PATHWISE HÖLDER CONVERGENCE OF THE IMPLICIT EULER SCHEME FOR SEMI-LINEAR SPDEs WITH MULTIPLICATIVE NOISE

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Abstract. In this article we prove pathwise Hölder convergence with optimal rates of the implicit Euler scheme for the abstract stochastic Cauchy problem

\[
\begin{aligned}
dU(t) &= AU(t) dt + F(t, U(t)) dt + G(t, U(t)) dW_H(t); & t \in [0, T], \\
U(0) &= x_0.
\end{aligned}
\]

Here \(A\) is the generator of an analytic \(C_0\)-semigroup on a \(\text{UMD}\) Banach space \(X\), \(W_H\) is a cylindrical Brownian motion in a Hilbert space \(H\), and the functions \(F : [0, T] \times X \to X\) and \(G : [0, T] \times X \to \mathcal{L}(H, X)\) satisfy appropriate (local) Lipschitz conditions. The results are applied to a class of second order parabolic SPDEs driven by multiplicative space-time white noise.

1. Introduction

In this article we prove pathwise Hölder convergence with optimal rates for various numerical schemes, including the implicit Euler scheme and the so-called splitting scheme, associated with parabolic stochastic partial differential equations (SPDE) driven by multiplicative Gaussian noise. Such equations can be written as abstract Cauchy problems of the form

\[
\begin{aligned}
dU(t) &= AU(t) dt + F(t, U(t)) dt + G(t, U(t)) dW_H(t), & t \in [0, T]; \\
U(0) &= x_0,
\end{aligned}
\]

where \(A\) is the generator of an analytic \(C_0\)-semigroup on a \(\text{UMD}\) Banach space \(X\) and \(W_H\) is a cylindrical Brownian motion in a Hilbert space \(H\) with respect to some given filtration \((\mathcal{F}_t)_{t \in [0, T]}\).

The functions \(F : [0, T] \times X \to X\) and \(G : [0, T] \times X \to \mathcal{L}(H, X)\) are assumed to satisfy appropriate (local) Lipschitz and linear growth conditions that will be specified in Section 2.4.1. Here, \(X_\theta\) denotes the fractional domain space (if \(\theta \geq 0\)) or extrapolation space (if \(\theta \leq 0\)) of \(A\) of order \(\theta\). The initial value \(x_0\) is an \(\mathcal{F}_0\)-measurable random variable taking values in \(X_\eta\) for some \(\eta \geq 0\).

Under the conditions on the exponents \(\theta_F\) and \(\theta_G\) stated in Theorem 1.1, the problem (1.1) admits a unique mild solution in a suitable Banach space of ‘weighted’ stochastically integrable \(X\)-valued processes (see Subsection 2.4 for the details).

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Our strategy is to first prove convergence of the various approximation schemes in this space for globally Lipschitz coefficients $F$ and $G$. These results imply uniform convergence of all $p^\text{th}$ moments. A Kolmogorov argument then allows us to deduce convergence in certain Hölder norms (Theorems 1.1 and 1.2). A standard Borel-Cantelli argument then gives almost sure Hölder convergence of the paths, and a localization argument allows us to extend the results to locally Lipschitz coefficients $F$ and $G$.

To the best of our knowledge, this is the first article proving convergence with respect to Hölder norms. A detailed comparison with known results in the literature is given below (see page 5).

At the end of the paper we apply our abstract results to a class of second order parabolic SPDEs driven by multiplicative space-time white noise. Further examples of SPDEs that fit into this framework can be found in [30, Section 10]. In these applications, typically one takes $X = L^p(D)$ (or a fractional domain space derived from it) with $D \subseteq \mathbb{R}^d$ an open domain; the choice $H = L^2(D)$ then corresponds to the case of space-time white noise on $D$.

