The vector-valued Loomis theorem for the half-line and individual stability of C_0−semigroups: a counterexample

J. M. A. M. van Neerven

Communicated by K. H. Hofmann

Abstract. We construct a bounded, uniformly continuous function $g : [0,\infty) \to l^2$ with the following properties:

(1) The Laplace transform $\mathcal{L}g(\cdot)$ has a holomorphic extension to a neighbourhood of $\{\mathrm{Re}\,\lambda \geq 0\}\backslash\{0\}$;
(2) The non-tangential strong limit $\lim_{\lambda\to 0}\mathcal{L}g(\lambda)$ exists;
(3) $\displaystyle\lim_{\tau\to\infty}\left\|\frac{1}{\tau}\int_0^\tau g(t)\,dt\right\| = 0$;
(4) $\lim_{t\to\infty}\langle g(t), x^*\rangle = 0$ for all $x^* \in l^2$;
(5) $\limsup_{t\to\infty}\|g(t)\| \geq 1$.

This function is then used to construct a C_0−semigroup $\{T(t)\}_{t\geq 0}$, with generator A, on a Banach space X with the following property. There exists an element $x \in X$ such that:

(i) The orbit $t \mapsto T(t)x$ is bounded and uniformly continuous;
(ii) The local resolvent $\lambda \mapsto R(\lambda, A)x$ has a holomorphic extension to a neighbourhood of $\{\mathrm{Re}\,\lambda \geq 0\}\backslash\{0\}$;
(iii) $\displaystyle\lim_{\tau\to\infty}\left\|\frac{1}{\tau}\int_0^\tau T(t)x\,dt\right\| = 0$;
(iv) There is a norming subspace $Z \subseteq X^*$ such that

$$\lim_{t\to\infty}\langle T(t)x, x^*\rangle = 0 \text{ for all } x^* \in Z;$$

(v) $\displaystyle\limsup_{t\to\infty}\|T(t)x\| \geq 1$.

This example shows that in the local version of the Arendt-Batty-Lyubich-Vũ stability theorem, obtained recently by Batty-van Neerven-Räbiger, the total ergodicity assumption cannot be weakened to ergodicity.

1991 Mathematics Subject Classification: 47D03, 44A10, 43A60, 43A65

0. Introduction

In this paper we present a counterexample related to the validity of the vector-valued Loomis theorem for the half-line $\mathbb{R}_+ = [0,\infty)$ and examine its consequences for the theory of individual stability of C_0−semigroups.

In order to motivate the questions studied here, let us first recall some well-known results for the real line. Let X be a complex Banach space and let $\mathbb{C}_-$ and $\mathbb{C}_+$ denote the sets $\{\lambda \in \mathbb{C} : \mathrm{Re}\,\lambda < 0\}$ and $\{\lambda \in \mathbb{C} : \mathrm{Re}\,\lambda > 0\}$, respectively. The *Carleman transform* of a function $f \in L^\infty(\mathbb{R}, X)$ is the holomorphic $X-$valued function $\hat{f} : \mathbb{C}_- \cup \mathbb{C}_+ \to X$ defined by

$$\hat{f}(\lambda) := \begin{cases} -\int_0^\infty e^{\lambda t} f(-t)\,dt, & \operatorname{Re}\lambda < 0; \\ \int_0^\infty e^{-\lambda t} f(t)\,dt, & \operatorname{Re}\lambda > 0. \end{cases}$$

A point $i\omega \in i\mathbb{R}$ is *regular* for $\hat{f}$ if $\hat{f}$ admits a holomorphic extension to some open neighbourhood of $\mathbb{C}_- \cup \mathbb{C}_+ \cup \{i\omega\}$. The *Carleman spectrum* of f, notation $\sigma_{\mathcal{C}}(f)$, is the set of all $i\omega \in i\mathbb{R}$ that are *singular*, i.e. not regular, for $\hat{f}$.

Let $C_b(\mathbb{R}, X)$ denote the Banach space of bounded continuous X-valued functions on $\mathbb{R}$ and let $AP(\mathbb{R}, X)$ be its closed subspace of all almost periodic $X-$valued functions on $\mathbb{R}$. Recall that a function $f \in C_b(\mathbb{R}, X)$ is *almost periodic* if it belongs to the closed linear span in $C_b(\mathbb{R}, X)$ of the set of trigonometric polynomials $\{e_{i\omega} \otimes x : \omega \in \mathbb{R}, x \in X\}$; here $(e_{i\omega} \otimes x)(t) := e^{i\omega t}x$, $t \in \mathbb{R}$. It is well-known that a function f is almost periodic if and only if the set of its translates $\{f_t : t \in \mathbb{R}\}$ is a relatively compact subset of $C_b(\mathbb{R}, X)$; here f_t is the function defined by $f_t(s) := f(t+s)$, $s \in \mathbb{R}$.

It is well-known that almost periodic $X-$valued functions are uniformly continuous and have countable Carleman spectrum. Conversely, a function $f \in BUC(\mathbb{R}, X)$, the Banach space of bounded uniformly continuous $X-$valued functions on $\mathbb{R}$, whose Carleman spectrum is countable, is almost periodic if at least one of the following four conditions is satisfied:

(i) X does not contain a closed subspace isomorphic to c_0;
(ii) $\sigma_{\mathcal{C}}(f)$ is discrete (i.e. consists of isolated points only);
(iii) f has relatively weakly compact range;
(iv) f is *totally ergodic*, i.e. $\lim_{\tau\to\infty} \frac{1}{2\tau}\int_{-\tau}^{\tau} e^{-i\omega t} f(t+s)\,dt$ exists, uniformly in $s \in \mathbb{R}$, for all $i\omega \in \sigma_{\mathcal{C}}(f)$.

The fact that a scalar-valued bounded uniformly continuous function with countable Carleman spectrum is almost periodic is known as Loomis's theorem. For the proof of its vector-valued versions we refer to [LZ, Theorem 6.4.4], [AB1] (for (i) and (iii)), [AS] (for (ii)), and [RV], [AB1] (for (iv)).

