UNIQUENESS OF INVARIANT MEASURES FOR THE STOCHASTIC CAUCHY PROBLEM IN BANACH SPACES

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ABSTRACT. We study uniqueness for invariant measures of the stochastic abstract Cauchy problem

$$du(t) = Au(t)\,dt + B\,dW_H(t), \quad t \geqslant 0,$$
$$u(0) = x,$$

where A is the generator of a C_0-semigroup $\{S(t)\}_{t\geqslant 0}$ on a separable real Banach space, $\{W_H(t)\}_{t\geqslant 0}$ is a cylindrical Wiener process with Cameron-Martin space H, and $B \in \mathscr{L}(H,E)$ is a bounded linear operator. Under a nondegeneracy assumption, it is shown that a Gaussian invariant measure, if one exists, is unique if the adjoint orbits $t \mapsto S^*(t)x^*$ are bounded for all $x^* \in \bigcap_{n\geqslant 1} \mathscr{D}(A^{*n})$.

1. INTRODUCTION

In this paper we study uniqueness of invariant measures for the stochastic abstract Cauchy problem

$$du(t) = Au(t)\,dt + B\,dW_H(t), \quad t \geqslant 0, \qquad (1.1)$$
$$u(0) = x,$$

where A is the generator of a C_0-semigroup $\mathbf{S} = \{S(t)\}_{t\geqslant 0}$ on a separable real Banach space E, $\{W_H(t)\}_{t\geqslant 0}$ is a cylindrical Wiener process with Cameron-Martin space H, and $B \in \mathscr{L}(H,E)$ is a bounded linear operator. For Hilbert spaces E, a variety of conditions is known that imply uniqueness of the invariant measure if one exists:

- $\mathbf{S}$ is strongly stable, i.e. $\lim_{t\to\infty} \|S(t)x\| = 0$ for all $x \in E$;
- $\mathbf{S}$ is compact;
- $\mathbf{S}$ is null controllable with respect to the control pair (B, H).

In the last two cases an additional nondegeneracy condition needs to be imposed; cf. Example 2.8 below. We refer to [7], [8], and the references cited there for the proofs and more detailed information.

We will give alternative, purely functional analytic proofs of these results which work for arbitrary separable real Banach spaces E. Among other things we prove uniqueness of the invariant measure under a nondegeneracy assumption and the following rather mild boundedness condition:

- For all $x^* \in \bigcap_{n\geqslant 1} \mathscr{D}(A^{*n})$, the orbit $t \mapsto S^*(t)x^*$ is bounded.

This result seems to be new also in the Hilbert space framework.

Finally we mention [4] and the papers cited there, where uniqueness of invariant measures in the finite-dimensional setting is studied from a more general perspective.

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2. THE LYAPUNOV EQUATION $AX + XA^* = -Q$

Let E be a real Banach space. An operator $Q \in \mathscr{L}(E^*, E)$ is called *positive* if $\langle Qx^*, x^*\rangle \geqslant 0$ for all $x^* \in E^*$, and *symmetric* if $\langle Qx^*, y^*\rangle = \langle Qy^*, x^*\rangle$ for all $x^*, y^* \in E^*$.

In our study of (1.1), we will be mostly interested the following example:

Example 2.1. If H is a real Hilbert space and $B \in \mathscr{L}(H, E)$ is a bounded linear operator, then $Q := B \circ B^* \in \mathscr{L}(E^*, E)$ is positive and symmetric.

It is easily checked that on the range of Q, the formula

$$[Qx^*, Qy^*]_{H_Q} := \langle Qx^*, y^*\rangle \tag{2.1}$$

defines an inner product $[\,\cdot\,,\cdot\,]_{H_Q}$. If $Qx^* = 0$, then $[Qx^*, Qy^*]_{H_Q} = \langle Qx^*, y^*\rangle = 0$, and if $Qy^* = 0$, then $[Qx^*, Qy^*]_{H_Q} = \langle Qy^*, x^*\rangle = \langle Qx^*, y^*\rangle = 0$ by the symmetry of Q. This shows that $[\,\cdot\,,\cdot\,]_{H_Q}$ is well defined.

We denote by H_Q the real Hilbert space obtained by completing the range of Q with respect to $[\,\cdot\,,\cdot\,]_{H_Q}$. This space is usually called the *reproducing kernel Hilbert space* associated with Q. From

$$\|Qx^*\|_{H_Q}^2 = \langle Qx^*, x^*\rangle = |\langle Qx^*, x^*\rangle| \leqslant \|Q\|_{\mathscr{L}(E^*,E)}\|x^*\|^2$$

we see that Q, as an operator from E^* into H_Q, is bounded with norm $\leqslant \|Q\|_{\mathscr{L}(E^*,E)}^{\frac12}$. From

$$|\langle Qx^*, y^*\rangle| \leqslant \|Qx^*\|_{H_Q}\|Qy^*\|_{H_Q} \leqslant \|Qx^*\|_{H_Q}\|Q\|_{\mathscr{L}(E^*,H_Q)}\|y^*\|$$

it then follows that

$$\|Qx^*\| \leqslant \|Q\|_{\mathscr{L}(E^*,H_Q)}\|Qx^*\|_{H_Q}.$$

Thus, the inclusion mapping from the range of Q into E is continuous with respect to the inner product $[\,\cdot\,,\cdot\,]_{H_Q}$ and extends to a bounded linear operator i_Q from H_Q into E of norm $\leqslant \|Q\|_{\mathscr{L}(E^*,H_Q)}$. We will show next that i_Q is an injection from H_Q into E and that

$$Q = i_Q \circ i_Q^*. \tag{2.2}$$

Given an element $x^* \in E^*$ we denote by h_{x^*} the element in H represented by Qx^*. With this notation we have

$$i_Q(h_{x^*}) = Qx^*$$

and (2.1) becomes

$$[h_{x^*}, h_{y^*}]_{H_Q} = \langle Qx^*, y^*\rangle.$$

For all $y^* \in E^*$ we then have

$$[h_{x^*}, h_{y^*}]_{H_Q} = \langle Qx^*, y^*\rangle = \langle i_Q(h_{x^*}), y^*\rangle = [h_{x^*}, i_Q^* y^*]_{H_Q}.$$

Since the elements h_{x^*} span a dense subspace of H_Q it follows that

$$h_{y^*} = i_Q^* y^*.$$

Therefore,

$$Qy^* = i_Q(h_{y^*}) = i_Q(i_Q^* y^*)$$

for all $y^* \in E^*$, and (2.2) follows. Finally if $i_Q g = 0$ for some $g \in H_Q$, then for all $y^* \in E^*$ we have

$$[g, h_{y^*}]_{H_Q} = [g, i_Q^* y^*]_{H_Q} = \langle i_Q g, y^*\rangle = 0,$$

and therefore $g = 0$. This proves that i_Q is injective.

Being an adjoint operator, i_Q^* is weak*-continuous, hence weak*-to-weakly continuous since H is reflexive. Since i_Q is continuous, hence weakly continuous, it follows that $Q =$

$i_Q \circ i_Q^*$ is weak*-to-weakly continuous. A similar argument shows that H_Q is separable whenever E is separable. These observations will be used repeatedly below.

If E is a Hilbert space, then upon identifting E^* with E we may regard Q as a positive selfadjoint operator on E and as such we have $i_Q(H_Q) = Q^{\frac{1}{2}}(E)$.

We refer to [17] and the references cited there for more details.

Throughout the rest of this paper, E is a *separable* real Banach space, $\mathbf{S} = \{S(t)\}_{t\geqslant 0}$ is a C_0-semigroup of bounded linear operators on E with infinitesimal generator A, and $Q \in \mathscr{L}(E^*, E)$ is a positive symmetric operator. For each $t > 0$ we define a positive symmetric operator $Q_t \in \mathscr{L}(E^*, E)$ by

$$Q_t x^* := \int_0^t S(s)QS^*(s)x^* \, ds, \qquad x^* \in E^*.$$

This integral is easily seen to exist as a Bochner integral; cf. [15]. For $t > 0$ and $h \geqslant 0$ we have the following algebraic identity:

$$Q_{t+h} = Q_t + S(t)Q_h S^*(t).$$

From [10] we recall the following result.