The implicit-linear Euler scheme. The main result of this article gives optimal pathwise Hölder convergence rates for the implicit-linear Euler scheme associated with (1.2). The scheme is defined as follows. Fixing a finite time horizon $0 < T < \infty$ and an integer $n \in \mathbb{N} = \{1, 2, 3, \ldots\}$, we set $V_0^{(n)} := x_0$ and, for $j = 1, \ldots, n$, define the random variables $V_j^{(n)}$ implicitly by the identity

$$V_j^{(n)} = V_{j-1}^{(n)} + \frac{T}{n} [AV_j^{(n)} + F(t_{j-1}^{(n)}, V_{j-1}^{(n)})] + G(t_{j-1}^{(n)}, V_{j-1}^{(n)})W_j^{(n)}.$$  

Here $$t_j^{(n)} = t \frac{j}{n}$$ and, formally, $W_j^{(n)} = W_H(t_j^{(n)}) - W_H(t_{j-1}^{(n)})$.

The rigorous interpretation of the term $G(t_{j-1}^{(n)}, V_{j-1}^{(n)})W_j^{(n)}$ is explained in Section 5. Note that this scheme is implicit only in its linear part. As a consequence of this, and noting that for large enough $n \in \mathbb{N}$ we have $\frac{T}{n} \in \mathcal{g}(A)$, the resolvent set of $A$, this identity may be rewritten in the explicit format

(1.2) \hspace{1cm} V_j^{(n)} = (1 - \frac{T}{n} A)^{-1} [V_{j-1}^{(n)} + \frac{T}{n} F(t_{j-1}^{(n)}, V_{j-1}^{(n)})] + G(t_{j-1}^{(n)}, V_{j-1}^{(n)})W_j^{(n)}).

For a sequence $x = (x_j)_{j=0}^\infty$ in $X$ (we think of the $x_j$ as the values $f(t_j^{(n)})$ of some function $f : [0, T] \to X$) and $0 \leq \gamma \leq 1$ we define the discrete Hölder norm $\| \cdot \|_{\nu, (0,T];X}$ as follows:

$$\|x\|_{\nu, (0,T];X} := \sup_{0 \leq j \leq n} \|x_j\|_X + \sup_{0 \leq i < j \leq n} \frac{\|x_j - x_i\|_X}{|t_j^{(n)} - t_i^{(n)}|^{\gamma}}.$$  

Set $u = (U(t_j^{(n)}))_{j=0}^n$, where $(U(t))_{t\in [0,T]}$ is the mild solution of (1.1), and $v^{(n)} = (V_j^{(n)})_{j=0}^n$. Under appropriate Lipschitz and linear growth conditions on the functions $F : [0, T] \times X \to X_{\theta_F}$ and $G : [0, T] \times X \to \mathcal{L}(H, X_{\theta_G})$ (conditions (F), (G) in Section 2.4.1 and conditions (F'), (G') in Section 5), the following result is obtained in Section 6.

Theorem 1.1 (Hölder convergence of the Euler scheme). Let $X$ be a UMD Banach space with Pisier’s property $(\alpha)$, and let $\tau \in (1,2]$ be the type of $X$. Let $\theta_F > \frac{1}{\tau}$ and
$-1 + \left(\frac{1}{p} - \frac{1}{2}\right)$, $\theta_G > -\frac{1}{2}$. Suppose $p \in (2, \infty)$ and $\gamma, \delta \in [0, \frac{1}{2})$ and $\eta > 0$ satisfy

$$\gamma + \delta + \frac{1}{p} < \min\{1 - \left(\frac{1}{p} - \frac{1}{2}\right) + (\theta_F \wedge 0), \frac{1}{2} + (\theta_G \wedge 0), \eta\},$$

and suppose that $x_0 \in L^p(\Omega, \mathcal{F}_0; X_n)$. There is a constant $C$, independent of $x_0$, such that for all large enough $n \in \mathbb{N}$,

$$\mathbb{E}\left\|u - v^{(n)}\right\|^p_{c^{(n)}_p([0,T]; X)} \leq Cn^{-\delta}(1 + \left\|x_0\right\|_{L^p(\Omega; X_n)}).$$

Examples of UMD spaces with property $(\alpha)$ are the Hilbert spaces and the Lebesgue spaces $L^p(\mu)$ with and $1 < p < \infty$ and $\mu$ a $\sigma$-finite measure (see the introductions of Sections 2 and 5 for more details).