The following simple example, which is included for reasons of completeness, shows that the condition 'uniformly in $s \geq 0$' cannot be omitted in (iv).

Example 0.1. Define $g \in BUC(\mathbb{R}, c_0)$ by

$$g(t) = (e^{it} - e^{it/2}, e^{it/2} - e^{it/4}, e^{it/4} - e^{it/8}, \ \dots\ ), \qquad t \in \mathbb{R}.$$

It is easily verified that $\sigma_{\mathcal{C}}(g) = \{i, i/2, i/4, i/8, \ \dots\ \} \cup \{0\}$ and that for all $i\omega \in \sigma_{\mathcal{C}}(g)$ the limit $\lim_{\tau\to\infty} \frac{1}{2\tau}\int_{-\tau}^{\tau} e^{-i\omega t} g(t)\,dt$ exists. Nevertheless g is readily seen not to be almost periodic. ∎

For functions on the half-line $\mathbb{R}_+$ the concept of Carleman spectrum breaks down and needs to be replaced by that of Laplace spectrum. Recall that the *Laplace transform* of a function $f \in L^\infty(\mathbb{R}_+, X)$ is the holomorphic $X-$valued function $\mathcal{L}f$ on $\mathbb{C}_+$ defined by

$$\mathcal{L}f(\lambda) = \int_0^\infty e^{-\lambda t} f(t)\,dt, \qquad \lambda \in \mathbb{C}_+.$$

A point $i\omega \in i\mathbb{R}$ is *regular* for $\mathcal{L}f$ if $\mathcal{L}f$ admits a holomorphic extension to some open neighbourhood of $\mathbb{C}_+ \cup \{i\omega\}$. The *Laplace spectrum* of f, notation $\sigma_{\mathcal{L}}(f)$, is the set of all $i\omega \in i\mathbb{R}$ that are *singular*, i.e. not regular, for $\mathcal{L}f$.

For a function $f \in BUC(\mathbb{R}_+, X)$ the Laplace spectrum $\sigma_{\mathcal{L}}(f|_{\mathbb{R}_+})$ of its restriction to $\mathbb{R}_+$ is usually much smaller than its Carleman spectrum $\sigma_{\mathcal{C}}(f)$. For instance if $f(t) = e^{-t^2}$, then $\sigma_{\mathcal{L}}(f|_{\mathbb{R}_+}) = \emptyset$ and $\sigma_{\mathcal{C}}(f) = i\mathbb{R}$.

A function $f \in C_b(\mathbb{R}_+, X)$ is called *almost periodic* if it is the restriction to $\mathbb{R}_+$ of an almost periodic function on $\mathbb{R}$, and *asymptotically almost periodic* if its set of left translates $\{f_t : t \geq 0\}$ is a relatively compact subset of $C_b(\mathbb{R}_+, X)$; we now define $f_t(s) := f(t+s)$, $s \geq 0$. The spaces of almost periodic functions and asymptotically almost periodic functions on $\mathbb{R}_+$ are denoted by $AP(\mathbb{R}_+, X)$ and $AAP(\mathbb{R}_+, X)$, respectively. As closed subspaces of $C_b(\mathbb{R}_+, X)$ we have the direct sum decomposition [RS1]

$$AAP(\mathbb{R}_+, X) = AP(\mathbb{R}_+, X) \oplus C_0(\mathbb{R}_+, X).$$

The following analogue of version (iv) of the vector-valued Loomis theorem for the half-line was obtained recently in [BNR2, Theorem 4.1] (where the result is stated in terms of Abel means) and [Ne, Theorem 5.3.5]:

Proposition 0.2. *Let $f \in BUC(\mathbb{R}_+, X)$ and assume that $\sigma_{\mathcal{L}}(f)$ is countable. If for all $i\omega \in \sigma_{\mathcal{L}}(f)$ the limit*

$$\lim_{\tau\to\infty} \frac{1}{\tau} \int_0^\tau e^{-i\omega t} f(t+s)\, dt$$

exists, uniformly in $s \geq 0$, then $f \in AAP(\mathbb{R}_+, X)$. If in addition we know that $\lim_{t\to\infty} \langle f(t), x^* \rangle = 0$ for all $x^* \in X^*$, then $f \in C_0(\mathbb{R}_+, X)$.

As is the case for the real line, the condition 'uniformly in $s \geq 0$' cannot be omitted from the first statement. This is shown by the following example, which also shows that there is no analogue for $\mathbb{R}_+$ of versions (i), (ii), and (iii) of the vector-valued Loomis theorem:

Example 0.3 [BNR, Example 4.2], [RV, Example 3.12], [St, p. 608]**.** Let $X = \mathbb{C}$ and consider the function $g(t) = \sin\sqrt{t}$, $t \geq 0$. Then $g \in BUC(\mathbb{R}_+)$ and its Laplace transform is given by

$$\mathcal{L}g(z) = \frac{\sqrt{\pi} e^{-1/(4z)}}{2z^{3/2}}, \qquad z \in \mathbb{C} \setminus (-\infty, 0].$$

Hence, $\sigma_{\mathcal{L}}(g) = \{0\}$. Moreover,

$$\lim_{\tau\to\infty} \left| \frac{1}{\tau} \int_0^\tau g(t)\, dt \right| = 0,$$

but g is not asymptotically almost periodic. ∎

Example 0.3 does not rule out the possibility that the condition 'uniformly in $s \geq 0$' may be dropped in Proposition 0.2 if in addition to the stated assumptions we have $\lim_{t\to\infty} \langle f(t), x^* \rangle = 0$ for all $x^* \in X^*$. In Section 1 we will show that this hope is unfounded by proving:

Theorem 0.4. *There exists a function* $g \in BUC(\mathbb{R}_+, l^2)$ *with the following properties:*

(1) $\sigma_{\mathcal{L}}(g) = \{0\}$;
(2) *the non-tangential strong limit* $\lim_{\lambda\to 0} \mathcal{L}g(\lambda)$ *exists;*
(3) $\left\| \frac{1}{\tau}\int_0^\tau g(t)\,dt \right\|_{l^2} \le \frac{C}{\sqrt[4]{\tau}}$ *for all* $\tau > 0$ *and some constant* $C > 0$;
(4) $\lim_{t\to\infty} \langle g(t), x^* \rangle = 0$ *for all* $x^* \in l^2$;
(5) $\limsup_{t\to\infty} \|g(t)\|_{l^2} \ge 1$.

