norm making $X_{\mathbb{C}}$ into a complex Banach space is induced by a complex-homogenous cross-norm on $X \otimes \mathbb{R}^2$ and conversely. Thus the study of complex norms of $X_{\mathbb{C}}$ is reduced to that of cross-norms on $X \otimes \mathbb{R}^2$.

This is applied to Banach lattices as follows. The complexification $E_{\mathbb{C}}$ of a real Banach lattice E is a complex Banach lattice in the norm $\|z\| := \|\, |z| \,\|$, where $|z|$ is the complex modulus of an element $z \in E_{\mathbb{C}}$, which is defined in Section 2 below. We show that this norm is induced by the l-norm on $E \otimes \mathbb{R}^2$. This is the cross-norm induced on $E \otimes \mathbb{R}^2$ by the operator ideal $\mathcal{L}^l(E^*; \mathbb{R}^2)$ of cone absolutely summing operators.

It is interesting to observe at this point that there exist complex Banach spaces which cannot be obtained as the complexification of a real Banach space. The existence of such a space was proved by Bourgain [B] using probabilistic arguments; the first explicit example was constructed by Kalton [K].

---

[1] This work was carried out while the author was affiliated at the Centre for Mathematics and Computer Science (CWI) in Amsterdam.

1. The complexification of a real Banach space

Let X be a real vector space. The *complexification* of X is the complex vector space $X_{\mathbb{C}} := X \otimes \mathbb{C}$, with scalar multiplication defined by $\alpha(x \otimes \beta) := x \otimes \alpha\beta$ ($\alpha, \beta \in \mathbb{C}$). Here and in the rest of this note, tensor products are *real*.

Let Y be a complex vector space, with scalar multiplication $\mu : Y \times \mathbb{C} \to Y$. Let μ' be the restriction of μ to $Y \times \mathbb{R}$. The *realification* of Y is the real vector space $Y_{\mathbb{R}} := Y$ with scalar multiplication μ'. Thus as a set, Y and $Y_{\mathbb{R}}$ are the same. If X is a real vector space, then $(X_{\mathbb{C}})_{\mathbb{R}}$ can be identified with $X \otimes \mathbb{R}^2$ by the natural map

$$x \otimes (a + bi) \mapsto (x \otimes (a, b)).$$

In turn, $X \otimes \mathbb{R}^2$ can be identified with $X \times X$ by the natural map

$$x \otimes (a, b) \mapsto (ax, bx).$$

We will use the somewhat informal notation $x+iy$ for the element $x \otimes 1 + y \otimes i \in X_{\mathbb{C}}$ and (x, y) for the element $x \otimes (1, 0) + y \otimes (0, 1) \in X \otimes \mathbb{R}^2 = (X_{\mathbb{C}})_{\mathbb{R}}$.

Following [R], a norm $\|\cdot\|_{\mathbb{C}}$ on $X_{\mathbb{C}}$ will be called *admissible* if for all $x, y \in X$ we have

$$\max\bigl(\|x\|, \|y\|\bigr) \leqslant \|x + iy\|_{\mathbb{C}} \leqslant \|x\| + \|y\|.$$

Two admissible norms $\|\cdot\|_\infty$ and $\|\cdot\|_1$ are of special interest. They are defined by

$$\|x + iy\|_\infty := \sup_{0 \leqslant \theta \leqslant 2\pi} \|x \cos\theta + y \sin\theta\|,$$

$$\|x + iy\|_1 := \inf \sum_r |a_r + b_r i| \; \|x_r\|,$$

where the infimum is taken over all finite sequences $(a_r, b_r, x_r) \in \mathbb{R} \times \mathbb{R} \times X$ such that $\sum_r a_r x_r = x$ and $\sum_r b_r x_r = y$.

The following two propositions, taken from [R], summarise some properties of admissible norms. The notation $\langle \cdot, \cdot \rangle$ is used for the pairing between the dual space X^* and X.

Proposition 1.1.

(i) If a norm $\|\cdot\|_{\mathbb{C}}$ satisfies $\max(\|x\|, \|y\|) \leqslant \|x + iy\|_{\mathbb{C}}$, then it is admissible if and only if $\|x\|_{\mathbb{C}} = \|x\|$ holds for all $x \in X$.