Proposition 2.2. *The following assertions are equivalent:*

(1) *There exists an operator $Q_\infty \in \mathscr{L}(E^*, E)$ such that*

$$\langle Q_\infty x^*, y^*\rangle = \lim_{t\to\infty} \langle Q_t x^*, y^*\rangle, \qquad x^*, y^* \in E^*;$$

(2) *There exists a positive symmetric solution $X \in \mathscr{L}(E^*, E)$ to the operator equation*

$$AX + XA^* = -Q.$$

In this situation Q_∞ is a positive symmetric solution of the equation $AX + XA^* = -Q$.

Here, we say that an operator $X \in \mathscr{L}(E^*, E)$ is a *solution* of $AX + XA^* = -Q$ if for all $x^* \in \mathscr{D}(A^*)$ we have $Xx^* \in \mathscr{D}(A)$ and $AXx^* + XA^*x^* = -Qx^*$.

For a systematic study of the so-called *Lyapunov equation*

$$AX + XB = C$$

we refer to [1], [18], and the references cited there. We will need the following observation; cf. [18], [10]:

Proposition 2.3. *If $X \in \mathscr{L}(E^*, E)$ is a positive symmetric solution of the equation $AX + XA^* = -Q$, then for all $t > 0$ we have*

$$X - S(t)XS^*(t) = Q_t.$$

In what follows, we shall say that *Hypothesis* $(\mathbf{H}Q_\infty)$ *holds* if the equivalent conditions of Proposition 2.2 are satisfied. Whenever we assume $(\mathbf{H}Q_\infty)$, this will be indicated by the notation '$(\mathbf{H}Q_\infty)$' in the statement of the results.

Hypothesis $(\mathbf{H}Q_\infty)$ trivially holds if $\mathbf{S}$ is *uniformly exponentially stable*, i.e. if there are constants $M \geqslant 1$ and $a > 0$ such that $\|S(t)\| \leqslant Me^{-at}$ for all $t \geqslant 0$. In this case Q_∞ is the *unique* positive symmetric solution of $AX + XA^* = -Q$. Indeed, by Proposition 2.3 for all $x^*, y^* \in E^*$ we have

$$\langle Q_\infty x^*, y^*\rangle = \lim_{t\to\infty} \langle Q_t x^*, y^*\rangle = \lim_{t\to\infty} \langle Xx^* - S(t)XS^*(t)x^*, y^*\rangle = \langle Xx^*, y^*\rangle.$$

The following result shows that conversely, Hypothesis $(\mathbf{H}Q_\infty)$ implies a stability property of the adjoint semigroup $\mathbf{S}^*$:

Lemma 2.4 ($\mathbf{H}Q_\infty$). *For all $x^* \in \mathscr{D}(A^*)$ we have*

$$\lim_{t\to\infty} \|i_Q^* S^*(t)x^*\|_{H_Q} = 0.$$

Proof. Let $x^* \in \mathscr{D}(A^*)$ be fixed. Put $y^* := (\lambda - A^*)x^*$, where λ is a sufficiently large fixed positive real number. Then by the Cauchy-Schwarz inequality,

$$\|i_Q^* S^*(t)x^*\|_{H_Q}^2 = \left\| \int_0^\infty e^{-\lambda s} i_Q^* S^*(t+s)y^*\,ds \right\|_{H_Q}^2 \leqslant \frac{1}{2\lambda}\int_0^\infty \|i_Q^* S^*(t+s)y^*\|_{H_Q}^2\,ds.$$

As $t \to \infty$, the right hand side tends to 0. ∎

We shall use the notation $\sigma_p(B)$ for the point spectrum of the complexification $B_{\mathbb{C}}$ of a linear operator B.

Proposition 2.5 ($\mathbf{H}Q_\infty$). *For all $\lambda \in \mathbb{C}$ with $\operatorname{Re}\lambda \geqslant 0$ we have*

$$\ker(\lambda - A_{\mathbb{C}}^*) \subseteq \ker((Q_\infty)_{\mathbb{C}}).$$

In particular, if Q_∞ is injective then

$$\sigma_p(A^*) \cap \{\operatorname{Re} z \geqslant 0\} = \varnothing.$$

Proof. First note that since by assumption Hypothesis ($\mathbf{H}Q_\infty$) holds for $\mathbf{S}$, for all $\alpha \geqslant 0$ it also holds for the rescaled semigroup $\mathbf{S}_\alpha = \{e^{-\alpha t}S(t)\}_{t\geqslant 0}$. Indeed, this follows from

$$\int_0^\infty e^{-2\alpha t}\langle QS^*(t)x^*, S^*(t)x^*\rangle\,dt \leqslant \int_0^\infty \langle QS^*(t)x^*, S^*(t)x^*\rangle\,dt, \qquad x^* \in E^*,$$

and polarization. Therefore it is enough to check that $\ker(i\omega - A_{\mathbb{C}}^*) \subseteq \ker((Q_\infty)_{\mathbb{C}})$ for all $\omega \in \mathbb{R}$.

Let $x_{\mathbb{C}}^* \in \ker(i\omega - A_{\mathbb{C}}^*)$ be fixed. By the spectral mapping theorem for the point spectrum [12, Theorem A.III.6.3] we have $S_{\mathbb{C}}^*(t)x_{\mathbb{C}}^* = e^{i\omega t}x_{\mathbb{C}}^*$ for all $t \geqslant 0$. Write $x_{\mathbb{C}}^* = y^* + iz^*$ with $y^*, z^* \in E^*$ and note that $y^*, z^* \in \mathscr{D}(A^*)$. By Lemma 2.4, applied to the elements $S^*(t)y^*, S^*(t)z^* \in \mathscr{D}(A^*)$,

$$\|(i_Q)_{\mathbb{C}}^* S_{\mathbb{C}}^*(t)x_{\mathbb{C}}^*\|_{(H_Q)_{\mathbb{C}}} = \lim_{s\to\infty} \|(i_Q)_{\mathbb{C}}^* S_{\mathbb{C}}^*(t+s)x_{\mathbb{C}}^*\|_{(H_Q)_{\mathbb{C}}} = 0.$$

It follows that $i_Q^* S^*(t)y^* = i_Q^* S^*(t)z^* = 0$ for all $t \geqslant 0$. By the identity

$$\int_0^\infty \|i_Q^* S^*(t)x^*\|_{H_Q}^2\,dt = \langle Q_\infty x^*, x^*\rangle, \qquad x^* \in E^*,$$

this implies

$$\langle Q_\infty y^*, y^*\rangle = \langle Q_\infty z^*, z^*\rangle = 0.$$

Since Q_∞ is positive and symmetric, it follows that $Q_\infty y^* = Q_\infty z^* = 0$ and therefore $(Q_\infty)_{\mathbb{C}}\, x_{\mathbb{C}}^* = 0$. ∎

Theorem 2.6 ($\mathbf{H}Q_\infty$). *If Q_∞ is injective and $S(t)$ is compact for all $t > 0$, then $\mathbf{S}$ is uniformly exponentially stable.*

Proof. Let

$$E^\odot := \bigl\{x^* \in E^* : \lim_{t\downarrow 0} \|S^*(t)x^* - x^*\| = 0\bigr\}.$$

This is a closed, $\mathbf{S}^*-$invariant linear subspace of E^*. The restriction $\mathbf{S}^\odot$ of $\mathbf{S}^*$ to $E^\odot$ is strongly continuous, and for the spectrum of its generator $A^\odot$ we have $\sigma(A) = \sigma(A^\odot)$; cf. [13, Chapter 1].