This function g is obtained through a construction which combines the essential features of Examples 0.1 and 0.3. We point out that this construction can be simplified somewhat to obtain an example of a c_0-valued function with the properties (1) - (5); this would suffice for the applications in Sections 2 and 3. But in the context of version (i) of Loomis's theorem it is interesting that the example can be realized for l^2-valued functions.

Note that (4) and (5) imply that g is not asymptotically almost periodic. In fact, g even fails to be Eberlein weakly almost periodic. Recall that a function $f \in C_b(\mathbb{R}_+, X)$ is called *Eberlein weakly almost periodic* if the set $\{f_t : t \ge 0\}$ is a relatively weakly compact subset of $C_b(\mathbb{R}_+, X)$ Indeed, if g were Eberlein weakly almost periodic, then (4) in combination with [BNR1, Theorem 6.1] and [BNR2, Theorem 4.1] would imply that $\lim_{t\to\infty} \|g(t)\|_{l^2} = 0$.

It is well-known that there is a close relationship between the theory of asymptotic almost periodicity on the one hand and the homogenous and inhomogenous abstract Cauchy problem on the other; we refer the reader to [AB], [AS], [Ba], [LZ], [RS], [RV] and the references given there. For more information on the general theory of C_0-semigroups, as well as for an explanation of the standard terminology and notation, we refer to the book [Pa]. Here we mention the fact that Proposition 0.2 implies an individual version of the celebrated Arendt-Batty-Lyubich-Vũ stability theorem. In order to state the precise result we need the following notation.

Let $\mathbf{T} = \{T(t)\}_{t\ge 0}$ be a C_0-semigroup on a Banach space X with generator A. Choose constants $M > 0$ and $\omega \in \mathbb{R}$ such that $\|T(t)\| \le Me^{\omega t}$ for all $t \ge 0$. Fix $x \in X$. The *local resolvent* of A at x is the $X-$valued holomorphic function $\lambda \mapsto R(\lambda, A)x := (\lambda - A)^{-1}x$, $\operatorname{Re}\lambda > \omega$. Let us assume that this function has a holomorphic extension F_x to $\mathbb{C}_+$. This happens, for instance, if the orbit $t \mapsto T(t)x$ is bounded. We then denote by $\sigma_{i\mathbb{R}}(A, x)$ the set of singular points of F_x on the imaginary axis. Recalling that the resolvent is given by the Laplace transform of the semigroup,

$$R(\lambda, A)x = \int_0^\infty e^{-\lambda t} T(t)x\,dt, \qquad \operatorname{Re}\lambda > \omega,$$

we see that $\sigma_{i\mathbb{R}}(A, x) = \sigma_{\mathcal{L}}(T(\cdot)x)$ if the orbit $t \mapsto T(t)x$ is bounded. Applying Proposition 0.2 to such orbits gives the following result [BNR2, Theorem 5.3] (cf. [Ne, Theorem 5.3.6]).

Proposition 0.5. *Let* $\mathbf{T} = \{T(t)\}_{t\ge 0}$ *be a* C_0-*semigroup on* X *with generator* A. *Let* $x \in X$ *be an element with the following properties:*

(1) *The orbit* $t \mapsto T(t)x$ *is bounded and uniformly continuous;*
(2) $\sigma_{i\mathbb{R}}(A, x)$ *is countable;*

(3) *For all* $i\omega \in \sigma_{i\mathbb{R}}(A,x)$ *the limit* $\lim_{\tau\to\infty} \frac{1}{\tau}\int_0^\tau e^{-i\omega t} T(t+s)x\,dt$ *exists, uniformly in* $s \geq 0$.

Then the orbit $t \mapsto T(t)x$ *is asymptotically almost periodic. If the limits in (3) equal* 0, *then*

$$\lim_{t\to\infty} \|T(t)x\| = 0.$$

The question whether the assumption 'uniformly in $s \geq 0$' can be omitted from this result was left open in [BNR2]. We mention the fact, proved in [BNR1], that it can be omitted indeed if $\mathbf{T}$ is a uniformly bounded semigroup.

Here we will show that in general the answer is negative:

Theorem 0.6. *There exists a* C_0-*semigroup* $\mathbf{T} = \{T(t)\}_{t\geq 0}$ *with generator* A *on a Banach space* X *with the following property: there is an element* $x \in X$ *such that*

(1) *The orbit* $t \mapsto T(t)x$ *is bounded and uniformly continuous;*

(2) $\sigma_{i\mathbb{R}}(A,x) = \{0\}$;

(3) $\lim_{\tau\to\infty} \left\| \frac{1}{\tau}\int_0^\tau T(t)x\,dt \right\| = 0$;

(4) *There is a norming subspace* $Z \subseteq X^*$ *such that* $\lim_{t\to\infty} \langle T(t)x, x^*\rangle = 0$ *for all* $x^* \in Z$;

(5) $\limsup_{t\to\infty} \|T(t)x\| \geq 1$.

Note that (4) implies that the limits in Proposition 0.5 (3), whenever they exist, are equal to 0.

The proofs of Theorems 0.4 and 0.6 are given in Sections 1 and 2, respectively. In the final Section 3 we use Theorem 0.4 to construct a translation invariant linear functional on the closed subspace of $BUC(\mathbb{R}_+, l^2)$ consisting of all functions converging to 0 scalarly.