(ii) The norms $\|\cdot\|_\infty$ and $\|\cdot\|_1$ are admissible. Moreover, if $\|\cdot\|_{\mathbb{C}}$ is any admissible norm on $X_{\mathbb{C}}$, then $\|\cdot\|_\infty \leqslant \|\cdot\|_{\mathbb{C}} \leqslant \|\cdot\|_1$;

(iv) $\|x + iy\|_\infty = \sup\bigl\{\bigl(\langle x^*, x\rangle^2 + \langle x^*, y\rangle^2\bigr)^{\frac{1}{2}} : x^* \in X^*, \; \|x^*\| \leqslant 1\bigr\}$.

The pairing

$$\langle x^* + iy^*, x + iy\rangle := \langle x^*, x\rangle - \langle y^*, y\rangle + i(\langle x^*, y\rangle + \langle y^*, x\rangle)$$

defines a natural vector space isomorphism $\psi : (X^*)_{\mathbb{C}} \to (X_{\mathbb{C}})^*$. If $\|\cdot\|$ is an admissible norm on $X_{\mathbb{C}}$, then ψ induces a norm on $(X^*)_{\mathbb{C}}$, for which we have the following.

Proposition 1.2. *The norm which is induced on $(X^*)_{\mathbb{C}}$ by ψ is admissible again.*

The norms $\|\cdot\|_\infty$ and $\|\cdot\|_1$ are dual to each other in the sense that ψ gives rise to isometrical isomorphisms $(X_{\mathbb{C}}, \|\cdot\|_1)^* = ((X^*)_{\mathbb{C}}, \|\cdot\|_\infty)$ and $(X_{\mathbb{C}}, \|\cdot\|_\infty)^* = ((X^*)_{\mathbb{C}}, \|\cdot\|_1)$.

Next we summarise some properties of cross-norms. For proofs and more information we refer to [DU]. Let X and Y be real Banach spaces. A norm $\|\cdot\|_\otimes$ on $X\otimes Y$ is said to be a *reasonable cross norm* (briefly, a *cross-norm*), if for all $x \in X, y \in Y, x^* \in X^*$ and $y^* \in Y^*$ we have
(i) $\|x \otimes y\|_\otimes = \|x\|\,\|y\|$;
(ii) $\|x^* \otimes y^*\|_\otimes = \|x^*\|\,\|y^*\|$.
Here $\|x^* \otimes y^*\|_\otimes$ is the norm of $x^* \otimes y^*$ regarded as the element of $(X \otimes Y, \|\cdot\|_\otimes)^*$ defined by

$$\langle x^* \otimes y^*, x \otimes y\rangle := \langle x^*, x\rangle\langle y^*, y\rangle.$$

Two cross-norms $\|\cdot\|_\varepsilon$ and $\|\cdot\|_\pi$ are of special interest. They are defined by

$$\|u\|_\varepsilon := \sup\big\{|\langle x^* \otimes y^*, u\rangle| : \|x^*\| \leqslant 1, \|y^*\| \leqslant 1\big\};$$
$$\|u\|_\pi := \inf \sum_n \|x_n\|\,\|y_n\|,$$

where the infimum is taken over all finite sequences $(x_n, y_n) \in X \times X$ such that u is representable as $u = \sum_n x_n \otimes y_n$. The following proposition is taken from [DU, Chapter 8].

Proposition 1.3. *Let $\|\cdot\|$ and $\|\cdot\|_\otimes$ be a norm resp. a cross-norm on $X \otimes Y$.*
(i) If for all x, y, x^* and y^* we have $\|x \otimes y\| \leqslant \|x\|\,\|y\|$ and $\|x^* \otimes y^*\| \leqslant \|x^*\|\,\|y^*\|$, then $\|\cdot\|$ is a cross-norm;
(ii) $\|\cdot\|_\varepsilon \leqslant \|\cdot\|_\otimes \leqslant \|\cdot\|_\pi$;
(iii) The norm on $X^*\otimes Y^*$, regarding it as a subspace of $(X\otimes Y, \|\cdot\|_\otimes)^*$, is a cross-norm again. In this way the norms $\|\cdot\|_\varepsilon$ and $\|\cdot\|_\pi$ are dual to each other.