The operators $S^{\odot}(t)$ being compact for all $t > 0$, by [12, Theorem A.II.1.25 and Proposition A.III.2.5] and the Riesz-Schauder theory of compact operators we have $\sigma(A^{\odot}) = \sigma_p(A^{\odot})$; cf. [11, Theorem 5.14.2]. Hence,

$$\sigma(A) = \sigma(A^{\odot}) = \sigma_p(A^{\odot}) = \sigma_p(A^*) \subset \{\mathrm{Re}\, z < 0\}.$$

Moreover, $\mathbf{S}$ is uniformly continuous for $t > 0$ by [12, A.II.1.25]. Hence by [12, Theorem A.II.1.20], for all $\varepsilon > 0$ the set $\{\lambda \in \sigma(A) : \mathrm{Re}\,\lambda \geqslant -\varepsilon\}$ is compact. It follows that the spectral bound $s(A) = \sup\{\mathrm{Re}\,\lambda : \lambda \in \sigma(A)\}$ is strictly negative. The spectral mapping theorem for eventually uniformly continuous semigroups [12, Theorem A.III.6.6] now implies the desired result. ∎

Corollary 2.7 ($\mathbf{H}Q_\infty$)**.** *If Q_∞ is injective and $S(t)$ is compact for all $t > 0$, then Q_∞ is the unique positive symmetric solution of the equation $AX - XA^* = -Q$.*

The following example shows that the injectivity of Q_∞ cannot be omitted:

Example 2.8. Let $E = \mathbb{R}^2$ and define $\mathbf{S}$ and Q by

$$S(t)(x, y) = (e^{-t}x, y), \quad t \geqslant 0,$$
$$Q(x, y) = (x, 0).$$

Note that Q is positive and symmetric. Hypothesis ($\mathbf{H}Q_\infty$) holds and we have

$$Q_\infty(x, y) = (-\tfrac{1}{2}x, 0).$$

For $c \geqslant 0$ define positive symmetric operators X_c by

$$X_c(x, y) := \big(\tfrac{1}{2}x, cy\big).$$

Since $A(x, y) = A^*(x, y) = (-x, 0)$ we have

$$AX_c(x, y) + X_cA^*(x, y) = \big(-\tfrac{1}{2}x, 0\big) + \big(-\tfrac{1}{2}x, 0\big) = -(x, 0) = -Q(x, y),$$

and therefore each X_c is a solution of $AX + XA^* = -Q$.

By [14, Proposition 5.1.15], for *bounded* semigroups we have $\sigma_p(A) \cap i\mathbb{R} \subseteq \sigma_p(A^*) \cap i\mathbb{R}$. For later use we need a slightly sharpened version of this result which we state next.

Proposition 2.9. *If there exists a weak*-dense linear subspace Y of E^* such that $t \mapsto S^*(t)x^*$ is bounded for all $x^* \in Y$, then $\sigma_p(A) \cap i\mathbb{R} \subseteq \sigma_p(A^*) \cap i\mathbb{R}$.*

Proof. Let $i\omega \in \sigma_p(A) \cap i\mathbb{R}$. By rescaling we may assume that $\omega = 0$. Let $x_0 \in D(A)$ be an eigenvector with eigenvalue 0. Then $S(t)x_0 = x_0$ for all $t \geqslant 0$. Here and in the rest of the paper, for the sake of notational simplicity we will suppress subscripts indicating that we are using complexifications.

Since Y is weak*-dense in E^* we may choose $y_0^* \in Y$ such that $\langle x_0, y_0^*\rangle = 1$. Then $t \mapsto \langle S(t)x, y_0^*\rangle$ defines an element of $C_b[0, \infty)$, the space of all bounded continuous functions on $[0, \infty)$. Let $\phi \in (C_b[0, \infty))^*$ be a left invariant mean and define $x_0^* \in X^*$ by

$$\langle x, x_0^*\rangle := \phi(\langle S(\cdot)x, y_0^*\rangle), \quad x \in X.$$

For all $x \in X$ we have

$$\langle x, S^*(t)x_0^*\rangle = \phi(\langle S(t + \cdot)x, y_0^*\rangle) = \phi(\langle S(\cdot)x, y_0^*\rangle) = \langle x, x_0^*\rangle.$$

Hence, $S^*(t)x_0^* = x_0^*$ for all $t \geqslant 0$, so $x_0^* \in D(A^*)$ and $A^*x_0^* = 0$. Also,

$$\langle x_0, x_0^*\rangle = \phi(\langle S(\cdot)x_0, y_0^*\rangle) = \phi(\langle x_0, y_0^*\rangle\mathbf{1}) = \phi(\mathbf{1}) = 1,$$

which shows that $x_0^* \neq 0$. This proves that x_0^* is an eigenvector of A^* with eigenvalue 0. ∎

Combining Propositions 2.5 and 2.9, we have proved:

Theorem 2.10 (**H**Q_∞)**.** *If Q_∞ is injective and the linear subspace of all $x^* \in E^*$ for which $t \mapsto S^*(t)x^*$ is bounded is weak*-dense in E^*, then $\sigma_p(A) \cap i\mathbb{R} = \varnothing$.*

Recall that if $S(t)$ is compact for all $t > 0$, then the resolvent operator $(\lambda - A)^{-1}$ is compact for all $\lambda \in \varrho(A)$. In the following result we prove uniqueness of positive symmetric solutions under a weaker compactness assumption. The price to pay is a mild boundedness condition.

Let

$$\mathscr{E} := \bigcap_{n \geqslant 1} \mathscr{D}(A^{*n})$$

denote the set of *entire vectors* for A^*. As is well known, $\mathscr{E}$ is weak*-dense in E^*.

Theorem 2.11. *Assume that Q_∞ is injective and that for all $x^* \in \mathscr{E}$ the orbit $t \mapsto S^*(t)x^*$ is bounded. Let R_1 and R_2 be positive symmetric solutions of the equation*

$$AX - XA^* = -Q.$$

If the operator $(\lambda - A)^{-1}(R_1 - R_2)$ is compact for some $\lambda \in \varrho(A)$, then $R_1 = R_2$.

Proof. Step 1. Fix $x^* \in \mathscr{E}$. For all $y^* \in E^*$ we have

$$\begin{aligned} |\langle Q_\infty y^*, S^*(t)x^*\rangle| &= \left| \int_0^\infty [i_Q^* S^*(s)y^*, i_Q^* S^*(t+s)x^*]_{H_Q}\, ds \right| \\ &\leqslant \left(\int_0^\infty \|i_Q^* S^*(s)y^*\|_{H_Q}^2\, ds \right)^{\frac12} \left(\int_t^\infty \|i_Q^* S^*(s)x^*\|_{H_Q}^2\, ds \right)^{\frac12}. \end{aligned}$$

It follows that

$$\lim_{t\to\infty} \langle Q_\infty y^*, S^*(t)x^*\rangle = 0, \qquad y^* \in E^*.$$

Since Q_∞ is injective and positive symmetric, its range is dense in E. The boundedness of $t \mapsto S^*(t)x^*$ therefore implies that

$$\lim_{t\to\infty} \langle x, S^*(t)x^*\rangle = 0, \qquad x \in E.$$

Step 2. It follows from Proposition 2.3 that

$$R_1 - R_2 = S(t)(R_1 - R_2)S^*(t), \qquad t \geqslant 0.$$

For $x^* \in \mathscr{E}$ define

$$V_{x^*} := \big\{ (R_1 - R_2)S^*(t)x^* :\ t \geqslant 0 \big\}.$$

We will show that V_{x^*} is relatively compact in E.

Let $y^* := (\lambda - A^*)x^*$. Then $y^* \in \mathscr{E}$, so $t \mapsto S^*(t)y^*$ is bounded. It follows that the set

$$W_{y^*} := \big\{ (R_1^* - R_2^*)(\lambda - A^*)^{-1} S^*(t)y^* : t \geqslant 0 \big\}$$

is relatively compact in E. But R_1^* and R_2^* map E^* into E, and as operators from E^* into E we have $R_1^* = R_1$ and $R_2^* = R_2$. Hence,

$$(R_1 - R_2)S^*(t)x^* = (R_1^* - R_2^*)(\lambda - A^*)^{-1} S^*(t)y^*,$$

and $V_{x^*} = W_{y^*}$ is relatively compact in E.

Step 3. For all $x^* \in \mathscr{E}$ and $y^* \in \mathscr{E}$ we have

$$\langle (R_1 - R_2)x^*, y^* \rangle = \langle (R_1 - R_2)S^*(t)x^*, S^*(t)y^* \rangle.$$

By Step 1, $\lim_{t\to\infty} S^*(t)y^* = 0$ weak*. By the boundedness of $t \mapsto S^*(t)y^*$, the weak*-convergence is uniform on compact subsets of E. In particular, it is uniform on the relatively compact set V_{x^*}. Hence by Step 2,

$$\lim_{t\to\infty} \langle (R_1 - R_2)S^*(t)x^*, S^*(t)y^* \rangle = 0.$$

This shows that $\langle (R_1 - R_2)x^*, y^* \rangle = 0$ for all $x^* \in \mathscr{E}$ and $y^* \in \mathscr{E}$.

