Vector measures of bounded γ-variation and stochastic integrals

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Abstract. We introduce the class of vector measures of bounded γ-variation and study its relationship with vector-valued stochastic integrals with respect to Brownian motions.

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1. Introduction

It is well known that stochastic integrals can be interpreted as vector measures, the identification being given by the identity

$$F(A) = \int_A \phi \, dB.$$

Here, the driving process B is a (semi)martingale (for instance, a Brownian motion), and ϕ is a stochastic process satisfying suitable measurability and integrability conditions. This observation has been used by various authors as the starting point of a theory of stochastic integration for vector-valued processes.

Let X be a Banach space. In [5] we characterized the class of functions $\phi : (0,1) \to X$ which are stochastically integrable with respect to a Brownian motion $(W_t)_{t\in[0,1]}$ as being the class of functions for which the operator $T_\phi : L^2(0,1) \to X$,

$$T_\phi f := \int_0^1 f(t)\phi(t)\, dt,$$

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belongs to the operator ideal $\gamma(L^2(0,1),X)$ of all γ-radonifying operators. Indeed, we established the Itô isomorphism

$$\mathbb{E}\Big\|\int_0^1 f\,dW\Big\|^2 = \|T_f\|^2_{\gamma(L^2(0,1),X)}.$$

The linear subspace of all operators in $\gamma(L^2(0,1),X)$ of the form $T=T_f$ for some function $f:(0,1)\to X$ is dense, but unless X has cotype 2 it is strictly smaller than $\gamma(L^2(0,1),X)$. This means that in general there are operators $T\in\gamma(L^2(0,1),X)$ which are not representable by an X-valued function. Since the space of test functions $\mathscr{D}(0,1)$ embeds in $L^2(0,1)$, by restriction one could still think of such operators as X-valued distributions. It may be more intuitive, however, to think of T as an X-valued vector measure. We shall prove (see Theorem 2.3 and the subsequent remark) that if X does not contain a closed subspace isomorphic to c_0, then the space $\gamma(L^2(0,1),X)$ is isometrically isomorphic in a natural way to the space of X-valued vector measures on $(0,1)$ which are of bounded γ-variation. This gives a 'measure theoretic' description of the class of admissible integrands for stochastic integrals with respect to Brownian motions. The condition $c_0 \not\subseteq X$ can be removed if we replace the space of γ-radonifying operators by the larger space of all γ-summing operators (which contains the space of all γ-radonifying operators isometrically as a closed subspace).

Vector measures of bounded γ-variation behave quite differently from vector measures of bounded variation. For instance, the question whether an X-valued vector measure of bounded γ-variation can be represented by an X-valued function is not linked to the Radon-Nikodým property, but rather to the type 2 and cotype 2 properties of X (see Corollaries 2.5 and 2.6).

In section 3 we consider yet another class of vector measures whose variation is given by certain random sums, and we show that a function $\phi:(0,1)\to X$ is stochastically integrable with respect to a Brownian motion $(W_t)_{t\in[0,1]}$ on a probability space $(\Omega,\mathbb{P})$ if and only if the formula $F(A):=\int_A \phi\,dW$ defines an $L^2(\Omega;X)$-valued vector measure F in this class.

2. Vector measures of bounded γ-variation

Let (S,Σ) be a measurable space, X a Banach space, and $(\gamma_n)_{n\geqslant 1}$ a sequence of independent standard Gaussian random variables defined on a probability space $(\Omega,\mathscr{F},\mathbb{P})$.

Definition 2.1. We say that a countably additive vector measure F has *bounded γ-variation* with respect to a probability measure μ on (S,Σ) if $\|F\|_{V_\gamma(\mu;X)}<\infty$, where

$$\|F\|_{V_\gamma(\mu;X)} := \sup\Big(\mathbb{E}\Big\|\sum_{n=1}^N \gamma_n\,\frac{F(A_n)}{\sqrt{\mu(A_n)}}\Big\|^2\Big)^{\frac12},$$

the supremum being taken over all finite collections of disjoint sets $A_1,\dots,A_N\in\Sigma$ such that $\mu(A_n)>0$ for all $n=1,\dots,N$.