We will now prove a theorem which relates admissible norms on $X_{\mathbb{C}}$ to cross-norms on its realification $X \otimes \mathbb{R}^2$. First note that since $X_{\mathbb{C}}$ and $(X_{\mathbb{C}})_{\mathbb{R}}$ have the same underlying set, a norm on $X_{\mathbb{C}}$ induces a norm on $X \otimes \mathbb{R}^2$. Conversely, a norm on $X \otimes \mathbb{R}^2$ induces a norm on $X_{\mathbb{C}}$ if and only if for all $x, y \in X$ and $a, b \in \mathbb{R}$ we have

$$\|(ax - by, bx + ay)\| = (a^2 + b^2)^{\frac{1}{2}}\|(x, y)\|. \qquad (*)$$

This is because in $X_{\mathbb{C}}$ this equation reads

$$\|(a + bi)(x + iy)\| = |a + bi|\;\|x + iy\|.$$

Let us call a norm on $X \otimes \mathbb{R}^2$ satisfying $(*)$ a *complex-homogeneous* norm.

Theorem 1.4. *A norm on $X_{\mathbb{C}}$ is admissible if and only if it is induced by a complex-homogeneous cross-norm on $X \otimes \mathbb{R}^2$.*

Proof: Let $\|\cdot\|_\otimes$ be a complex-homogeneous cross-norm on $X \otimes \mathbb{R}^2$. We must show that the norm $\|\cdot\|_{\mathbb{C}}$ on $X_{\mathbb{C}}$ given by $\|x+iy\|_{\mathbb{C}} := \|(x,y)\|_\otimes$ is admissible. Since by convention $(x,y) = x \otimes (1,0) + y \otimes (0,1)$ we have

$$\|x+iy\|_{\mathbb{C}} = \|(x,y)\|_\otimes \leqslant \|x\| \; \|(1,0)\| + \|y\| \; \|(0,1)\| = \|x\| + \|y\|.$$

Also, by Proposition 1.3 (ii),

$$\begin{aligned}\|x+iy\|_{\mathbb{C}} &= \|(x,y)\|_\otimes \geqslant \|(x,y)\|_\varepsilon = \sup\big\{|a\langle x^*,x\rangle + b\langle x^*,y\rangle| : \|x^*\| \leqslant 1, |(a,b)| \leqslant 1\big\} \\ &\geqslant \sup\big\{|\langle x^*,x\rangle| : \|x^*\| \leqslant 1\big\} = \|x\|.\end{aligned}$$

The inequality $\|x+iy\|_{\mathbb{C}} \geqslant \|y\|$ is proved similarly.

Conversely, let $\|\cdot\|_{\mathbb{C}}$ be admissible. Then the induced norm $\|\cdot\|_\otimes$ on $X \otimes \mathbb{R}^2$ is complex-homogeneous, and we have by Proposition 1.1 (i)

$$\|x \otimes (a,b)\|_\otimes = \|(ax,bx)\|_\otimes = \|(a+bi)x\|_{\mathbb{C}} = |a+bi| \; \|x\|_{\mathbb{C}} = |a+bi| \; \|x\|.$$

Also,

$$\begin{aligned}\|x^* \otimes (a,b)\|_\otimes &= \sup\big\{|a\langle x^*,x\rangle + b\langle x^*,y\rangle| : \|(x,y)\|_\otimes = 1\big\} \\ &= \sup\big\{|a\langle x^*,x\rangle + b\langle x^*,y\rangle| : \|x+iy\|_{\mathbb{C}} = 1\big\} \\ &= \sup\big\{|\mathrm{Re}\langle (a-bi)x^*, x+iy\rangle| : \|x+iy)\|_{\mathbb{C}} = 1\big\} \\ &\leqslant \|(a-bi)x^*\|_{\mathbb{C}} = |a-bi| \; \|x^*\|_{\mathbb{C}} = |a-bi| \; \|x^*\| \\ &= |a+bi| \; \|x^*\|.\end{aligned}$$

Here we used the fact that the dual norm of $\|\cdot\|_{\mathbb{C}}$ is admissible in tandem with Proposition 1.1 (i). Thus we have shown that $\|x^* \otimes (a,b)\|_\otimes \leqslant \|x^*\| \; \|(a,b)\|$. Therefore by Proposition 1.3 (i) the norm $\|\cdot\|_\otimes$ is a cross-norm. ∎

By now, the following theorem should not come as a surprise.