1. Proof of Theorem 0.4

We start with a simple estimate.

Lemma 1.1. $\sup_{\tau>0}\ \sup_{\lambda\geq 0} \left| \frac{1}{\tau}\int_0^\tau e^{-\lambda s}\sin\sqrt{s}\,ds \right| \leq \frac{6}{\sqrt{\tau}}.$

Proof. Fix $\tau > 0$ and $\lambda \geq 0$. By a change of variable,

$$\frac{1}{\tau}\int_0^\tau e^{-\lambda s}\sin\sqrt{s}\,ds = \frac{1}{\tau}\int_0^{\sqrt{\tau}} 2te^{-\lambda t^2}\sin t\,dt,$$

and by partial integration we have

$$\begin{aligned}\left|\int_0^{\sqrt{\tau}} te^{-\lambda t^2}\sin t\,dt\right| &= \left| -\sqrt{\tau}e^{-\lambda\tau}\cos\sqrt{\tau} + \int_0^{\sqrt{\tau}} (1-2\lambda t^2)e^{-\lambda t^2}\cos t\,dt \right| \\ &\leq \sqrt{\tau} + \int_0^{\sqrt{\tau}} \left|(1-2\lambda t^2)e^{-\lambda t^2}\right| dt \\ &\leq \sqrt{\tau} + \left(1+\frac{2}{e}\right)\sqrt{\tau};\end{aligned}$$

in the last step we used the inequality $0 \leq ue^{-u} \leq 1/e$, $u \geq 0$. ∎

Proof of Theorem 0.4: Let $0 < \lambda_1 \le 1$ so small that

$$\int_0^{2\pi} (1 - e^{-\lambda_1 s})\, ds \le 1$$

and choose $t_1 > 0$ such that $t_1 = ((2n_1 + \frac{1}{2})\pi)^2$ for some $n_1 \in \mathbb{N}$ and $e^{-\lambda_1 t_1} \le \frac{1}{2}$. Let $0 < \lambda_2 \le \frac{1}{2}$ be so small that $\lambda_2 < \lambda_1$,

$$1 - e^{-2\lambda_2} \le \frac{1}{2}, \quad e^{-\lambda_2 t_1} \ge 1 - \frac{1}{2} = \frac{1}{2}, \quad \text{and} \quad \int_0^{8\pi} (1 - e^{-\lambda_2 s})\, ds \le \frac{1}{2}.$$

Choose $t_2 > t_1$ such that $t_2 = ((2n_2 + \frac{1}{2})\pi)^2$ for some $n_2 \in \mathbb{N}$ and $e^{-\lambda_2 t_2} \le \frac{1}{4}$. Continuing in the obvious way we obtain sequences (λ_n) and (t_n) of positive real numbers satisfying:

(i) $0 < t_1 < t_2 < \ldots \to \infty$ and for all $j = 1, 2, \ldots$ we have $t_j = ((2n_j + \frac{1}{2})\pi)^2$ for some $n_j \in \mathbb{N}$;
(ii) $\lambda_1 > \lambda_2 > \ldots \downarrow 0$ and $\lambda_j \le \frac{1}{j}$ for all $j = 1, 2, \ldots$;
(iii) $1 - e^{-j\lambda_j} \le \frac{1}{j}$ for all $j = 1, 2, \ldots$;
(iv) $e^{-\lambda_j t_j} \le 2^{-j}$ for all $j = 1, 2, \ldots$;
(v) $e^{-\lambda_{j+1} t_j} \ge 1 - 2^{-j}$ for all $j = 1, 2, \ldots$;
(vi) $\int_0^{2\pi j^2} (1 - e^{-\lambda_j s})\, ds \le \frac{1}{j}$ for all $j = 1, 2, \ldots$

The reader will notice some reduncancy in these condition; trying to avoid this would just complicate the construction below.

For $n = 1, 2, \ldots$ we define $f_n : \mathbb{R}_+ \to \mathbb{C}$ by

$$f_n(t) = e^{-\lambda_{n+1} t} - e^{-\lambda_n t}, \qquad t \ge 0.$$

Let $f : \mathbb{R}_+ \to l^2$ be defined by

$$f(t) = (f_1(t), f_2(t), \ldots\,).$$

Finally let $\phi(t) := \sin \sqrt{t}$, $t \ge 0$, and define $g_n : \mathbb{R}_+ \to \mathbb{C}$ and $g : \mathbb{R}_+ \to l^2$ by

$$g_n(t) := \phi(t) f_n(t), \quad g(t) := \phi(t) f(t), \qquad t \ge 0.$$

First we check that indeed $f(t) \in l^2$, and hence $g(t) \in l^2$, for all $t \ge 0$. To this end let $t \ge 0$ be fixed and let k denote the smallest positive integer such that $t \le t_k$. Then $t \in [t_{k-1}, t_k]$, with the convention that $t_0 = 0$. If $m \ge k$, then $0 \le t \le t_k \le t_m$ implies $1 \ge e^{-\lambda_{m+1} t} \ge e^{-\lambda_{m+1} t_m} \ge 1 - 2^{-m}$. Therefore, for $n \ge k+1$ we have $1 \ge e^{-\lambda_{n+1} t} \ge e^{-\lambda_n t} \ge 1 - 2^{-n+1}$ and thus

$$|f_n(t)| \le |e^{-\lambda_{n+1} t} - e^{-\lambda_n t}| \le 2^{-n+1}.$$

If $1 \le m \le k-1$, then $e^{-\lambda_m t} \le e^{-\lambda_m t_{k-1}} \le e^{-\lambda_m t_m} \le 2^{-m}$, and therefore, for $1 \le n \le k-2$ we have $e^{-\lambda_n t} \le e^{-\lambda_{n+1} t} \le 2^{-n-1}$ and thus

$$|f_n(t)| \le |e^{-\lambda_{n+1} t} - e^{-\lambda_n t}| \le 2^{-n-1}.$$

It follows that

$$\|f(t)\|_{l^2}^2 \le \left(\sum_{n=1}^{k-2} 2^{-2n-2} \right) + 1 + 1 + \left(\sum_{n=k+1}^{\infty} 2^{-2n+2} \right) \le 2 + \sum_{n=1}^{\infty} 2^{-2n+2} \le 4.$$