It is routine to check (e.g. by an argument similar to [4, Proposition 5.2]) that the space $V_\gamma(\mu;X)$ of all countably additive vector measures $F:\Sigma\to X$ which have bounded γ-variation with respect to μ is a Banach space with respect to the norm $\|\cdot\|_{V_\gamma(\mu;X)}$. Furthermore, every vector measure which is of bounded γ-variation is of bounded 2-semivariation.

In order to give a necessary and sufficient condition for a vector measure to have bounded γ-variation we need to introduce the following terminology. A bounded operator $T:H\to X$, where H is a Hilbert space, is said to be γ-*summing* if there exists a constant C such that for all finite orthonormal systems $\{h_1,\dots,h_N\}$ in H one has

$$\mathbb{E}\Big\|\sum_{n=1}^N \gamma_n\, T h_n\Big\|^2 \leqslant C^2.$$

The least constant C for which this holds is called the γ-*summing norm* of T, notation $\|T\|_{\gamma_\infty(H,X)}$. With respect to this norm, the space $\gamma_\infty(H,X)$ of all γ-summing operators from H to X is a Banach space which contains all finite rank operators from H to X. In what follows we shall make free use of the elementary properties of γ-summing operators. For a systematic exposition of these we refer to [2, Chapter 12] and the lecture notes [4].

Theorem 2.2. *Let $\mathscr{A}$ be an algebra of subsets of S which generates the σ-algebra Σ, and let $F:\mathscr{A}\to X$ be a finitely additive mapping. If, for some $1\leqslant p<\infty$, $T:L^p(\mu)\to X$ is a bounded operator such that*

$$F(A)=T1_A,\quad A\in\mathscr{A},$$

then F has a unique extension to a countably additive vector measure on Σ which is absolutely continuous with respect to μ. If $T:L^2(\mu)\to X$ is γ-summing, then the extension of F has bounded γ-variation with respect to μ and we have

$$\|F\|_{V_\gamma(\mu;X)}\leqslant \|T\|_{\gamma_\infty(L^2(\mu),X)}.$$

Proof. We define the extension $F:\Sigma\to X$ by $F(A):=T1_A$, $A\in\Sigma$. To see that F is countably additive, consider a disjoint union $A=\bigcup_{n\geqslant 1}A_n$ with $A_n, A\in\Sigma$. Then $\lim_{N\to\infty}1_{\bigcup_{n=1}^N A_n}=1_A$ in $L^p(\mu)$ and therefore

$$\lim_{N\to\infty}\sum_{n=1}^N F(A_n)=\lim_{N\to\infty}T\sum_{n=1}^N 1_{A_n}=T1_A=F(A).$$

The absolute continuity of F is clear. To prove uniqueness, suppose $\tilde F:\Sigma\to X$ is another countably additive vector measure extending F. For each $x^*\in X^*$, $\langle\tilde F,x^*\rangle$ and $\langle F,x^*\rangle$ are finite measures on Σ which agree on $\mathscr{A}$, and therefore by Dynkin's lemma they agree on all of Σ. This being true for all $x^*\in X^*$, it follows that $\tilde F=F$ by the Hahn-Banach theorem.

Suppose next that $T:L^2(\mu)\to X$ is γ-summing, and consider a finite collection of disjoint sets $A_1,\dots,A_N$ in Σ such that $\mu(A_n)>0$ for all $n=1,\dots,N$.

The functions $f_n = 1_{A_n}/\sqrt{\mu(A_n)}$ are orthonormal in $L^2(\mu)$ and therefore

$$\mathbb{E}\Big\| \sum_{n=1}^N \gamma_n \frac{F(A_n)}{\sqrt{\mu(A_n)}}\Big\|^2 = \mathbb{E}\Big\| \sum_{n=1}^N \gamma_n \, Tf_n\Big\|^2 \leqslant \|T\|^2_{\gamma_\infty(L^2(\mu),X)}.$$

It follows that F has bounded γ-variation with respect to μ and that $\|F\|_{V_\gamma(\mu;X)} \leqslant \|T\|_{\gamma_\infty(L^2(\mu),X)}$. □

Theorem 2.3. *For a countably additive vector measure $F : \Sigma \to X$ the following assertions are equivalent:*