Theorem 1.5. *$\|\cdot\|_\varepsilon$ induces $\|\cdot\|_\infty$ and $\|\cdot\|_\pi$ induces $\|\cdot\|_1$.*

Proof: Let us prove the first assertion.

$$\begin{aligned}\|(x,y)\|_\varepsilon &= \sup_{\|x^*\|=1} \sup_{\|(a,b)\|=1} |a\langle x^*,x\rangle + b\langle x^*,y\rangle| \\ &= \sup_{\|x^*\|=1} \sup_{0 \leqslant \theta \leqslant 2\pi} |\langle x^*,x\rangle \cos\theta + \langle x^*,y\rangle \sin\theta| \\ &= \sup_{\|x^*\|=1} \left(\langle x^*,x\rangle^2 + \langle x^*,y\rangle^2\right)^{\frac{1}{2}} \\ &= \|x+iy\|_\infty.\end{aligned}$$

The proof of the other statement is also quite formal and omitted. ∎

Of course, we could also prove the above theorem by showing the $\|\cdot\|_\pi$- and the $\|\cdot\|_\varepsilon$-norms to be complex-homogeneous.

We close this section with some easy examples.

Example 1.6. $C_0(\Omega;\mathbb{C}) = (C_0(\Omega;\mathbb{R})_{\mathbb{C}}, \|\cdot\|_\infty)$ and $L^1(\mu;\mathbb{C}) = (L^1(\mu;\mathbb{R})_{\mathbb{C}}, \|\cdot\|_1)$.

Indeed, since $C_0(\Omega;\mathbb{R}^2) = (C_0(\Omega;\mathbb{R}) \otimes \mathbb{R}^2, \|\cdot\|_\varepsilon)$ and $L^1(\mu;\mathbb{R}^2) = (L^1(\mu;\mathbb{R}) \otimes \mathbb{R}^2, \|\cdot\|_\pi)$ (cf. [DU]), this follows from Theorem 1.5.

Example 1.7. Let H be a real Hilbert space. On the complexification $H_{\mathbb{C}} = H \otimes_{\mathbb{C}} \mathbb{R}^2$ there is a natural inner product $\langle\cdot,\cdot\rangle_{\mathbb{C}}$ given by

$$\langle x_0 + ix_1, y_0 + iy_1\rangle := \langle x_0, y_0\rangle + \langle x_1, y_1\rangle + i(\langle x_1, y_0\rangle - \langle x_0, y_1\rangle).$$

This inner product turns $H_{\mathbb{C}}$ into a complex Hilbert space with (admissible) norm

$$\|x_0 + ix_1\| = (\|x_0\| + \|x_1\|)^2.$$

On the other hand, if H and G are real Hilbert spaces, on the tensor product $H \otimes G$ we can define the inner product

$$\langle x_0 \otimes y_0, x_1 \otimes y_1\rangle = \langle x_0, x_1\rangle\langle y_0, y_1\rangle,$$

and consequently the completion $H\tilde{\otimes}G$ of $H \otimes G$ is a real Hilbert space. In the particular case $G = \mathbb{R}^2$, the norm on $H\tilde{\otimes}\mathbb{R}^2$ is given by

$$\|x_0 \otimes (1,0) + x_1 \otimes (0,1)\|^2 := (\|x_0\|^2 + \|x_1\|^2)^{\frac{1}{2}}.$$