By a Borel-Cantelli argument, (1.3) implies that for almost all $\omega \in \Omega$ there exists a constant $C$ depending on $\omega$ but independent of $n$, such that:

$$\sup_{j \in \{0, \ldots, n\}} \left\|u(\omega) - v^{(n)}(\omega)\right\|_{c^{(n)}_p([0,T]; X)} \leq Cn^{-\delta}.$$

By a localization argument, we can now obtain almost sure pathwise convergence, with the same rates, for the case that $x_0 : \Omega \rightarrow X_n$ is merely $\mathcal{F}_0$-measurable and the coefficients $F$ and $G$ are locally Lipschitz continuous with linear growth (see Section 7).

The splitting scheme. Theorem 1.1 will be deduced from the corresponding convergence result for a splitting scheme, essentially by using the classical Trotter-Kato formula

$$S(t)x = \lim_{n \rightarrow \infty} \left(I - \frac{t}{n} A\right)^{-n}x$$

to pass from the resolvent of $A$ to the semigroup $S(t)$ generated by $A$.

We shall use a scheme which is an appropriate modification of the ‘classical’ splitting scheme, which allows us to cover the case of negative fractional indices $\theta_F$ and $\theta_G$. It is defined by setting $U_0^{(n)}(0) := x_0$ and then successively solving, for $j = 1, \ldots, n$, the problem

$$dU_j^{(n)}(t) = S(\frac{t}{n})\left[F(t, U_j^{(n)}(t)) + G(t, U_j^{(n)}(t))dW_H(t)\right], \quad t \in \left[t_{j-1}^{(n)}, t_j^{(n)}\right],$$

$$U_j^{(n)}(t_{j-1}^{(n)}) = S(\frac{t_{j-1}^{(n)}}{n})U_{j-1}^{(n)}(t_{j-1}^{(n)}).$$

Set $u = (U(t_j^{(n)}))_{j=0}^n$ and $u^{(n)} = (U_j^{(n)}(t_j^{(n)}))_{j=0}^n$. In Section 6 the following theorem is obtained from Theorem 3.1, assuming that $F$ and $G$ satisfy the Lipschitz and linear growth conditions (F) and (G) as stated in Section 2.4.1.

Theorem 1.2 (Hölder convergence of the splitting scheme). Let $X$ be a UMD Banach space and let $\tau \in (1, 2]$ be the type of $X$. Let $\theta_F > -1 + \left(\frac{1}{p} - \frac{1}{2}\right)$, $\theta_G > -\frac{1}{2}$. Suppose $p \in [2, \infty)$ and $\gamma, \delta \in [0, 1)$ and $\eta > 0$ satisfy

$$\gamma + \delta + \frac{1}{p} < \min\{1 - \left(\frac{1}{p} - \frac{1}{2}\right) + \theta_F, \frac{1}{2} + \theta_G, \eta, 1\},$$

and suppose that $x_0 \in L^p(\Omega, \mathcal{F}_0; X_n)$. There is a constant $C$, independent of $x_0$, such that for all $n \in \mathbb{N}$,

$$\mathbb{E}\left\|u - u^{(n)}\right\|^p_{c^{(n)}_p([0,T]; X)} \leq Cn^{-\delta}(1 + \left\|x_0\right\|_{L^p(\Omega; X_n)}).$$
it is possible to obtain almost sure Hölder convergence of the paths for the case that \( F \) and \( G \) are locally Lipschitz. However, this requires some tedious extra arguments that will be presented in [7] (see also Remark 7.1).

In the special case of fractional indices \( \theta_F \geq 0 \) and \( \theta_G \geq 0 \) we can compare the above scheme with the ‘classical’ splitting scheme defined by solving the problems
\[
\begin{aligned}
\{ & dU_j^{(n)}(t) = F(t, U_j^{(n)}(t)) \, dt + G(t, U_j^{(n)}(t)) \, dW_H(t), \quad t \in [t_j^{(n)}-1, t_j^{(n)}], \\
U_j^{(n)}(t_j^{(n)}-1) & = S(U_j^{(n)}(t_j^{(n)}-1)).
\end{aligned}
\]

The conditions \( \theta_F \geq 0 \) and \( \theta_G \geq 0 \) guarantee the existence of unique \( X \)-valued mild solutions to these problems; in the modified splitting scheme the additional term \( S(U_j^{(n)}) \) provides the smoothing needed in the case of negative fractional indices. We shall prove that for \( \theta_F \geq 0 \) and \( \theta_G \geq 0 \) one obtains the same convergence rates as in Theorem 1.2.