This shows that f, and hence also g, is a bounded l^2-valued function; in fact,

$$\sup_{t\geq 0} \|g(t)\|_{l^2} \leq \sup_{t\geq 0} \|f(t)\|_{l^2} \leq 2.$$

Next we check that the l^2-valued function g is uniformly continuous on $\mathbb{R}_+$. Once more it suffices to prove this for f. Let $0 \leq s \leq t$ be fixed. We have

$$\begin{aligned}\|f(t)-f(s)\|_{l^2} &= \left(\sum_{n=1}^{\infty} |(e^{-\lambda_{n+1}t} - e^{-\lambda_n t}) - (e^{-\lambda_{n+1}s} - e^{-\lambda_n s})|^2\right)^{\frac{1}{2}} \\ &\leq \left(\sum_{n=1}^{\infty} |e^{-\lambda_{n+1}s} - e^{-\lambda_{n+1}t}|^2\right)^{\frac{1}{2}} + \left(\sum_{n=1}^{\infty} |e^{-\lambda_n s} - e^{-\lambda_n t}|^2\right)^{\frac{1}{2}}. \\ &\leq \left(\sum_{n=1}^{\infty} |1 - e^{-\lambda_{n+1}(t-s)}|^2\right)^{\frac{1}{2}} + \left(\sum_{n=1}^{\infty} |1 - e^{-\lambda_n (t-s)}|^2\right)^{\frac{1}{2}}.\end{aligned}$$

By (iii) the last two sums are finite and depend only on the difference $t-s$. Moreover, as $t-s \downarrow 0$ these sums tend to 0 by monotone convergence. It follows that f, as an l^2-valued function, is uniformly continuous and we conclude that $f \in BUC(\mathbb{R}_+, l^2)$.

We will check that g has the properties (1) - (5) stated in Theorem 0.4.

The Laplace transform of g_n is given by

$$\mathcal{L}g_n(z) = \mathcal{L}\phi(\lambda_{n+1}+z) - \mathcal{L}\phi(\lambda_n + z), \qquad \operatorname{Re} z > 0,$$

with

$$\mathcal{L}\phi(\zeta) = \frac{\sqrt{\pi}e^{-1/(4\zeta)}}{2\zeta^{3/2}}, \qquad \zeta \in \mathbb{C}\setminus(-\infty, 0].$$

We will show that $\mathcal{L}g$ extends holomorphically to $D := \mathbb{C}\setminus(-\infty,0]$. Let $z \in D$ be fixed and choose $\varepsilon > 0$ so small that dist $(z, \partial D) < \varepsilon$. By analyticity we can choose a constant $C > 0$ such that

$$|\mathcal{L}\phi(z_0) - \mathcal{L}\phi(z_1)| \leq C|z_0 - z_1|$$

whenever $|z - z_0| \leq \varepsilon$ and $|z - z_1| \leq \varepsilon$. Since $\lambda_{j+1} \leq \lambda_j \leq \frac{1}{j}$ for all $j \geq 1$, for $n > 1/\varepsilon$ we have

$$|\mathcal{L}g_n(z)| = |\mathcal{L}\phi(\lambda_{n+1}+z) - \mathcal{L}\phi(\lambda_n+z)| \leq C|\lambda_{n+1} - \lambda_n| \leq \frac{1}{n}.$$

It follows that

$$\mathcal{L}g(z) := (\mathcal{L}g_1(z), \mathcal{L}g_2(z), \ldots)$$

defines an element in l^2. The function $z \mapsto \mathcal{L}g(z)$ is coordinatewise holomorphic on D, from which it is easily seen to be weakly holomorphic, and hence holomorphic.

Having obtained a holomorphic extension of $\mathcal{L}g$ to D, it follows that $\sigma_{\mathcal{L}}(g) \subset \{0\}$. But since the singularities of $\mathcal{L}g_n$ accumulate in 0, this is a singular point of $\mathcal{L}g$. This proves that $\sigma_{\mathcal{L}}(g) = \{0\}$, which is (1).

Next we check that the non-tangential strong limit $\lim_{\lambda\to 0}\mathcal{L}g(\lambda)$ exists. Fix $\theta\in[0,\frac{\pi}{2})$ and let

$$\Sigma_\theta := \{z\in\mathbb{C} : \operatorname{Re} z>0,\ |\arg z|<\theta\}.$$

Define

$$\psi(\zeta) := \mathcal{L}\phi(\zeta) = \frac{\sqrt{\pi}e^{-1/(4\zeta)}}{2\zeta^{-3/2}}, \qquad \zeta\in\Sigma_\theta.$$

For $r>0$ we put $\Sigma_{r,\theta} := \{\zeta\in\Sigma_\theta : |\zeta|<r\}$ and let

$$C_{r,\theta} := \sup_{\zeta\in\Sigma_{r,\theta}} |\psi'(\zeta)|.$$

It is easy to check that this number is finite for each $r>0$ and that $\lim_{r\to 0} C_{r,\theta} = 0$. Now fix $r>0$. If n is so large that $\lambda_n<r$, then for all $n'\geq n$ and $z\in\Sigma_{r,\theta}$ we have $\lambda_{n'}+z\in\Sigma_{2r,\theta}$, and the mean-value theorem then gives

$$\begin{aligned}|\mathcal{L}g_n(z)-\mathcal{L}g_n(0)| &\leq |\psi(\lambda_{n+1}+z)-\psi(\lambda_n+z)| + |\psi(\lambda_{n+1})-\psi(\lambda_n)|\\ &\leq 2C_{2r,\theta}|\lambda_{n+1}-\lambda_n| \leq \frac{2C_{2r,\theta}}{n},\end{aligned}$$

where in the last inequality we used (ii). Defining $\hat{g}(0)\in l^2$ by

$$\hat{g}(0) := (\mathcal{L}g_1(0),\mathcal{L}g_2(0),\ldots)$$

it follows that

$$\|\mathcal{L}g(z)-\hat{g}(0)\|_{l^2}^2 \leq \sum_{\lambda_n\geq r} |\mathcal{L}g_n(z)-\mathcal{L}g_n(0)|^2 + 4C_{2r,\theta}^2 \sum_{\lambda_n<r}\frac{1}{n^2}.$$