Therefore, the identity map on H induces a natural isometrical isomorphism

$$(H_{\mathbb{C}})_{\mathbb{R}} \simeq H\tilde{\otimes}\mathbb{R}^2.$$

Example 1.8. Let X be a real Banach space. The identity map on $\mathcal{L}(X)$ induces a natural algebraic isomorphism $(\mathcal{L}(X))_{\mathbb{C}} \simeq \mathcal{L}(X_{\mathbb{C}})$. Suppose we are given a complex norm on $X_{\mathbb{C}}$ which turns it into a complex Banach space. Let $\|\cdot\|$ denote the associated complex-homogenous cross-norm on $X \otimes \mathbb{R}^2$. The elements of the algebraic tensor product $\mathcal{L}(X) \otimes \mathbb{R}^2$ act in a natural way as bounded linear opertors on $X \otimes \mathbb{R}^2$ by

$$\begin{aligned}\big(T_0 \otimes (1,0) + &T_1 \otimes (0,1)\big)\big(x_0 \otimes (1,0) + x_1 \otimes (0,1)\big) \\ &:= \big((T_0x_0 - T_1x_1) \otimes (1,0)\big) + \big((T_0x_1 + T_1x_0) \otimes (0,1)\big).\end{aligned}$$

The idea behind this is that we regard $(1,0)$ as 'multiplication by 1', i.e. the identity operator on $\mathbb{R}^2$, and $(0,1)$ as 'multiplication by i', i.e. the operator $\begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix}$ on $\mathbb{R}^2$. The norm $\|\cdot\|$ on $\mathcal{L}(X) \otimes \mathbb{R}^2$ induced by $\mathcal{L}(X \otimes \mathbb{R}^2, \|\cdot\|)$ is easily checked to be complex-homogenous, and satisfies

$$\max\{\|T_0\|, \|T_1\|\} \leqslant \|T_0 \otimes (1,0) + T_1 \otimes (0,1)\| \leqslant \|T_0\| + \|T_1\|.$$

On the other hand, giving $(\mathcal{L}(X))_{\mathbb{C}}$ the norm of $\mathcal{L}(X_{\mathbb{C}})$, we have

$$\begin{aligned}\|T_0 + iT_1\| &= \sup\{|\langle (T_0 + iT_1)(x_0 + ix_1)\rangle| : \ \|x_0 + ix_1\| \leqslant 1\} \\ &= \sup\{|\langle T_0x_0 - T_1x_1 + i(T_0x_1 + T_1x_0)\rangle| : \ \|x_0 + ix_1\| \leqslant 1\} \\ &= \sup\{|\langle (T_0x_0 - T_1x_1) \otimes (1,0) + (T_0x_1 + T_1x_0) \otimes (0.1)\rangle| : \\ &\qquad\qquad \|x_0 \otimes (1,0) + x_1 \otimes (0,1)\| \leqslant 1\} \\ &= \|T_0 \otimes (1,0) + T_1 \otimes (0,1)\|_{\mathcal{L}(X \otimes \mathbb{R}^2, \|\cdot\|)}.\end{aligned}$$

Thus, the norm on $(\mathcal{L}(X))_{\mathbb{C}}$ is admissible and we have a natural isometrical isomorphism

$$((\mathcal{L}(X))_{\mathbb{C}})_{\mathbb{R}} \simeq (\mathcal{L}(X) \otimes \mathbb{R}^2, \|\cdot\|).$$

Example 1.9. Let A be a *real* commutative C^*-algebra, i.e. a real commutative Banach algebra such that $\|x^2\| = \|x\|^2$ for all $x \in A$. On the complexification $A_{\mathbb{C}}$, which is a commutative algebra under the multiplication $(a+ib)(c+id) = (ac-bd) + i(ac+bd)$, the norm

$$\|a+ib\|_{\mathbb{C}} := \|a^2+b^2\|^{\frac{1}{2}}$$

defines an algebra norm which coincides with the original norm on the real part A. Commutativity is used to prove the triangle inequality. Moreover, with the natural involution on $A_{\mathbb{C}}$, $(a+ib)^* = a - ib$, we have

$$\|(a+ib)^*(a+ib)\| = \|a^2+b^2\| = \|a+ib\|^2,$$

so $A_{\mathbb{C}}$ is a complex commutative C^*-algebra. Therefore, A is isomorphic to a space $C_0(\Omega)$ with Ω locally compact Hausdorff, and to a space $C(K)$, K compact Hausdorff, if A has a unit. It follows that A is isomorphic to the real part of these spaces, i.e. to the space of real-valued continuous functions on Ω or K.