(1) *F has bounded γ-variation with respect to μ;*
(2) *There exists a γ-summing operator $T : L^2(\mu) \to X$ such that*

$$F(A) = T1_A, \quad A \in \Sigma.$$

In this situation we have

$$\|F\|_{V_\gamma(\mu;X)} = \|T\|_{\gamma_\infty(L^2(\mu),X)}.$$

Proof. (1)⇒(2): Suppose that F has bounded γ-variation with respect to μ. For a simple function $f = \sum_{n=1}^N c_n 1_{A_n}$, where the sets $A_n \in \Sigma$ are disjoint and of positive μ-measure, define

$$Tf := \sum_{n=1}^N c_n F(A_n).$$

By the Cauchy-Schwarz inequality, for all $x^* \in X^*$ we have

$$\begin{aligned}
|\langle Tf, x^*\rangle| &= \Big|\mathbb{E} \sum_{m=1}^N \gamma_m \, c_m \sqrt{\mu(A_m)} \cdot \sum_{n=1}^N \gamma_n \frac{\langle F(A_n), x^*\rangle}{\sqrt{\mu(A_n)}}\Big| \\
&\leqslant \Big(\mathbb{E}\Big|\sum_{n=1}^N \gamma_n \, c_n \sqrt{\mu(A_n)}\Big|^2\Big)^{\frac12} \Big(\mathbb{E}\Big|\sum_{n=1}^N \gamma_n \frac{\langle F(A_n), x^*\rangle}{\sqrt{\mu(A_n)}}\Big|^2\Big)^{\frac12} \\
&\leqslant \Big(\sum_{n=1}^N |c_n|^2 \mu(A_n)\Big)^{\frac12} \|F\|_{V_\gamma(\mu;X)} \, \|x^*\| \\
&= \|f\|_{L^2(\mu)} \|F\|_{V_\gamma(\mu;X)} \, \|x^*\|.
\end{aligned}$$

It follows that T is bounded and $\|T\|_{\mathscr{L}(L^2(\mu),X)} \leqslant \|F\|_{V_\gamma(\mu;X)}$. To prove that T is γ-summing we shall first make the simplifying assumption that the σ-algebra Σ is countably generated. Under this assumption there exists an increasing sequence of finite σ-algebras $(\Sigma_n)_{n\geqslant 1}$ such that $\Sigma = \bigvee_{n\geqslant 1} \Sigma_n$. Let P_n be the orthogonal projection in $L^2(\mu)$ onto $L^2(\Sigma_n,\mu)$ and put $T_n := T \circ P_n$. These operators are of finite rank and we have $\lim_{n\to\infty} T_n \to T$ in the strong operator topology of $\mathscr{L}(L^2(\mu), X)$.

Fix an index $n \geqslant 1$ for the moment. Since Σ_n is finitely generated there exists a partition $S = \bigcup_{j=1}^N A_j$, where the disjoint sets $A_1, \dots, A_N$ generate Σ_n. Assuming that $\mu(A_j) > 0$ for all $j = 1, \dots, M$ and $\mu(A_j) = 0$ for $j = M+1, \dots, N$, the functions $g_j = 1_{A_j}/\sqrt{\mu(A_j)}$, $j = 1, \dots, M$, form an orthonormal basis for $L^2(\Sigma_n, \mu)$ and

$$\begin{aligned}\|T_n\|^2_{\gamma_\infty(L^2(\mu),X)} &= \|T_n\|^2_{\gamma_\infty(L^2(\Sigma_n,\mu),X)} \\ &= \mathbb{E}\Big\|\sum_{j=1}^{M} \gamma_j\, T g_j\Big\|^2 = \mathbb{E}\Big\|\sum_{j=1}^{M} \gamma_j\, \frac{F(A_j)}{\sqrt{\mu(A_n)}}\Big\|^2 \leqslant \|F\|^2_{V_\gamma(\mu;X)},\end{aligned}$$

the first identity being a consequence of [4, Corollary 5.5] and the second of [4, Lemma 5.7]. It follows that the sequence $(T_n)_{n\geqslant 1}$ is bounded in $\gamma_\infty(L^2(\mu), X)$. By the Fatou lemma, if $\{f_1, \dots, f_k\}$ is any orthonormal family in $L^2(\mu)$, then

$$\mathbb{E}\Big\|\sum_{j=1}^{k} \gamma_j T f_j\Big\|^2 \leqslant \liminf_{n\to\infty} \mathbb{E}\Big\|\sum_{j=1}^{k} \gamma_j T_n f_j\Big\|^2 \leqslant \|T_n\|^2_{\gamma_\infty(L^2(\mu),X)} \leqslant \|F\|^2_{V_\gamma(\mu;X)}.$$

This proves that T is γ-summing and $\|T\|_{\gamma_\infty(L^2(\mu),X)} \leqslant \|F\|_{V_\gamma(\mu;X)}$.