The first sum on the right hand side extends over finitely many n and tends to 0 as $z\to 0$. Keeping $r>0$ fixed and letting $z\to 0$ in Σ_θ, we obtain

$$\limsup_{z\in\Sigma_\theta,\, z\to 0} \|\mathcal{L}g(z)-\hat{g}(0)\|_{l^2}^2 \leq 4C_{2r,\theta}^2\sum_{n=1}^\infty \frac{1}{n^2}.$$

Because $\lim_{r\to 0} C_{2r,\theta}=0$ this gives (2).

We now check (3). First note that since g is bounded, it suffices to prove that there is a constant $C>0$ such that $\left\|\frac{1}{\tau}\int_0^\tau g(t)\,dt\right\|_{l^2}\leq C\tau^{-\frac14}$ for all $\tau\geq(2\pi)^2$.

Fix $\tau\geq(2\pi)^2$ and an integer $k\geq 1$ such that $\tau\in[\tau_k,\tau_{k+1}]$, where $\tau_j := (2\pi j)^2$. If $n\geq k+1$, then by (vi),

$$\begin{aligned}\left|\frac{1}{\tau}\int_0^\tau g_n(s)\,ds\right| &\leq \frac{1}{\tau}\int_0^{(2\pi n)^2}(e^{-\lambda_{n+1}s}-e^{-\lambda_n s})\,ds\\ &\leq \frac{1}{\tau}\int_0^{(2\pi n)^2}(1-e^{-\lambda_n s})\,ds \leq \frac{1}{n\tau}.\end{aligned}$$

On the other hand, if $1\leq n\leq k$ then by Lemma 1.1,

$$\left|\frac{1}{\tau}\int_0^\tau g_n(s)\,ds\right| \leq \frac{12}{\sqrt{\tau}}.$$

Hence,

$$\begin{aligned}\left\|\frac{1}{\tau}\int_0^\tau g(s)\,ds\right\|_{l^2}^2 &\le \sum_{n=1}^{k}\frac{144}{\tau}+\sum_{n=k+1}^{\infty}\frac{1}{n^2\tau^2}\\ &\le k\cdot\frac{144}{\tau}+\sum_{n=k}^{\infty}\frac{1}{n^2\tau^2}\\ &\le \frac{144}{2\pi\sqrt{\tau}}+\frac{2}{\tau^2}\\ &\le \left(\frac{144}{2\pi}+2\right)\frac{1}{\sqrt{\tau}},\end{aligned}$$

where in the last inequality we recall that $\tau \ge (2\pi)^2 > 1$. Taking square roots on both sides gives (3).

Let $e_n = (0,0,\ldots,0,1,0,\ldots)$ denote the $n-$th unit vector in l^2. Then

$$\lim_{t\to\infty}\langle g(t), e_n\rangle = \lim_{t\to\infty} g_n(t) = 0,$$

and since the linear span of the sequence (e_n) is dense in l^2 and g is bounded, (4) follows from this.

Finally, noting that by (i) we have $\phi(t_n)=1$ all $n\ge 1$, it follows that

$$\|g(t_n)\|_{l^2} \ge |g_n(t_n)| = e^{-\lambda_{n+1}t_n} - e^{-\lambda_n t_n} \ge (1-2^{-n}) - 2^{-n} = 1-2^{-n+1},$$

which gives (5). ■

2. Proof of Theorem 0.6

In this section we will prove Theorem 0.6, thus giving a negative answer to the question, mentioned in the introduction, that was raised in [BNR2].

We start with a lemma.

Lemma 2.1. *Suppose $f \in L^\infty(\mathbb{R}_+, X)$ satisfies*

$$\left\|\frac{1}{\tau}\int_0^\tau f(t)\,dt\right\| \le C\tau^{-\frac{1}{4}}, \qquad \forall \tau > 0,$$

for some constant $C>0$. Then,

$$\left\|\frac{1}{\tau}\int_0^\tau f(t+s)\,dt\right\| \le C\tau^{-\frac{1}{4}}(1+2s^{\frac{3}{4}}), \qquad \forall \tau \ge 1,\ s\ge 0.$$

Proof. For $s=0$ the estimate is trivial. Fix $\tau\ge 1$ and $s>0$. We estimate

$$\begin{aligned}\left\|\frac{1}{\tau}\int_0^\tau f(t+s)\,dt\right\| &\le \frac{\tau+s}{\tau}\left\|\frac{1}{\tau+s}\int_0^{\tau+s} f(t)\,dt\right\| + \frac{s}{\tau}\left\|\frac{1}{s}\int_0^s f(t)\,dt\right\|\\ &\le \frac{\tau+s}{\tau}\cdot\frac{C}{(\tau+s)^{\frac{1}{4}}}+\frac{s}{\tau}\cdot\frac{C}{s^{\frac{1}{4}}}\\ &= \frac{C}{(\tau+s)^{\frac{1}{4}}}+\frac{Cs}{\tau(\tau+s)^{\frac{1}{4}}}+\frac{Cs^{\frac{3}{4}}}{\tau}\\ &\le \frac{C}{\tau^{\frac{1}{4}}}+\frac{Cs^{\frac{3}{4}}}{\tau}+\frac{Cs^{\frac{3}{4}}}{\tau},\end{aligned}$$