Let us now show how the norm $\|\cdot\|$ arises in a natural way from a complex-homogenous cross-norm on $A \otimes \mathbb{R}^2$. Given two complex C^*-algebras A_0 and A_1 acting on Hilbert spaces H_0 and H_1, respectively, the algebraic tensor product acts in a natural way on $H_0 \otimes H_1$ by the formula

$$(a_0 \otimes a_1)(h_0 \otimes h_1) = a_0(h_0)\, a_1(h_1).$$

The operator norm on $\mathcal{L}(H_0 \tilde{\otimes} H_1)$ turns the completion of $A_0 \otimes A_1$ into a complex C^*-algebra $A_0 \tilde{\otimes}_\sigma A_1$, the *spatial tensor product* of A_0 and A_1. In the case of two abstract complex C^*-algebras, one can do the same via faithful representations; the spatial tensor product so obtained is independent of the choice of the representations. If A_0 and A_1 are real commutative C^*-algebras, we complexify as above and consider $A_0 \otimes A_1$ as a real-linear subspace of $(A_0)_{\mathbb{C}} \tilde{\otimes}_\sigma (A_1)_{\mathbb{C}}$. Then we define the spatial tensor product $A_0 \tilde{\otimes}_\sigma A_1$ as the closure of $A_0 \otimes A_1$ in $(A_0)_{\mathbb{C}} \tilde{\otimes}_\sigma (A_1)_{\mathbb{C}}$. In this way, one can check that for real commutative C^*-algebras we have the isomorphism

$$(A_{\mathbb{C}})_{\mathbb{R}} \simeq A \tilde{\otimes}_\sigma \mathbb{R}^2.$$

We do not know whether a similar argument can be given for arbitrary real C^*-algebras; in fact, it is not obvious how to define these in the right way.

2. The norm of a complex Banach lattice

In this section, we turn to a somewhat less trivial illustration of our ideas and show how to obtain the norm of a complex Banach lattice from a cross-norm of real Banach lattices.

Let E be a real Banach lattice. We will define an admissible norm on $E_{\mathbb{C}}$ as follows. For $z = x + iy \in E_{\mathbb{C}}$ define

$$|z| := \sup_{0 \leqslant \theta \leqslant 2\pi} |x \cos\theta + y \sin\theta|.$$

This supremum exists in E, and we define

$$\|z\| := \|\ |z|\ \|.$$

The complex Banach space $E_{\mathbb{C}}$ with this structure is called a *complex Banach space.* For more details, we refer to [LZ] and [S]. In fact one can show [MW] that $|\cdot|$ is the *unique* extension of the modulus function of E to a function $E_{\mathbb{C}} \to E_+$ satisfying $|\alpha z| = |\alpha|\,|z|, (\alpha \in \mathbb{C})$ (complex-homogenity) and $|z_1 + z_2| \leqslant |z_1| + |z_2|$ (subadditivity). Thus one can talk about $E_{\mathbb{C}}$ as *the* complex Banach lattice associated to E. The function $|\cdot|$ on $E_{\mathbb{C}}$ will be called the *modulus function* of $E_{\mathbb{C}}$.

The following result is due to de Schipper [Sch] and Schaefer [S].

Proposition 2.1. *Let $E_{\mathbb{C}}$ be a complex Banach lattice. Under the natural identification $\psi : (E^*)_{\mathbb{C}} \simeq (E_{\mathbb{C}})^*$, the Banach space $(E_{\mathbb{C}})^*$ is a complex Banach lattice again.*

Since the norm of a complex Banach lattice $E_{\mathbb{C}}$ is admissible, Theorem 1.4 shows that it must be induced by a cross-norm on $E \otimes_{\mathbb{R}} \mathbb{R}^2$. The rest of this section is devoted to identifying this cross-norm as the l-norm. First we recall its definition.

Let E be a real Banach lattice and Y a real Banach space.