It remains to remove the assumption that Σ is countably generated. The preceding argument shows that if we define T in the above way, then its restriction to $L^2(\Sigma', \mu)$ is γ-summing for every countably generated σ-algebra $\Sigma' \subseteq \Sigma$, with a uniform bound

$$\|T\|_{\gamma_\infty(L^2(\Sigma',\mu),X)} \leqslant \|F\|_{V_\gamma(\mu;X)}.$$

Since every finite orthonormal family $\{f_1, \dots, f_k\}$ in $L^2(\mu)$ is contained in $L^2(\Sigma', \mu)$ for some countably generated σ-algebra $\Sigma' \subseteq \Sigma$, we see that

$$\mathbb{E}\Big\|\sum_{j=1}^{k} \gamma_j\, T f_j\Big\|^2 \leqslant \|T\|^2_{\gamma_\infty(L^2(\Sigma',\mu),X)} \leqslant \|F\|^2_{V_\gamma(\mu;X)}.$$

It follows that T is γ-summing and $\|T\|_{\gamma_\infty(L^2(\mu),X)} \leqslant \|F\|_{V_\gamma(\mu;X)}$.

(2)⇒(1): This implication is contained in Theorem 2.2. □

By a theorem of Hoffmann-Jørgensen and Kwapień [3, Theorem 9.29], if X is a Banach space not containing an isomorphic copy of c_0, then for any Hilbert space H one has

$$\gamma_\infty(H, X) = \gamma(H, X),$$

where by definition $\gamma(H, X)$ denotes the closure in $\gamma_\infty(H, X)$ of the finite rank operators from H to X. Since any operator in this closure is compact we obtain:

Corollary 2.4. *If X does not contain an isomorphic copy of c_0 and $F : \Sigma \to X$ has bounded γ-variation with respect to μ, then F has relatively compact range.*

Using the terminology of [5], a theorem of Rosiński and Suchanecki [6] asserts that if X has type 2 we have a continuous inclusion $L^2(\mu; X) \hookrightarrow \gamma(L^2(\mu), X)$ and that if X has cotype 2 we have a continuous inclusion $\gamma_\infty(L^2(\mu), X) \hookrightarrow L^2(\mu; X)$. In both cases the embedding is contractive, and the relation between the operator T and the representing function ϕ is given by

$$Tf = \int_S f\phi\, d\mu, \quad f \in L^2(\mu).$$

If $\dim L^2(\mu) = \infty$, then in the converse direction the existence of a continuous embedding $L^2(\mu; X) \hookrightarrow \gamma_\infty(L^2(\mu), X)$ (respectively $\gamma(L^2(\mu), X) \hookrightarrow L^2(\mu; X)$) actually implies the type 2 property (respectively the cotype 2 property) of X.

Corollary 2.5. *Let X have type 2. For all $\phi \in L^2(\mu; X)$ the formula*

$$F(A) := \int_A \phi \, d\mu, \quad A \in \Sigma,$$

defines a countably additive vector measure $F : \Sigma \to X$ which has bounded γ-variation with respect to μ. Moreover,

$$\|F\|_{V_\gamma(\mu;X)} \leqslant \|\phi\|_{L^2(\mu;X)}.$$

If $\dim L^2(\mu) = \infty$, this property characterises the type 2 property of X.

Proof. By the theorem of Rosiński and Suchanecki, ϕ represents an operator $T \in \gamma(L^2(\mu), X)$ such that $T1_A = \int_A \phi \, d\mu = F(A)$ for all $A \in \Sigma$. The result now follows from Theorem 2.2. The converse direction follows from Theorem 2.3 and the preceding remarks. □

Corollary 2.6. *Let X have cotype 2. If $F : \Sigma \to X$ has bounded γ-variation with respect to μ, there exists a function $\phi \in L^2(\mu; X)$ such that*

$$F(A) = \int_A \phi \, d\mu, \quad A \in \Sigma.$$

Moreover,

$$\|\phi\|_{L^2(\mu;X)} \leqslant \|F\|_{V_\gamma(\mu;X)}.$$

If $\dim L^2(\mu) = \infty$, this property characterises the cotype 2 property of X.