which gives the desired result. ■

Let

$$\psi(t) := 1 + t^{\frac{3}{4}}, \qquad t \geq 0,$$

and denote by $BUC_\psi(\mathbb{R}_+, X)$ the space of all functions $f : \mathbb{R}_+ \to X$ such that $f/\psi \in BUC(\mathbb{R}_+, X)$. This is a Banach space with respect to the norm

$$\|f\|_{BUC_\psi(\mathbb{R}_+,X)} := \|f/\psi\|_{BUC(\mathbb{R}_+,X)}.$$

For a function $f \in BUC_\psi(\mathbb{R}_+, X)$ and a real number $t \geq 0$ we denote by $f_t : \mathbb{R}_+ \to X$ the function

$$f_t(s) := f(s+t), \qquad s \geq 0.$$

It is easy to see that $f_t \in BUC_\psi(\mathbb{R}_+, X)$; in fact:

Lemma 2.2. *For all* $f \in BUC_\psi(\mathbb{R}_+, X)$ *we have*

$$\lim_{t\downarrow 0} \|f - f_t\|_{BUC_\psi(\mathbb{R}_+,X)} = 0.$$

Proof. For all $s, t \geq 0$,

$$\begin{aligned}
\left\| \frac{f(s+t)}{1+s^{\frac{3}{4}}} - \frac{f(s+t)}{1+(s+t)^{\frac{3}{4}}} \right\| &= \frac{((s+t)^{\frac{3}{4}} - s^{\frac{3}{4}})\|f(s+t)\|}{(1+s^{\frac{3}{4}})(1+(s+t)^{\frac{3}{4}})} \\
&\leq \|f\|_{BUC_\psi(\mathbb{R}_+,X)} \frac{((s+t)^{\frac{3}{4}} - s^{\frac{3}{4}})}{1+s^{\frac{3}{4}}}. \\
&\leq \|f\|_{BUC_\psi(\mathbb{R}_+,X)}((s+t)^{\frac{3}{4}} - s^{\frac{3}{4}}) \\
&\leq \|f\|_{BUC_\psi(\mathbb{R}_+,X)} \cdot t^{\frac{3}{4}}
\end{aligned}$$

Using this estimate and the strong continuity of translation in $BUC(\mathbb{R}_+, X)$ we obtain

$$\begin{aligned}
\limsup_{t\downarrow 0} \|f - f_t\|_{BUC_\psi(\mathbb{R}_+,X)} &= \limsup_{t\downarrow 0} \left(\sup_{s\geq 0} \frac{1}{1+s^{\frac{3}{4}}} \|f(s) - f(s+t)\| \right) \\
&\leq \lim_{t\downarrow 0} \left(\sup_{s\geq 0} \left\| \frac{f(s)}{1+s^{\frac{3}{4}}} - \frac{f(s+t)}{1+(s+t)^{\frac{3}{4}}} \right\| \right) \\
&\qquad + \lim_{t\downarrow 0} \left(\sup_{s\geq 0} \left\| \frac{f(s+t)}{1+(s+t)^{\frac{3}{4}}} - \frac{f(s+t)}{1+s^{\frac{3}{4}}} \right\| \right) \\
&= 0.
\end{aligned}$$

It follows that $\lim_{t\downarrow 0} \|f - f_t\|_{BUC_\psi(\mathbb{R}_+,X)}$ exists and equals 0. ■

This lemma shows that the left translation semigroup $\mathbf{S}_\psi$ on $BUC_\psi(\mathbb{R}_+, X)$ defined by

$$S_\psi(t)f := f_t, \qquad f \in BUC_\psi(\mathbb{R}_+, X), \quad t \geq 0,$$

is strongly continuous. From

$$\begin{aligned}
\|S_\psi(t)f\|_{BUC_\psi(\mathbb{R}_+,X)} &= \sup_{s\geq 0} \frac{f(s+t)}{1+s^{\frac{3}{4}}} \\
&\leq \sup_{s\geq 0} \frac{1+(s+t)^{\frac{3}{4}}}{1+s^{\frac{3}{4}}} \|f\|_{BUC_\psi(\mathbb{R}_+,X)} \\
&= (1+t^{\frac{3}{4}})\|f\|_{BUC_\psi(\mathbb{R}_+,X)}
\end{aligned}$$

we see that
$$\|S_\psi(t)\| \le \psi(t), \qquad \forall t \ge 0.$$
Denote the inclusion map $BUC(\mathbb{R}_+, X) \hookrightarrow BUC_\psi(\mathbb{R}_+, X)$ by i_ψ, and let $\mathbf{S}$ denote the left translation semigroup on $BUC(\mathbb{R}_+, X)$. Then $i_\psi \circ S(t) = S_\psi(t) \circ i_\psi$ for all $t \ge 0$. Denoting by B and B_ψ the generators of $\mathbf{S}$ and $\mathbf{S}_\psi$, respectively, it also follows that $\{\mathrm{Re}\,\lambda > 0\} \subset \varrho(B_\psi)$, the resolvent set of B_ψ, and $i_\psi \circ R(\lambda, B) = R(\lambda, B_\psi) \circ i_\psi$ for all $\mathrm{Re}\,\lambda > 0$.

The final ingredient for the proof of Theorem 0.6 is taken from [BNR2]:

Proposition 2.3. *Let X be a Banach space and let $f \in BUC(\mathbb{R}_+, X)$. Then $\sigma_{\mathcal{L}}(f) = \sigma_{i\mathbb{R}}(B, f)$, where B is the generator of the left translation semigroup on $BUC(\mathbb{R}_+, X)$.*

Proof of Theorem 0.6: We will now show that the semigroup $\mathbf{S}_\psi$ on $Y := BUC_\psi(\mathbb{R}_+, l^2)$ and the element $i_\psi g \in Y$, where $g \in BUC(\mathbb{R}_+, l^2)$ is the function constructed in Theorem 0.4, have the required properties.