Definition 2.2. An operator $T \in \mathcal{L}(E;Y)$ is *cone absolutely summing (c.a.s)* if

$$\|T\|_l := \sup\left\{ \sum_{n=1}^{N} \|Tx_n\| : (x_n)_n \subset E_+ \text{ finite}, \left\|\sum_n x_n\right\| = 1 \right\} < \infty.$$

The subspace of $\mathcal{L}(E;Y)$ of all c.a.s. operators is denoted by $\mathcal{L}^l(E;Y)$. Each $u = \sum_n x_n \otimes y_n \in E \otimes Y$ defines an operator $T_u \in \mathcal{L}^l(E^*;Y)$ by

$$T_u x^* := \sum_n \langle x^*, x_n \rangle y_n.$$

In particular, for $Y = \mathbb{R}^2$ this reduces to

$$T_{(x,y)} x^* = (\langle x^*, x\rangle, \langle x^*, y\rangle).$$

On $E \otimes \mathbb{R}^2$ we define a norm by

$$\|(x,y)\|_l := \|T_{(x,y)}\|_l.$$

Lemma 2.3. *The norm $\|\cdot\|_l$ is a complex-homogeneous cross-norm on $E \otimes_{\mathbb{R}} \mathbb{R}^2$.*

Proof: That it is a cross-norm is proved in [S]. We check that $\|\cdot\|_l$ is complex-homogeneous. We have

$$\begin{aligned}
\|(ax-by,bx+ay)\|_l &= \sup\left\{\sum_n (\langle x_n^*, ax-by\rangle^2 + \langle x_n^*, bx-ay\rangle^2)^{\frac{1}{2}}\right\}\\
&= \sup\left\{\sum_n |\langle x_n^*, ax-by\rangle + i\langle x_n^*, bx+ay\rangle|\right\}\\
&= \sup\left\{\sum_n |(a+bi)(\langle x_n^*, x\rangle + i\langle x_n^*, y\rangle)|\right\}\\
&= |a+bi|\cdot\sup\left\{\sum_n (\langle x_n^*, x\rangle^2 + \langle x_n^*, y\rangle^2)^{\frac{1}{2}}\right\}\\
&= (a^2+b^2)^{\frac{1}{2}}\|(x,y)\|_l.
\end{aligned}$$

■

By Theorem 1.4, $\|\cdot\|_l$ induces a norm, also denoted by $\|\cdot\|_l$, on $E_{\mathbb{C}}$. This norm is self-dual in the following sense.

Lemma 2.4. *The natural vector space isomorphism $\psi : (E^*)_{\mathbb{C}} \simeq (E_{\mathbb{C}})^*$ induces an isometrical isomorphism $((E^*)_{\mathbb{C}}, \|\cdot\|_l) \simeq ((E_{\mathbb{C}}, \|\cdot\|_l)^*$.*

Proof: First we recall [S] that there is a natural isometrical isomorphism

$$(E\otimes\mathbb{R}^2, \|\cdot\|_l)^* \simeq \mathcal{L}^l(E;\mathbb{R}^2).$$

Using this, the fact that $\|x+iy\|_l = \|x-iy\|_l$ and Goldstine's theorem we see that

$$\begin{aligned}
&\|x^*+iy^*\|_{((E^*)_{\mathbb{C}},\|\cdot\|_l)} = \|(x^*,y^*)\|_{(X^*\otimes_{\mathbb{R}}\mathbb{R}^2,\|\cdot\|_l)}\\
&= \sup\left\{\sum_n (\langle x_n^{**}, x^*\rangle^2 + \langle x_n^{**}, y^*\rangle^2)^{\frac{1}{2}} : (x_n^{**}) \subset E_+^{**} \text{ finite}, \left\|\sum_n x_n^{**}\right\| = 1\right\}\\
&= \sup\left\{\sum_n (\langle x^*, x_n\rangle^2 + \langle y^*, x_n\rangle^2)^{\frac{1}{2}} : (x_n) \subset E_+ \text{ finite}, \left\|\sum_n x_n\right\| = 1\right\}\\
&= \|T_{(x^*,y^*)}\|_{\mathcal{L}^l(E,\mathbb{R}^2)}\\
&= \|(x^*,y^*)\|_{(E\otimes\mathbb{R}^2,\|\cdot\|_l)^*}\\
&= \sup_{\|(x,y)\|_l=1} |\langle x^*, x\rangle + \langle y^*, y\rangle|\\
&= \sup_{\|x-iy\|_l=1} |\mathrm{Re}\langle x^*+iy^*, x-iy\rangle|\\
&= \|x^*+iy^*\|_{(E_{\mathbb{C}},\|\cdot\|_l)^*}.
\end{aligned}$$