Since $\mathscr{E}$ is weak*-dense in E^*, it follows that $(R_1 - R_2)x^* = 0$ for all $x^* \in \mathscr{E}$. But the operator $R_1 - R_2$ is weak*-to-weakly continuous, being the difference of two positive symmetric operators, and therefore $R_1 - R_2 = 0$. ∎

The following example shows that without any compactness assumption, non-uniqueness may occur. At the same time it shows that Hypothesis ($\mathbf{H}Q_\infty$) need not imply any strong stability of the orbits of $\mathbf{S}^*$ itself; compare with Lemma 2.4.

Example 2.12. Let $E = L^2[0,\infty)$ and let $\mathbf{S}$ be the left shift semigroup on E:

$$S(t)g(s) := g(s+t), \qquad s,t \geqslant 0.$$

This is a strongly stable C_0-semigroup on E with generator A given by $Af = f'$ for $f \in \mathscr{D}(A) = W^{1,2}[0,\infty)$. Its adjoint is the right shift semigroup,

$$S^*(t)f(s) = \begin{cases} 0, & 0 \leqslant s < t; \\ f(s-t), & s \geqslant t. \end{cases}$$

This is a semigroup of partial isometries and for $f \in \mathscr{D}(A^*) = W_0^{1,2}[0,\infty)$ we have $A^*f = -f'$.

Let ψ be a nonnegative measurable function on $[0,\infty)$. Let $Q \in \mathscr{L}(E)$ be the operator of multiplication with ψ. Then Q is positive and selfadjoint, hence positive and symmetric in the sense defined at the beginning of this section. Defining

$$\psi_t(s) := \int_0^t \psi(\tau + s)\, d\tau,$$

an easy calculation shows that

$$Q_t f = \int_0^t S(\tau)QS^*(\tau)f\, d\tau = \psi_t f.$$

Hence ($\mathbf{H}Q_\infty$) holds if and only if the function $\psi_\infty : [0,\infty) \to [0,\infty]$ defined by

$$\psi_\infty(s) := \int_0^\infty \psi(\tau + s)\, d\tau$$

belongs to $L^\infty[0,\infty)$. Since ψ is nonnegative, ψ_∞ is nonincreasing, and therefore $\psi_\infty \in L^\infty[0,\infty)$ if and only if

$$\int_0^\infty \psi(\tau)\, d\tau < \infty,$$

i.e., if and only if $\psi \in L^1[0,\infty)$.

Let us from now on assume that $\psi \in L^1[0,\infty)$, and let ψ_∞ be defined as above. Then $Q_\infty f = \psi_\infty f$. Since ψ_∞ is nonincreasing, Q_∞ is injective if and only if ψ_∞ is strictly positive, which happens if and only if ψ is not compactly supported.

For $c \geqslant 0$ let us define

$$X_c f := (Q_\infty + c)f = (\psi_\infty + c)f, \qquad f \in L^2[0,\infty).$$

This is a positive selfadjoint operator on $L^2[0,\infty)$. For all $f \in \mathscr{D}(A^*)$ we have $Q_\infty f \in \mathscr{D}(A)$ (because Q_∞ is a solution of $AX + XA^* = -Q$) and $cf \in \mathscr{D}(A)$ (because $\mathscr{D}(A^*) \subseteq \mathscr{D}(A)$). Therefore $X_c f \in \mathscr{D}(A)$ for all $f \in \mathscr{D}(A^*)$ and

$$\begin{aligned} AX_c f + X_c A^* f &= \big((\psi_\infty + c)f\big)' - (\psi_\infty + c)f' \\ &= \big(\psi_\infty f\big)' - \psi_\infty f' \\ &= AQ_\infty f + Q_\infty A^* f = -Qf. \end{aligned}$$

Thus for all $c \geqslant 0$ the operator X_c is a positive selfadjoint solution of $AX + XA^* = -Q$.

Theorem 2.11 applies in particular to semigroups with compact resolvent. Compact semigroups always have a compact resolvent, but for such semigroups we can apply Corollary 2.7. On the other hand, *eventually* compact semigroups need not have a compact resolvent; cf. [12, Remark A.II.1.26]. The following result gives a sufficient condition implying uniqueness for eventually compact semigroups.

Corollary 2.13 ($\mathbf{H}Q_\infty$). *Assume that Q_∞ is injective and that for all $x^* \in \mathscr{E}$ the orbit $t \mapsto S^*(t)x^*$ is bounded. Let R_1 and R_2 be positive symmetric solutions of the equation*

$$AX - XA^* = -Q.$$

If $\mathbf{S}$ is eventually compact, then $R_1 = R_2$.

Proof. For all $t > 0$ we have $(R_1 - R_2) - S(t)(R_1 - R_2)S^*(t) = 0$. Taking t sufficiently large, it follows that $R_1 - R_2$ is compact. We may now apply Theorem 2.11. ∎

We conclude this section with an example where ($\mathbf{H}Q_\infty$) is satisfied with Q_∞ injective, even though $\mathbf{S}$ is unbounded.

Example 2.14. First let $\mathbf{S}$ be an arbitrary C_0-semigroup on E. Choose a fixed $x_0 \in E$ and let

$$Q := x_0 \otimes x_0;$$

cf. [7, Example 11.9]. The operator Q is positive and symmetric, and

$$Q_t x^* = \int_0^t \langle S(s)x_0, x^*\rangle S(s)x_0 \, ds, \qquad t > 0,\ x^* \in E^*.$$

In particular,

$$\langle Q_t x^*, x^*\rangle = \int_0^t \langle S(s)x_0, x^*\rangle^2 \, ds,$$

and therefore ($\mathbf{H}Q_\infty$) holds if and only if

$$\int_0^\infty \langle S(s)x_0, x^*\rangle^2 \, ds < \infty, \qquad x^* \in E^*.$$

Let E_0 denote the closed linear span in E of the orbit of x_0. Clearly, $Q_\infty x^* \in E_0$ for all $x^* \in E^*$. Hence $E_0 = E$ if Q_∞ has dense range. If Q_∞ fails to have dense range, then by the Hahn-Banach theorem there exists a non-zero $x^* \in E^*$ such that

$$\langle Q_\infty x^*, x^*\rangle = \int_0^\infty \langle S(s)x_0, x^*\rangle^2 \, ds = 0.$$

Hence $\langle S(s)x_0, x^*\rangle = 0$ for almost all $s \geqslant 0$, and then by continuity $\langle S(s)x_0, x^*\rangle = 0$ for all $s \geqslant 0$. It follows that x^* annihilates E_0, showing that $E_0 \neq E$. On the other hand, since Q_∞ is positive and symmetric, Q_∞ is injective if and only if Q_∞ has dense range. Summarizing, Q_∞ is injective if and only if the orbit of x_0 spans a dense linear subspace of E.

We will now give an example of an unbounded semigroup $\mathbf{S}$ with a square integrable orbit whose span is dense. It is a minor modification of [2, Example 3.4].

Take $E = C_0[0,\infty)$, the space of continuous functions on $[0,\infty)$ vanishing at infinity. Define

$$S(t)f(s) := e^t f(t+s), \qquad t,s \geqslant 0, \;\; f \in C_0[0,\infty).$$

Let $\{f^{(k)} : k = 1,2,\dots\}$ be a dense subset of the unit ball of

$$C_{00}[0,\infty) := \{g \in C_0[0,\infty) : \; g(0) = 0\}$$

such that each $f^{(k)}$ has support in $[0, 2^k]$, and let

$$\begin{aligned} f(t) = {} & \exp\left(-(2^{n+1}+2^{k+1})^2\right) f^{(k)}(t - 2^{n+1} - 2^k), \\ & \qquad 2^{n+1} + 2^k \leqslant t < 2^{n+1} + 2^{k+1}; \; k = 1,\dots,n; \; n = 1,2,\dots, \end{aligned}$$

with $f(t) = 0$ for all remaining $t \geqslant 0$. The construction of f implies that closed linear span of the translates of f contains each $f^{(k)}$. It follows that $C_{00}[0,\infty)$ is contained in the closed linear span of the translates of f, and the same is true for the the closed linear span of the orbit of f. Noting that $C_{00}[0,\infty)$ has codimension one in $C_0[0,\infty)$ and that the orbit of f also contains elements that do not vanish at 0, it follows that the closed linear span of the orbit of f is all of $C_0[0,\infty)$.