(1): The orbit $t \mapsto S(t)g$ is clearly bounded and uniformly continuous. Hence $t \mapsto i_\psi S(t)g = S_\psi(t) i_\psi g$ is bounded and uniformly continuous as well.

(2): By Theorem 0.4 (1) and Proposition 2.3 the local resolvent $\lambda \mapsto R(\lambda, B)g$ extends holomorphically across $i\mathbb{R}\setminus\{0\}$; let $F(\cdot)$ be such an extension. Then $i_\psi F(\cdot)$ is a holomorphic extension of $\lambda \mapsto R(\lambda, B_\psi) i_\psi g$ across $i\mathbb{R}\setminus\{0\}$. Hence $\sigma_{i\mathbb{R}}(B_\psi, i_\psi g) \subset \{0\}$. But $\sigma_{i\mathbb{R}}(B_\psi, i_\psi g)$ cannot be empty, since this would imply $\lim_{t\to\infty} \|S_\psi(t) i_\psi g\|_Y = 0$ by [Ne, Corollary 5.3.7], contradicting (5) below.

(3): By Lemma 2.1 for $\tau \ge 1$ we have
$$\begin{aligned}\left\| \frac{1}{\tau} \int_0^\tau S_\psi(t) i_\psi g \, dt \right\|_Y &= \sup_{s \ge 0} \frac{1}{1 + s^{\frac34}} \left\| \frac{1}{\tau} \int_0^\tau g(t+s)\, dt \right\| \\ &\le \sup_{s\ge 0} \frac{1}{1+s^{\frac34}} \cdot C\tau^{-\frac14}(1 + 2s^{\frac34}) \le 2C\tau^{-\frac14},\end{aligned}$$
where $C > 0$ is the constant of Theorem 0.4.

(4): Since $\lim_{t\to\infty} \langle g(t), x^*\rangle = 0$ for all $x^* \in l^2$ we may take $Z \subseteq Y^*$ to be the linear span of $\{\delta_t \otimes x^* : \ t \ge 0, \ x^* \in l^2\}$. Noting that $\delta_t \otimes x^*$ is a bounded linear form on Y of norm $\le \psi(t)\|x\|$, we see that Z is indeed norming.

(5): This follows from
$$\limsup_{t\to\infty} \|S_\psi(t) i_\psi g\|_Y \ge \limsup_{t\to\infty} \|(S_\psi(t) i_\psi g)(0)\| = \limsup_{t\to\infty} \|g(t)\| \ge 1.$$
∎

3. Translation invariant functionals

In this section we will use the function g of Theorem 0.4 to prove the existence of a translation invariant functional on the closed subspace $C_0^{weak}(\mathbb{R}_+, l^2)$ of $BUC(\mathbb{R}_+, l^2)$ consisting of all functions f such that $\lim_{t\to\infty}\langle f(t), x^*\rangle = 0$ for all $x^* \in l^2$.

Let X be a Banach space and let F be a closed subspace of $BUC(\mathbb{R}_+, X)$ which is invariant under left translations. A *translation invariant functional* on F is a non-zero element $L \in F^*$ such that $\langle f_t, L\rangle = \langle f, L\rangle$ for all $f \in F$. In terms of the left translation semigroup $\mathbf{S}$ on $BUC(\mathbb{R}_+, X)$, the assumptions can be reformulated as saying that F is a closed $\mathbf{S}-$invariant subspace and that $\langle S(t)f, L\rangle = \langle f, L\rangle$ for all $t \ge 0$. In other words, L is a fixed point of the semigroup $\mathbf{S}_F^*$, the adjoint of the restricted semigroup $\mathbf{S}_F = \mathbf{S}|_F$. Denoting by B_F the generator of $\mathbf{S}_F$, this, in turn, is equivalent to the condition $L \in D(B_F^*)$ and $B_F^* L = 0$.

Theorem 3.1. *There exists a translation invariant functional on* $C_0^{weak}(\mathbb{R}_+, l^2)$.

Proof. Let $F := C_0^{weak}(\mathbb{R}_+, l^2)$. By the observations just made we need to show that $0 \in \sigma_p(B_F^*)$, the point spectrum of B_F^*.

Suppose, for a contradiction, that $0 \notin \sigma_p(B_F^*)$. Let E denote the closed linear span in $BUC(\mathbb{R}_+, l^2)$ of the $\mathbf{S}-$orbit of g, where $g \in BUC(\mathbb{R}_+, l^2)$ is the function of Theorem 0.4. By Theorem 0.4 (4) we have $g \in C_0^{weak}(\mathbb{R}_+, l^2)$ and consequently $E \subseteq F$. The extension lemma for the purely imaginary point spectrum of an adjoint generator [Ne, Lemma 5.5.6] then implies that $0 \notin \sigma_p(B_E^*)$. Since $\sigma_{i\mathbb{R}}(B_E, g) = \sigma_{i\mathbb{R}}(B, g) = \sigma_{\mathcal{L}}(g) = \{0\}$, from [Ne, Lemma 5.1.8] we infer that $\sigma_p(B_E^*) \cap i\mathbb{R} = \emptyset$. Then by [BNR1, Proposition 3.2 and Theorem 3.4] or [Ne, Lemma 5.1.9 and Theorem 5.1.11], applied to $\mathbf{S}_E$, it follows that

$$\lim_{t\to\infty} \|g(t)\| = \lim_{t\to\infty} \|S_E(t)g\| = 0,$$

a contradiction. ∎

Acknowledgement. I am indebted to Ben de Pagter for suggesting an improvement in Section 2.

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Department of Mathematics
TU Delft
P. O. Box 5031
NL 2600 GA Delft
The Netherlands
J.vanNeerven@twi.tudelft.nl