■

Lemma 2.5. *Let E be a real Banach lattice and let $z^* = x^* + iy^*$ be an element of the complex Banach lattice $(E^*)_{\mathbb{C}}$. Then $\|z^*\|_l = \|z^*\|$.*

Proof: The proof uses the following two facts [S, p. 234-5]: Firstly, for $0 \leqslant x \in E$ we have

$$\langle |z^*|, x\rangle = \sup_{|z|\leqslant x} |\langle z^*, z\rangle| \leqslant \sup \Big|\sum_n \langle z^*, \alpha_n x_n\rangle\Big|,$$

where the second supremum is over all finite sequences $(\alpha_n, x_n) \in \mathbb{C} \times E_+$ such that $|\alpha_n| \leqslant 1$ and $\sum_n x_n = x$. Secondly,

$$\Big|\sum_n \langle z^*, \alpha_n x_n\rangle\Big| \leqslant \sum_n |\alpha_n|\ |\langle z^*, x_n\rangle| \leqslant \langle |z^*|, x\rangle.$$

Combining these facts, noting that the supremum is taken by $|\alpha_n| = 1$, and by taking the supremum of all $0 \leqslant x \in E$ of norm one, we find that

$$\|\ |z^*|\ \| = \sup\left\{\sum_n |\langle z^*, x_n\rangle| : (x_n) \subset E_+ \text{ finite}, \Big\|\sum_n x_n\Big\| = 1\right\} = \|z^*\|_l.$$

Note that we used Goldstine's theorem in the last identity. Since the norm on $(E^*)_{\mathbb{C}}$ satisfies $\|\ |z^*|\ \| = \|z^*\|$, it follows that $\|z^*\| = \|z^*\|_l$. ∎

Theorem 2.6. *The norm of a complex Banach lattice $E_{\mathbb{C}}$ agrees with its l-norm.*

Proof: The dual norms on $(E^*)_{\mathbb{C}}$ of $\|\cdot\|$ and $\|\cdot\|_l$ are again $\|\cdot\|$ and $\|\cdot\|_l$ (Proposition 2.1 and Lemma 2.4), and since they agree (Lemma 2.5), again by 2.1 and 2.4 it follows that $\|\cdot\|$ and $\|\cdot\|_l$ agree on $(E^{**})_{\mathbb{C}}$. Hence, letting $j : E_{\mathbb{C}} \to (E^{**})_{\mathbb{C}}$ be the natural map, we see that for all $z \in E_{\mathbb{C}}$,

$$\|z\| = \|jz\| = \|jz\|_l = \|z\|_l.$$

∎

3. References

[B] J. Bourgain, Real isomorphic complex Banach spaces need not be complex isomorphic, Proc. Am. Math. Soc. **96** (1986), 221-226.

[DU] J. Diestel and J.J. Uhl, *Vector measures*, Math. Surveys nr. 15, Amer. Math. Soc., Providence, R. I. (1977).

[K] N.J. Kalton, An elementary example of a Banach space not isomorphic to its complex conjugate, Canad. Math. Bull. **38** (1995), 218-222.

[LZ] W.A.J. Luxemburg and A.C. Zaanen, The linear modulus of an order bounded linear transformation I, Indag. Math. **33** (1971), 422-434.

[MW] G. Mittelmeyer and M. Wolff, Über den Absolutbetrag auf komplexen Vektorverbänden, Math. Z. **137** (1974), 87-92.

[R] A.F. Ruston, *Fredholm theory in Banach spaces*, Cambridge U.P., Cambridge (1986).

[S] H.H. Schaefer, *Banach Lattices and Positive Operators*, Springer Verlag, Berlin-Heidelberg-New York (1974).

[Sch] W.J. de Schipper, A note on the modulus of an order bounded linear operator between complex Banach lattices, Indag. Math. **35** (1973), 355-367.