Proof. By Theorem 2.3 there exists an operator $T \in \gamma_\infty(L^2(\mu), X)$ such that $F(A) = T1_A$ for all $A \in \Sigma$. Since X has cotype 2, X does not contain an isomorphic copy of c_0 and therefore the theorem of Hoffmann-Jørgensen and Kwapień implies that $T \in \gamma(L^2(\mu), X)$. Now the theorem of Rosiński and Suchanecki shows that T is represented by a function $\phi \in L^2(\mu; X)$. The converse direction follows from Theorem 2.2 and the remarks preceding Corollary 2.5. □

3. Vector measures of bounded randomised variation

Let (S, Σ) be a measurable space and $(r_n)_{n\geqslant 1}$ a Rademacher sequence, i.e., a sequence of independent random variables with $\mathbb{P}(r_n = \pm 1) = \frac{1}{2}$.

Definition 3.1. A countably additive vector measure $F : \Sigma \to X$ is of *bounded randomised variation* if $\|F\|_{V^{\mathrm{r}}(\mu;X)} < \infty$, where

$$\|F\|_{V^{\mathrm{r}}(\mu;X)} = \sup \Big(\mathbb{E}\Big\| \sum_{n=1}^N r_n \, F(A_n)\Big\|^2\Big)^{\frac{1}{2}},$$

the supremum being taken over all finite collections of disjoint sets $A_1, \dots, A_N \in \Sigma$.

Clearly, if F is of bounded variation, then F is of bounded randomised variation. The converse fails; see Example 1. If X has finite cotype, standard comparison results for Banach space-valued random sums [2, 3] imply that an equivalent norm is obtained when the Rademacher variables are replaced by Gaussian variables.

It is routine to check that the space $V^{\mathrm{r}}(\mu; X)$ of all countably additive vector measures $F : \Sigma \to X$ of bounded randomised variation is a Banach space with respect to the norm $\|\cdot\|_{V^{\mathrm{r}}(\mu;X)}$.

In Theorem 3.2 below we establish a connection between measures of bounded randomised variation and the theory of stochastic integration. For this purpose we need the following terminology. A *Brownian motion* on $(\Omega, \mathscr{F}, \mathbb{P})$ indexed by another probability space (S, Σ, μ) is a mapping $W : \Sigma \to L^2(\Omega)$ such that:

(i) For all $A \in \Sigma$ the random variable $W(A)$ is centred Gaussian with variance

$$\mathbb{E}(W(A))^2 = \mu(A);$$

(ii) For all disjoint $A, B \in \Sigma$ the random variables $W(A)$ and $W(B)$ are independent.

A strongly μ-measurable function $\phi : S \to X$ is *stochastically integrable* with respect to W if for all $x^* \in X^*$ we have $\langle \phi, x^* \rangle \in L^2(\mu)$ (i.e, f *belongs to* $L^2(\mu)$ *scalarly*) and for all $A \in \Sigma$ there exists a strongly measurable random variable $Y_A : \Omega \to X$ such that for all $x^* \in X^*$ we have

$$\langle Y_A, x^* \rangle = \int_A \langle \phi, x^* \rangle \, dW$$

almost surely. Note that each Y_A is centred Gaussian and therefore belongs to $L^2(\Omega; X)$ by Fernique's theorem; the above equality then holds in the sense of $L^2(\Omega)$. We define the *stochastic integral* of ϕ over A by $\int_A \phi \, dW := Y_A$. For more details and various equivalent definitions we refer to [5].