Finally, from

$$|f(t)| \leqslant \exp\left(-(2^{n+1}+2^{k+1})^2\right), \qquad 2^{n+1} + 2^k \leqslant t < 2^{n+1} + 2^{k+1},$$

it follows that $|f(t)| \leqslant e^{-t^2}$ for all $t \geqslant 0$. Therefore,

$$\int_0^\infty \|S(t)f\|^2 \, dt = \int_0^\infty (e^t f(t))^2 \, dt \leqslant \int_0^\infty e^{2t-2t^2} \, dt < \infty.$$

3. Cylindrical invariant measures

In what follows, we will assume the reader to be familiar with elements of the theory of cylindrical Gaussian measures. For more information and an explanation of the terminology we are using we refer to [17]. We just recall the following elementary facts.

An operator $R \in \mathscr{L}(E^*, E^{**})$ is called *positive* if $\langle Rx^*, x^*\rangle \geqslant 0$ for all $x^* \in E^*$ and *symmetric* if $\langle Rx^*, y^*\rangle = \langle Ry^*, x^*\rangle$ for all $x^*, y^* \in E^*$. The Fourier transform establishes a one-to-one correspondence between centred cylindrical Gaussian measures μ on E and positive symmetric operators R from $\mathscr{L}(E^*, E^{**})$ via the formula

$$\widehat{\mu}(x^*) := \int_E \exp(i\langle x, x^*\rangle)\, d\mu(x) = \exp\left(-\frac{1}{2}\langle Rx^*, x^*\rangle\right), \qquad x^* \in E^*.$$

Let $m \in E$. We say that μ is a *cylindrical Gaussian measure with mean* m if there exists a positive symmetric $R \in \mathscr{L}(E^*, E^{**})$ such that the Fourier transform of μ is given by

$$\widehat{\mu}(x^*) = \exp\left(i\langle m, x^*\rangle - \frac{1}{2}\langle Rx^*, x^*\rangle\right), \qquad x^* \in E^*.$$

Now let us return to the setting we have been considering so far. For each $t > 0$, the operator $Q_t \in \mathscr{L}(E^*, E)$ is positive and symmetric, and as such it is the covariance of a unique centred cylindical Gaussian measure on E, which we denote by μ_t. Similarly, the operator $Q_\infty \in \mathscr{L}(E^*, E)$, if it exists, is the covariance of a unique centred cylindical

Gaussian measure μ_∞ on E. The following result describes the relationship between the μ_t and μ_∞:

Proposition 3.1. *The following assertions are equivalent:*

(1) *There exists a centred cylindrical Gaussian measure μ_∞ on E such that for all bounded continuous cylindrical functions $f : E \to \mathbb{R}$ we have*

$$\lim_{t\to\infty} \int_E f(x)\,d\mu_t(x) = \int_E f(x)\,d\mu_\infty(x);$$

(2) *Hypothesis $(\mathbf{H}Q_\infty)$ holds.*

In this situation, Q_∞ is the covariance of μ_∞.

Proof. $1. \Rightarrow 2.$: Let $Q_\infty \in \mathscr{L}(E^*, E^{**})$ denote the covariance operator of μ_∞. By considering the real- and imaginary parts of the function $x \mapsto \exp(i\langle x, x^*\rangle)$ separately, it follows from 1. that

$$\begin{aligned}\lim_{t\to\infty} \exp\left(-\frac{1}{2}\langle Q_t x^*, x^*\rangle\right) &= \lim_{t\to\infty} \int_E \exp(i\langle x, x^*\rangle)\,d\mu_t(x)\\ &= \int_E \exp(i\langle x, x^*\rangle)\,d\mu_\infty(x) = \exp\left(-\frac{1}{2}\langle Q_\infty x^*, x^*\rangle\right).\end{aligned}$$

Hence,

$$\lim_{t\to\infty} \langle Q_t x^*, x^*\rangle = \langle Q_\infty x^*, x^*\rangle, \qquad x^* \in E^*.$$

By polarization, it follows that

$$\lim_{t\to\infty} \langle Q_t x^*, y^*\rangle = \langle Q_\infty x^*, y^*\rangle, \qquad x^*, y^* \in E^*.$$

By [5, Proposition 2.2], this implies that Q_∞ actually takes values in E.

$2. \Rightarrow 1.$: Let μ_∞ be the unique centred cylindrical Gaussian measure on E whose covariance operator is Q_∞.

Fix $x^* \in E^*$. Since $\lim_{t\to\infty} \langle Q_t x^*, x^*\rangle = \langle Q_\infty x^*, x^*\rangle$, for the Fourier transforms of μ_t and μ_∞ we have

$$\lim_{t\to\infty} \widehat{\mu_t}(x^*) = \widehat{\mu_\infty}(x^*).$$

By Levi's theorem [16, Corollary 2.8], this implies 1. ∎

If $f : E \to \mathbb{R}$ is a bounded continuous cylindrical function, then the function $P(t)f : E \to \mathbb{R}$ defined by

$$P(t)f(x) := \int_E f(S(t)x + y)\,d\mu_t(y)$$

is a bounded continuous cylindrical function as well. Hence it makes sense to call a cylindrical measure ν on E $P(t)-$*invariant* if for all bounded continuous cylindrical $f : E \to \mathbb{R}$ we have

$$\int_E P(t)f(x)\,d\nu(x) = \int_E f(x)\,d\nu(x). \tag{3.1}$$

We say that ν is $\mathbf{P}-$invariant, or briefly *invariant*, if ν is $P(t)-$invariant for all $t > 0$. We have the following simple observation:

Proposition 3.2 $(\mathbf{H}Q_\infty)$**.** *The centred cylindrical Gaussian measure μ_∞, whose covariance operator is Q_∞, is invariant.*

Proof. The algebraic identity $Q_\infty = Q_t + S(t)Q_\infty S^*(t)$ implies $\mu_\infty = \mu_t * S(t)\mu_\infty$; here $S(t)\mu_\infty$ denotes the image cylindrical measure of μ_∞ under the operator $S(t)$.

Now let $f : E \to \mathbb{R}$ be a bounded continuous cylindrical function. Then,

$$\begin{aligned}\int_E P(t)f(x)\,d\mu_\infty(x) &= \int_E\int_E f(S(t)x+y)\,d\mu_t(y)\,d\mu_\infty(x)\\ &= \int_E\int_E f(\xi+y)\,d\mu_t(y)\,d(S(t)\mu_\infty)(\xi)\\ &= \int_E f(\eta)\,d(\mu_t * S(t)\mu_\infty)(\eta)\\ &= \int_E f(\eta)\,d\mu(\eta).\end{aligned}$$

■

In the converse direction we have the following result:

Theorem 3.3. *Let ν be a cylindrical Gaussian measure on E with covariance operator $Q_\nu \in \mathscr{L}(E^*, E)$ and mean $m \in E$. If ν is invariant, then Q_ν is a solution of the equation $AX - XA^* = -Q$ and its mean satisfies $S(t)m = m$ for all $t \geqslant 0$.*

Proof. Applying (3.1) to the real- and imaginary parts of the functions $\exp(i\langle x, x^*\rangle)$, it follows that the Fourier transform of ν satisfies

$$\widehat{\nu}(x^*) = \widehat{\nu}(S^*(t)x^*)\exp\left(-\frac{1}{2}\langle Q_t x^*, x^*\rangle\right), \qquad t > 0,\ x^* \in E^*;$$

cf. [7, p. 307]. Because of $\widehat{\nu}(x^*) = \exp(i\langle m, x^*\rangle) - \frac{1}{2}\langle Q_\nu x^*, x^*\rangle)$, for $t \geqslant 0$ this implies

$$\langle m, x^*\rangle = \langle m, S^*(t)x^*\rangle, \qquad x^* \in E^*,$$

and

$$\langle Q_\nu x^*, x^*\rangle = \langle Q_\nu S^*(t)x^*, S^*(t)x^*\rangle + \langle Q_t x^*, x^*\rangle, \qquad x^* \in E^*.$$