**An example: the 1D stochastic heat equation.** In Section 8 we present a detailed application of our results to the stochastic heat equation with multiplicative noise in space dimension one. We consider the problem
\[
\begin{aligned}
\{ & \frac{\partial u}{\partial t}(\xi, t) = \alpha_2(\xi) \frac{\partial^2 u}{\partial \xi^2}(\xi, t) + \alpha_1(\xi) \frac{\partial u}{\partial \xi}(\xi, t) \\
& + f(t, \xi, u(\xi, t)) + g(t, \xi, u(\xi, t)) \frac{\partial u}{\partial \xi}(\xi, t); \quad \xi \in (0, 1), \; t \in (0, T], \\
& u(0, \xi) = u_0(\xi); \quad \xi \in [0, 1],
\end{aligned}
\]

with Dirichlet or Neumann boundary conditions. Assuming \( \alpha_2 \in C[0, 1] \) to be bounded away from 0 and \( \alpha_1 \in C[0, 1] \), and assuming standard Lipschitz and linear growth assumptions on the nonlinearities \( f \) and \( g \), it is shown that if \( p > 4, \; q > 2, \; \alpha > 0, \; \beta \in (\frac{1}{2q}, \frac{1}{4}) \), and \( \gamma, \delta \geq 0 \) satisfy
\[
\beta + \gamma + \delta + \frac{1}{p} < \min\{\frac{1}{2}, \alpha\},
\]

then for initial values \( u_0 \in L^p(\Omega; H^{2\alpha,q}(0, 1)) \) the modified splitting scheme \( u^{(n)} = (U_j^{(n)}(t_j^{(n)}))_{j=0}^n \), defined by (1.5), and the implicit Euler scheme \( v^{(n)} = (V_j^{(n)})_{j=0}^n \), defined by (1.2), satisfy:
\[
\begin{aligned}
&E\|u - u^{(n)}\|_{L^p(\Omega; H^{\beta,\gamma,q}(0, 1))}^p \lesssim n^{-\delta}(1 + \|u_0\|_{L^p(\Omega; H^{2\alpha,q}(0, 1))}), \\
&E\|u - v^{(n)}\|_{L^p(\Omega; H^{\beta,\gamma,q}(0, 1))}^p \lesssim n^{-\delta}(1 + \|u_0\|_{L^p(\Omega; H^{2\alpha,q}(0, 1))}).
\end{aligned}
\]

Let us take \( \gamma = 0 \). By a Borel-Cantelli argument and the Sobolev embedding theorem, among other things we shall deduce that for large enough \( p \) and \( q \) and initial values in \( L^p(\Omega; H^{\beta,\gamma,q}(0, 1)) \) we obtain
\[
\begin{aligned}
&\sup_{0 \leq j \leq n} \|u(t_j^{(n)}) - u_j^{(n)}\|_{C^\lambda[0, 1]} \lesssim n^{-\delta}(1 + \|u_0\|_{L^\infty(\Omega; H^{\beta,\gamma,q}(0, 1))}), \\
&\sup_{0 \leq j \leq n} \|u(t_j^{(n)}) - v_j^{(n)}\|_{C^\lambda[0, 1]} \lesssim n^{-\delta}(1 + \|u_0\|_{L^p(\Omega; H^{\beta,\gamma,q}(0, 1))}).
\end{aligned}
\]

whenever \( \lambda \geq 0 \) and \( \delta \geq 0 \) satisfy \( \lambda + 2\delta < \frac{1}{2} \).
Concerning the optimality of the rate. For end-point estimates (i.e., a weaker type of estimates than the type we consider, which are pathwise) it is proven in [9] that the optimal convergence rate of a time discretization for the heat equation in one dimension with additive space-time white noise based on \( n \) equidistant time steps is \( n^{-\frac{1}{4}} \) (see Remark 8.2). In that sense our results on the convergence for the heat equation are optimal.