Theorem 3.2. *Let $W : \Sigma \to L^2(\Omega)$ be a Brownian motion. For a strongly μ-measurable function $\phi : S \to X$ the following assertions are equivalent:*

(1) *ϕ is stochastically integrable with respect to W;*

(2) *ϕ belongs to $L^2(\mu)$ scalarly and there exists a countably additive vector measure $F : \Sigma \to X$, of bounded γ-variation with respect to μ, such that for all $x^* \in X^*$ we have*

$$\langle F(A), x^* \rangle = \int_A \langle \phi, x^* \rangle \, d\mu, \quad A \in \Sigma;$$

(3) *ϕ belongs to $L^2(\mu)$ scalarly and there exists a countably additive vector measure $G : \Sigma \to L^2(\Omega; X)$ of bounded randomised variation such that for all $x^* \in X^*$ we have*

$$\langle G(A), x^* \rangle = \int_A \langle \phi, x^* \rangle \, dW, \quad A \in \Sigma.$$

In this situation we have

$$\|F\|_{V_\gamma(\mu;X)} = \|G\|_{V^{\mathrm{r}}(\mu;L^2(\Omega;X))} = \Big(\mathbb{E}\Big\|\int_S \phi\,dW\Big\|^2\Big)^{\frac12}.$$

Proof. (1)⇔(2): This equivalence is immediate from Theorem 2.3 and the fact, proven in [5], that ϕ is stochastically integrable with respect to W if and only there exists an operator $T \in \gamma(L^2(\mu), X)$ such that

$$Tf = \int_S f\phi\,d\mu, \quad f \in L^2(\mu).$$

In this case we also have

$$\|T\|_{\gamma(L^2(\mu),X)} = \Big(\mathbb{E}\Big\|\int_S \phi\,dW\Big\|^2\Big)^{\frac12}.$$

In view of Theorem 2.3, this proves the identity

$$\|F\|_{V_\gamma(\mu;X)} = \Big(\mathbb{E}\Big\|\int_S \phi\,dW\Big\|^2\Big)^{\frac12}.$$

(1)⇒(3): Define $G : \Sigma \to L^2(\Omega;X)$ by

$$G(A) := \int_A \phi\,dW, \quad A \in \Sigma.$$

By the γ-dominated convergence theorem [5], G is countably additive. To prove that G is of bounded randomised variation we consider disjoint sets $A_1, \dots, A_N \in \Sigma$. If $(\tilde r_n)_{n\geqslant 1}$ is a Rademacher sequence on a probability space $(\tilde\Omega, \tilde{\mathscr F}, \tilde{\mathbb P})$, then by randomisation we have

$$\begin{aligned}\tilde{\mathbb E}\Big\|\sum_{n=1}^N \tilde r_n\, G(A_n)\Big\|^2_{L^2(\Omega;X)} &= \tilde{\mathbb E}\mathbb{E}\Big\|\sum_{n=1}^N \tilde r_n \int_{A_n}\phi\,dW\Big\|^2 \\ &= \mathbb{E}\Big\|\sum_{n=1}^N \int_{A_n}\phi\,dW\Big\|^2 \leqslant \mathbb{E}\Big\|\int_S \phi\,dW\Big\|^2,\end{aligned}$$

with equality if $\bigcup_{n=1}^N A_n = S$. In the second identity we used that the X-valued random variables $\int_{A_n}\phi\,dW$ are independent and symmetric. The final inequality follows by, e.g., covariance domination [5] or an application of the contraction principle. It follows that G is a countably additive vector measure of bounded randomised variation and

$$\|G\|_{V^{\mathrm{r}}(\mu;X)} = \Big(\mathbb{E}\Big\|\int_S \phi\,dW\Big\|^2\Big)^{\frac12}.$$

(3)⇒(1): This is immediate from the definition of stochastic integrability. □

Example 1. If W is a standard Brownian motion on $(\Omega, \mathscr F, \mathbb P)$ indexed by the Borel interval $([0,1], \mathscr B, m)$, then W is a countably additive vector measure with values in $L^2(\Omega)$ which is of bounded randomised variation, but of unbounded variation. The first claim follows from Theorem 3.2 since $W(A) = \int_A 1\,dW$ for all

Borel sets A. To see that W is of unbounded variation, note that for any partition $0 = t_0 < t_1 < \cdots < t_{N-1} < t_N = 1$ we have

$$\sum_{n=1}^{N} \|W((t_{n-1}, t_n))\|_{L^2(\Omega)} = \sum_{n=1}^{N} \sqrt{t_n - t_{n-1}}.$$

The supremum over all possible partitions of $[0, 1]$ is unbounded.

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