The first of these identities implies that $S(t)m = m$ for all $t \geqslant 0$. By polarization, the second gives

$$\langle Q_\nu x^*, y^*\rangle = \langle Q_\nu S^*(t)x^*, S^*(t)y^*\rangle + \langle Q_t x^*, y^*\rangle, \qquad x^*, y^* \in E^*.$$

Now take $x^*, y^* \in \mathscr{D}(A^*)$, differentiate both sides with respect to t and evaluate at $t = 0$. It follows that

$$\langle Q_\nu x^*, A^*y^*\rangle + \langle Q_\nu A^*x^*, y^*\rangle = -\langle Qx^*, y^*\rangle.$$

This shows that $Q_\nu x^* \in \mathscr{D}(A)$ for all $x^* \in \mathscr{D}(A^*)$ and that Q_ν is a solution of $AX + XA^* = -Q$. ■

From Theorem 2.10 and the fact that the subspace $\mathscr{E}$ of entire vectors for A^* is weak*-dense in E^* we infer:

Corollary 3.4 (**H**Q_∞)**.** *Assume that Q_∞ is injective and $t \mapsto S^*(t)x^*$ is bounded for all $x^* \in \mathscr{E}$. If ν is an invariant cylindrical Gaussian measure with covariance $Q_\nu \in \mathscr{L}(E^*, E)$ and mean $m \in E$, then $m = 0$, i.e. ν is centred.*

We say that a cylindrical Gaussian measure γ is $\mathbf{S}-$*invariant* if $S(t)\gamma = \gamma$ for all $t \geqslant 0$; here $S(t)\gamma$ denotes the image cylindrical measure of γ under the linear transformation $S(t)$. We now have the following representation theorem extending [7, Theorem 11.7], [8, Proposition 7.2.3].

Proposition 3.5. *Let ν be an invariant cylindrical Gaussian measure on E with covariance operator $Q_\nu \in \mathscr{L}(E^*, E)$ and mean m. Then $(\mathbf{H}Q_\infty)$ holds and there exists a unique cylindrical Gaussian measure γ such that*

$$\nu = \mu_\infty * \gamma,$$

where μ_∞ is the cylindrical Gaussian measure on E whose covariance operator is Q_∞. Moreover, γ is $\mathbf{S}-$invariant and has mean m.

Proof. Let $R := Q_\nu - Q_\infty$. By Theorem 3.3, the positive symmetric operator Q_ν solves the equation $AX - XA^* = -Q$. Hence,

$$\langle Q_\nu x^*, x^*\rangle \geqslant \langle Q_\nu x^*, x^*\rangle - \langle Q_\nu S^*(t)x^*, S^*(t)x^*\rangle = \langle Q_t x^*, x^*\rangle.$$

It follows that

$$\langle Rx^*, x^*\rangle = \lim_{t\to\infty} \big(\langle Q_\nu x^*, x^*\rangle - \langle Q_t x^*, x^*\rangle\big) \geqslant 0, \qquad x^* \in E^*.$$

Since R is also symmetric, there exists a unique cylindrical Gaussian measure γ whose covariance operator is R and whose mean is m. From

$$\begin{aligned}(\widehat{\mu_\infty * \gamma})(x^*) &= \exp(-\tfrac{1}{2}\langle Q_\infty x^*, x^*\rangle)\exp(i\langle m, x^*\rangle - \tfrac{1}{2}\langle Rx^*, x^*\rangle)\\ &= \exp(i\langle m, x^*\rangle - \tfrac{1}{2}\langle Q_\nu x^*, x^*\rangle) = \widehat{\nu}(x^*), \qquad x^* \in E^*,\end{aligned}$$

it follows that $\nu = \mu_\infty * \gamma$. For all $t \geqslant 0$ we have

$$S^*(t)RS(t) = S^*(t)(Q_\nu - Q_\infty)S(t) = Q_\nu - Q_\infty = R,$$

because both Q_ν and Q_∞ solve $AX + XA^* = -Q$. Since $S^*(t)RS(t)$ is the covariance of $S(t)\gamma$, and $S(t)m = m$ is the mean of $S(t)\gamma$, this proves the $\mathbf{S}-$invariance of γ.

Finally, if γ' is a cylindrical Gaussian measure for which $\nu = \mu_\infty * \gamma'$, then by taking Fourier transforms we see that its covariance operator R' is equal to $Q_\nu - Q_\infty = R$ and that it has a mean equal to m. It follows that $\gamma' = \gamma$. ∎

We conclude with the following converse of Proposition 3.5; the proof is similar and is left to the reader.

Proposition 3.6 **($\mathbf{H}Q_\infty$).** *If γ is an $\mathbf{S}-$invariant cylindrical Gaussian measure with covariance operator $R \in \mathscr{L}(E^*, E)$ and mean $m \in E$, then $\mu_\infty * \gamma$ is invariant and has mean m.*

4. Uniqueness of invariant measures

As before let A be the generator of a C_0-semigroup $\mathbf{S}$ on a separable real Banach space E. Let H be a separable real Hilbert space and let $B \in \mathscr{L}(H, E)$ be bounded. In this section we will apply our results to obtain uniqueness results for invariant measures of the stochastic abstract Cauchy problem (1.1),

$$\begin{aligned} du(t) &= Au(t)\,dt + B\,dW_H(t), \quad t \geqslant 0,\\ u(0) &= x,\end{aligned}$$

where $\{W_H(t)\}_{t\geqslant 0}$ is a cylindrical Wiener process with Cameron-Martin space H. For a precise meaning of these concepts we refer to [5].

The operator $Q := B \circ B^* \in \mathscr{L}(E^*, E)$ is positive and symmetric. As a subset of E, the reproducing kernel Hilbert space H_Q associated with Q can be identified with range B; the inner product in H_Q is given by

$$[Bg, Bh]_{H_Q} = [Pg, Ph]_H, \qquad g, h \in H,$$

where P is the orthogonal projection in H onto $(\ker B)^{\perp}$. We define the operators $Q_t \in \mathscr{L}(E^*, E)$ as before:

$$Q_t x^* := \int_0^t S(s) Q S^*(s) x^* \, ds, \qquad x^* \in E^*.$$

We have the following existence and uniqueness result [5]:

Proposition 4.1. *The problem* (1.1) *has a weak solution* $\{u(t,x)\}_{t \geqslant 0}$ *if and only if for all* $t > 0$ *the operator* Q_t *is the covariance operator of a centred Gaussian Borel measure* μ_t *on* E*. In this case the solution is unique and* μ_t *is the distribution of the random variable* $u(t,0)$*. Moreover,* $\{u(t,x)\}_{t \geqslant 0}$ *is a Markov process and the semigroup* $\mathbf{P}$ *is its transition semigroup:*

$$P(t) f(x) = \mathbb{E}\left(f(u(t,x))\right), \qquad t \geqslant 0, \; x \in E,$$

for all bounded Borel functions $f : E \to \mathbb{R}$*. If* $(\mathbf{H}Q_\infty)$ *holds and* Q_∞ *is the covariance operator of a centred Gaussian Borel measure* μ_∞ *on* E*, then* μ_∞ *is an invariant measure.*

In what follows we will consider the following Hypotheses:

- $(\mathbf{H}\mu_t)$: For all $t > 0$, the operator Q_t is the covariance of a centred Gaussian Borel measure μ_t on E.
- $(\mathbf{H}\mu_\infty)$: Hypothesis $(\mathbf{H}Q_\infty)$ holds, and the operator Q_∞ is the covariance of a centred Gaussian Borel measure μ_∞ on E.

Trivially, $(\mathbf{H}\mu_\infty)$ implies $(\mathbf{H}Q_\infty)$, and by standard tightness arguments $(\mathbf{H}\mu_\infty)$ implies $(\mathbf{H}\mu_t)$. Conversely, if $(\mathbf{H}\mu_t)$ and $(\mathbf{H}Q_\infty)$ hold, it easily follows from Levi's continuity theorem that the centred cylindrical Gaussian measure μ_∞ whose covariance operator is Q_∞ extends uniquely to a Gaussian Borel measure on E if and only if the family of measures $\{\mu_t : t \geqslant 1\}$ is tight. In this situation, $\lim_{t\to\infty} \mu_t = \mu_\infty$ weakly.