For ‘trace class noise’ (which corresponds to taking \( \theta_G = 0 \) in our framework) it is known that the critical convergence rate \( n^{-\frac{1}{2}} \) is in a sense the best possible even in the simpler setting of ordinary stochastic differential equations with globally Lipschitz coefficients. To be precise, it is shown in [5] that there exist examples of equations of the form

\[
\begin{align*}
    dX(t) &= f(X(t))\,dt + g(X(t))\,dW(t), \quad t \in [0, T], \\
    X(0) &= x_0,
\end{align*}
\]

with \( x_0 \in \mathbb{R}^d \) and \( W \) a Brownian motion in \( \mathbb{R}^d \), whose solution \( X \) satisfies the endpoint estimate

\[
(\mathbb{E}|X(T) - \mathbb{E}(X(T)|\mathcal{P}_n)|^2)^{\frac{1}{2}} = n^{-\frac{1}{2}} \sqrt{\frac{1}{2} T}.
\]

Here \( \mathcal{P}_n = \sigma \{ W(t_j^{(n)}) : j = 1, \ldots, n \} \). Thus, if \( X \) is approximated by a sequence of processes \( X^{(n)} \) whose definition only depends on knowing \( \{ W(t_j^{(n)}) : j = 1, \ldots, n \} \) (such is the case for the implicit Euler scheme), the convergence rate cannot be better than \( n^{-\frac{1}{4}} \).

In the special case \( \theta_G = 0 \), Theorems 1.1 and 1.2 can be applied with any \( \gamma, \delta \geq 0 \) such that \( \gamma + \delta < \frac{1}{2} \), provided \( \theta_F \geq -\frac{1}{2} \), \( x_0 \) takes values in \( X_\eta \) with \( \eta \geq \frac{1}{2} \), and \( p \) is taken large enough. For bounded \( \mathcal{F}_0 \)-measurable initial values \( x_0 \) in \( X_{\frac{1}{2}} \) and taking \( \gamma = 0 \), this leads to pathwise uniform convergence of order \( n^{-\delta} \) for arbitrary \( \delta \in [0, \frac{1}{2}) \).

Our methods do not produce the critical convergence rate \( n^{-\frac{1}{2}} \). In the two examples in the literature that we know of where this rate is obtained ([12, 22]), the operator \( A \) has the property of ‘stochastic maximal regularity’ (see [28]) and the underlying space has type 2, and in neither of these results the convergence is pathwise. In the present work, we do obtain pathwise convergence under the weaker assumption that \( A \) generates an analytic semigroup, but in this more general framework we do not expect to attain the critical rate. We plan to study the maximally regular case in the type 2 setting in a forthcoming publication.

Related work on pathwise convergence. The literature on convergence rates for numerical schemes for stochastic evolution equations and SPDEs is extensive. For an overview we refer the reader to the excellent review paper [19].

To the best of our knowledge, this is the first article proving pathwise convergence with respect to Hölder norms for numerical schemes for stochastic evolution equations with (locally) Lipschitz coefficients. The works [11, 12, 13, 15, 16, 17, 18, 22, 26, 34, 37] consider frameworks which are amenable to a comparison with our Theorems 1.1 and 1.2; quite likely this list is far from complete. All these papers exclusively consider Hilbert spaces \( X \), the only exception being [17] where \( X \) is taken to be of martingale type 2. In particular, in all these papers \( X \) has type 2. Let us also mention the paper [14], where convergence is proved, for \( p = 2 \) and \( X \)
Hilbertian, for the implicit Euler scheme under monotonicity assumptions on the operator $A$.

Most of these papers cited above give endpoint convergence rates only. The first pathwise uniform convergence result of the implicit Euler scheme seems to be due to Gyöngy [12], who obtains convergence rate $n^{-\frac{1}{8}}$ for the 1D stochastic heat equation with multiplicative space-time white noise. Pathwise uniform convergence of the implicit Euler schemes (with convergence in probability) has been obtained by Printems [34] for the Burgers equation with rate $n^{-\gamma}$ for any $\gamma < \frac{1}{4}$.