The measure μ_∞, if it exists, is invariant.

Theorem 4.2 $(\mathbf{H}\mu_\infty)$**.** *If* $\mathbf{S}$ *is strongly stable, then* μ_∞ *is the unique invariant Gaussian Borel measure on* E*.*

Proof. First we recall that Gaussian measures have a mean and that their covariance operator is compact; this follows from [3, Theorem 3.2.3 and Corollary 3.2.4] and (2.2).

Let ν be in invariant Gaussian measure. By Theorem 3.3 and the strong stability of $\mathbf{S}$, ν is centred. Denote by Q_ν the covariance operator of ν. From

$$Q_\nu - S(t) Q_\nu S^*(t) = Q_t, \qquad t > 0,$$

and the compactness of Q_ν we obtain

$$\langle Q_\nu x^*, y^* \rangle = \lim_{t\to\infty} \langle S(t) Q_\nu S^*(t) x^* + Q_t x^*, y^* \rangle = \lim_{t\to\infty} \langle Q_t x^*, y^* \rangle = \langle Q_\infty x^*, y^* \rangle;$$

we used here that the strong stability of $\mathbf{S}$ is uniform on compacts. ∎

We call μ_∞ *nondegenerate* if $\mu_\infty(O) \neq 0$ for every open subset $O \subseteq E$. As is well known [3, Theorem 3.6.1], this happens if and only if the covariance operator Q_∞ of μ_∞ is injective.

The next result is an immediate consequence of Corollary 2.7, Theorem 3.3 and Corollary 3.4.

Theorem 4.3 $(\mathbf{H}\mu_\infty)$**.** *If* μ_∞ *is nondegenerate and* $\mathbf{S}$ *is compact, then* $\mathbf{S}$ *is uniformly exponentially stable and* μ_∞ *is the unique invariant Gaussian Borel measure on* E*.*

Example 2.8 shows that the injectivity of Q_∞ cannot be omitted.

From Theorem 2.11, Corollary 3.4 and the fact that covariance operators of Gaussian measures are compact, we obtain the following uniqueness result:

Theorem 4.4 ($\mathbf{H}\mu_\infty$). *If μ_∞ is non-degenerate and $t \mapsto S^*(t)x^*$ is bounded for all entire vectors $x^* \in \mathscr{E}$, then μ_∞ is the unique invariant Gaussian Borel measure on E.*

This result is applicable in the following situation:

Corollary 4.5 ($\mathbf{H}\mu_\infty$). *If $\mathscr{D}(A^n) \subseteq$ range B for some $n \geqslant 1$, then μ_∞ is the unique invariant Gaussian Borel measure on E.*

Proof. By the strong continuity of $\mathbf{S}$ and density of $\mathscr{D}(A^n)$ in E, range $B = H_Q$ is dense in E. It follows that Q is injective, and this easily implies that Q_∞ is injective; cf. [9]. Hence, μ_∞ is nondegenerate.

Fix $\lambda \in \varrho(A)$ and define $T \in \mathscr{L}(E, H_Q)$ by

$$Tx := (\lambda - A)^{-n}x \in \mathscr{D}(A^n) \subseteq \text{range}\, B = H_Q.$$

Fix $x^* \in \mathscr{E}$ and put $y^* := (\lambda - A^*)^n x^*$. Noting that $y^* \in \mathscr{D}(A^*)$, it follows from Lemma 2.4 that

$$\lim_{t\to\infty} \langle S^*(t)x^*, x\rangle = \lim_{t\to\infty} \langle S^*(t)y^*, (\lambda - A)^{-n}x\rangle = \lim_{t\to\infty} [i_Q^* S^*(t)y^*, Tx]_{H_Q} = 0$$

for all $x \in E$. Hence by the uniform boundedness theorem, the orbit $t \mapsto S^*(t)x^*$ is bounded. Now apply Theorem 4.4. ∎

For reasons of completeness we state the following $\sigma-$additive analogue of Proposition 3.5 and its converse:

Proposition 4.6. *Let ν be an invariant Gaussian Borel measure on E. Then ($\mathbf{H}\mu_\infty$) holds and there exists a unique Gaussian Borel measure γ such that*

$$\nu = \mu_\infty * \gamma.$$

Moreover, γ is $\mathbf{S}-$invariant and has the same mean as ν. Conversely, if ($\mathbf{H}\mu_\infty$) holds and γ is an $\mathbf{S}-$invariant Gaussian Borel measure, then $\mu_\infty * \gamma$ is invariant.

Proof. Let γ be the unique invariant cylindrical Gaussian measure from Proposition 3.5, and let R be its covariance operator. Let Q_ν be the covariance operator of ν. From

$$0 \leqslant \langle Rx^*, x^*\rangle = \langle (Q_\nu - Q_\infty)x^*, x^*\rangle \leqslant \langle Q_\nu x^*, x^*\rangle, \qquad x^* \in E^*$$

and the fact that Q_ν is the covariance of a Gaussian Borel measure, it follows that R is the covariance of a Gaussian Borel measure; cf. [17, Proposition VI.3.4]. This measure is the unique extension of γ to a Gaussian Borel measure on E. The converse follows immediately from Proposition 3.5. ∎

5. Null controllability

Let A be the generator of a C_0-semigroup $\mathbf{S}$ on a separable real Banach space E, let H be a separable real Hilbert space and let $B \in \mathscr{L}(H, E)$ be a bounded linear operator. We consider the inhomogeneous Cauchy problem

$$\begin{aligned} u'(t) &= Au(t) + Bf(t), \qquad 0 \leqslant t \leqslant T, \\ u(0) &= x, \end{aligned} \tag{5.1}$$

where $x \in E$ and $f \in L^2([0,T];H)$. The unique mild solution of this problem is given by

$$u(t,x;f) := S(t)x + \int_0^t S(t-s)Bf(s)\,ds, \qquad 0 \leqslant t \leqslant T.$$

We say that $\mathbf{S}$ is *null controllable* with respect to the *control pair* (B,H) if for all $x \in E$ and $T > 0$ there exists a function $f \in L^2([0,T];H)$ such that

$$u(T,x;f) = 0.$$

As before we let $Q = B \circ B^*$. For $0 < t \leqslant \infty$ let H_t denote the reproducing kernel Hilbert space associated with the positive symmetric operator Q_t; thus, H_t is Hilbert space completion of the range of Q_t with respect to the inner product $[Q_t x^*, Q_t y^*]_{H_t} := \langle Q_t x^*, y^* \rangle$. The inclusion mapping from H_t into E will be denoted by i_t. Whenever it is convenient, we will identify H_t with its image $i_t(H_t)$ in E.

We have the following characterization of null controllability in terms of the spaces H_t:

Proposition 5.1. $\mathbf{S}$ *is null controllable with respect to the control pair* (B,H) *if and only if*

$$S(t)E \subseteq H_t, \qquad t > 0.$$

This is an immediate consequence of the following simple lemma; cf. [21], [7, Appendix B]. For the convenience of the reader we include the proof.

Lemma 5.2. *For all* $t > 0$ *we have*

$$H_t = \left\{ \int_0^t S(t-s)Bf(s)\,ds : \ f \in L^2([0,t];H) \right\}.$$

Proof. Define $A_t : L^2([0,t];H) \to E$ by

$$A_t f := \int_0^t S(t-s)Bf(s)\,ds, \qquad f \in L^2([0,t];H).$$

It is trivially checked that

$$A_t^* x^* = B^* S(t-\cdot)x^*, \qquad x^* \in E^*.$$

Hence,

$$A_t(A_t^* x^*) = \int_0^t S(t-s)QS^*(t-s)x^*\,ds = Q_t x^* = i_t(i_t^* x^*)$$

from which it follows that $A_t(A_t^* x^*) \in i_t(H_t)$ for all $x^* \in E^*$. We have

$$\begin{aligned} \|A_t(A_t^* x^*)\|_{H_t}^2 &= \|i_t^* x^*\|_{H_t}^2 = \langle Q_t x^*, x^* \rangle \\ &= \|B^* S(t-\cdot)x^*\|_{L^2([0,t];H)}^2 = \|A_t^* x^*\|_{L^2([0,t];H)}^2. \end{aligned}$$

Hence A_t maps $\overline{\text{range}\, A_t^*}$ isometrically into H_t, and since the range of i_t^* is dense in H_t, this mapping is onto. Finally, for $f \perp \text{range}\, A_t^*$ it is clear that $A_t f = 0$. ∎

We refer to [6], [7], and [21] for a further discussion of the concept of null controllability.

Next we show that for differentiable semigroups $\mathbf{S}$, the assumption of Corollary 4.5 implies null controllability.

Proposition 5.3. *If* $\mathscr{D}(A^n) \subseteq$ *range* B *for some* $n \geqslant 1$ *and* $\mathbf{S}$ *is a differentiable semigroup, then* $\mathbf{S}$ *is null controllable with respect to the control pair* (B,H).

Proof. We will prove that $D(A^n) \subseteq \operatorname{range} B$ implies $D(A^{n+1}) \subseteq H_t$ for all $t > 0$; this does not require any regularity of the semigroup. If $\mathbf{S}$ is differentiable, it follows that $S(t)E \subseteq \mathscr{D}(A^{n+1}) \subseteq H_t$ for all $t > 0$.

Fix $t > 0$. First note that for all $\omega \in \mathbb{R}$, and $h \in H$ we have

$$\int_0^t e^{-\omega s} S(s)Bh\, ds = \int_0^t S(t-s)Bf_{\omega,h}(s)\, ds,$$

where $s \mapsto f_{\omega,h}(s) := e^{-\omega(t-s)} \otimes h \in L^2([0,t];H)$. Hence by Lemma 5.2,

$$\int_0^t e^{-\omega s} S(s)Bh\, ds \in H_t.$$

Choose $\omega \in \varrho(A) \cap \mathbb{R}$ so large that $\|e^{-\omega t} S(t)|_{D(A^n)}\|_{\mathscr{L}(D(A^n))} < 1$. Then the restriction to $D(A^n)$ of $I - e^{-\omega t}S(t)$ is invertible in $D(A^n)$, and for $x \in D(A^n)$ we have

$$\begin{aligned}(\omega - A)^{-1}x &= (I - e^{-\omega t}S(t))(\omega - A)^{-1}(I|_{D(A^n)} - e^{-\omega t}S(t)|_{D(A^n)})^{-1}x \\ &= -\int_0^t e^{-\omega s}S(s)(I|_{D(A^n)} - e^{-\omega t}S(t)|_{D(A^n)})^{-1}x\, ds \in H_t,\end{aligned}$$

noting that

$$(I|_{D(A^n)} - e^{-\omega t}S(t)|_{D(A^n)})^{-1}x = \sum_{n=0}^{\infty} e^{-n\omega t}S(nt)x \in D(A^n) \subseteq \operatorname{range} B,$$

the sum being absolutely convergent in $D(A^n)$. ∎

In particular, if in this situation ($\mathbf{H}\mu_\infty$) holds, then μ_∞ is the unique invariant Gaussian Borel measure on E by Corollary 4.5. More generally we have:

Theorem 5.4 **($\mathbf{H}\mu_\infty$).** *If $\mathbf{S}$ is null controllable with respect to the control pair (B,H), then $\mathbf{S}$ is uniformly exponentially stable and μ_∞ is the unique invariant Gaussian Borel measure on E.*

Proof. Let $t > 0$ be fixed. Denoting by $T(t)$ the operator $S(t)$, regarded as a map from E into H_t, we have

$$S(t) = i_t \circ T(t).$$

Since the triple (i_t, H_t, E) is an abstract Wiener space, the inclusion operator i_t is compact [3, Corollary 3.2.4]. Therefore $S(t)$ is compact as well.

It remains to check that μ_∞ is nondegenerate. But from $S(t)E \subseteq H_t \subseteq H_\infty$ (the second inclusion follows from [7, Appendix B]) and the strong continuity of $\mathbf{S}$ it follows that H_∞ is dense in E, and therefore Q_∞ is injective. This implies that μ_∞ is nondegenerate. ∎

For Hilbert spaces E, uniqueness of the invariant measure in the null controllable case is well known; cf. [20], [7, Theorem 11.3]. Note that more generally, by a similar argument the uniqueness of μ_∞ follows if we only assume that $S(t)E \subseteq H_\infty$ for all $t > 0$.

6. STABILIZABILITY

We call the problem (5.1) *stabilizable* if there exists a bounded linear operator $K : E \to H$ such that the operator

$$A_K := A + B \circ K$$

generates a uniformly exponentially stable semigroup.

If E is a *Hilbert space*, null controllability of $\mathbf{S}$ implies stabilizability of (5.1), and stabilizability of (5.1) together with ($\mathbf{H}Q_\infty$) implies uniform exponential stability of $\mathbf{S}$; see [6] and [19]. In particular, if (5.1) is stabilizable and ($\mathbf{H}\mu_\infty$) holds, then μ_∞ is the unique invariant Gaussian Borel measure on E.

The proofs of the results just cited are Hilbert space theoretic and as far as we know they do not extend to Banach spaces. Nevertheless, as an application of Theorem 4.4 we can prove directly:

Theorem 6.1 ($\mathbf{H}\mu_\infty$). *If μ_∞ is nondegenerate and the problem* (5.1) *is stabilizable, then μ_∞ is the unique invariant Gaussian Borel measure on E.*

We will prove that for all $x^* \in \mathscr{D}(A^*)$ the orbit $t \mapsto S^*(t)x^*$ is bounded. By Theorem 4.4, this will give the result.

Let $\mathbf{S}_K$ denote the semigroup generated by A_K and choose constants $M \geqslant 0$ and $\omega > 0$ such that $\|S_K(t)\| \leqslant Me^{\omega t}$ for all $t \geqslant 0$. Then by dualizing the variation of constants formula,

$$S_K^*(t)x^* = S^*(t)x^* + \int_0^t S_K^*(t-s)K^*B^*S^*(s)x^*\,ds.$$

Now let $x^* \in \mathscr{D}(A^*)$. Then by Lemma 2.4,

$$\lim_{t\to\infty} \|B^*S^*(t)x^*\|_H = \lim_{t\to\infty} \|i_Q^*S^*(t)x^*\|_{H_Q} = 0.$$

If $\varepsilon > 0$ is given, then we may choose $t_0 \geqslant 0$ so large that $\|B^*S^*(t)x^*\|_H \leqslant \varepsilon$ for all $t \geqslant t_0$. Then for $t \geqslant t_0$ we have

$$\begin{aligned}
&\left\|\int_0^t S_K^*(t-s)K^*B^*S^*(s)x^*\,ds\right\| \\
&\quad\leqslant \int_0^{t_0} \|S_K^*(t-s)K^*B^*S^*(s)x^*\|\,ds + \int_{t_0}^t \|S_K^*(t-s)K^*B^*S^*(s)x^*\|\,ds \\
&\quad\leqslant \left(\sup_{t\geqslant 0}\|B^*S^*(t)x^*\|_H\right)\int_0^{t_0}\|S_K^*(t-s)K^*\|\,ds + \varepsilon\int_{t_0}^t \|S_K^*(t-s)K^*\|\,ds \\
&\quad\leqslant \left(\sup_{t\geqslant 0}\|B^*S^*(t)x^*\|_H\right)\cdot\frac{M\|K\|}{\omega}\left(e^{-\omega(t-t_0)} - e^{-\omega t}\right) + \varepsilon\,\frac{M\|K\|}{\omega}.
\end{aligned}$$

Since $\varepsilon > 0$ was arbitrary and $\mathbf{S}_K$ is uniformly exponentially stable, it follows that for all $x^* \in \mathscr{D}(A^*)$,

$$\lim_{t\to\infty}\|S^*(t)x^*\| = \lim_{t\to\infty}\left\|S_K^*(t)x^* - \int_0^t S_K^*(t-s)K^*B^*S^*(s)x^*\,ds\right\| = 0.$$

∎

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