Pathwise uniform convergence of the splitting scheme, with rate $n^{-1}$ (which is the rate corresponding to exponents $\theta_F \geq \frac{1}{p} - \frac{1}{2}$ and $\theta_G \geq \frac{1}{2}$ in our framework, provided we take $\eta \geq 1$), has been obtained by Gyöngy and Krylov [13] for again a different splitting scheme, namely one for which the solution of (1.5) is guaranteed by adding (roughly speaking) a term $\varepsilon AU(t) dt$ to the right-hand side of (1.5).

Outline of the paper. Section 2 contains the preliminaries on stochastic analysis in UMD Banach spaces and its applications to stochastic evolution equations in such spaces. The precise assumptions on the operator $A$ and the nonlinearities $F$ and $G$ in (1.1), which are assumed to hold throughout the article, are stated in Subsection 2.4.

The first main result of this article, Theorem 3.1, concerns the convergence of the so-called modified splitting scheme. Section 3 deals entirely with the proof of this result and a comparison with the classical splitting scheme. Section 4 is dedicated to obtaining quantitative bounds for the rate of convergence in the Trotter-Kato formula (1.4). Our second main result, Theorem 5.2, is presented in Section 5. It provides optimal convergence rates for certain abstract time discretization schemes including the implicit-linear Euler scheme.

The theorems presented in the Sections 3 and 5 concerning convergence rates do not provide rates in the Hölder norm, but in a class of spaces $\mathcal{F}^\infty_{\infty}([0,T] \times \Omega; X)$, $p \in (2, \infty]$, introduced in Section 2. A Kolmogorov type argument then allows us to obtain pathwise Hölder convergence rates. This is demonstrated in Section 6, thereby completing the proofs of Theorems 1.1 and 1.2. By a Borel-Cantelli argument, we also give an almost sure version of the main theorems. In Section 7 we show how to extend the almost sure pathwise convergence of Section 6 to the case that $F$ and $G$ are locally Lipschitz continuous.

The final section 8 contains the application of our results to a class of SPDEs driven by multiplication space-time white noise.

In the set-up of this paper, the convergence of the Euler scheme is deduced from the convergence of the splitting scheme. A more streamlined proof for the Euler scheme would be possible, but we have chosen the present indirect route for the following reason. In the splitting scheme, the semigroup is discretized, but not the noise. In the Euler scheme, both the semigroup and the noise are discretized. Because of this, it is not possible to derive the correct rates for the splitting scheme from those of the Euler scheme. The present arrangement gives the optimal rates for both schemes.

In Sections A and B of the Appendix, we prove some technical lemmas whose proofs would interrupt the flow of the main text. In Section C we state and prove an existence and uniqueness result for mild solutions to the problem (1.1).
2. Preliminaries

Throughout this paper, we use $H$ denotes a real Hilbert space with inner product $[\cdot, \cdot]$ and $X$ a real Banach space with duality $\langle \cdot, \cdot \rangle$ between $X$ and its dual $X^*$. Our work relies on the theory of stochastic integration in UMD Banach spaces developed in [29]. For an overview of the theory of UMD spaces we refer to [4] and the references given therein. Examples of UMD spaces are Hilbert spaces and the spaces $L^p(\mu)$ with $1 < p < \infty$ and $\mu$ a $\sigma$-finite measure. We shall frequently use the following well-known facts:

- Banach spaces isomorphic to a closed subspace of a UMD space are UMD;
- If $X$ is UMD, $1 < p < \infty$, and $\mu$ is a $\sigma$-finite measure, then $L^p(\mu; X)$ is UMD;
- Every UMD space is $K$-convex. Hence, by a theorem of Pisier [32], every UMD space has non-trivial type $\tau \in (1, 2)$.

For a thorough treatment of the notion of type and the dual notion of cotype we refer to the monograph of Albiac and Kalton [2].

The results of [29] make use of the concept of $\gamma$-radonifying operators in an essential way. We refer to [27] for a survey on this topic and shall use the notations and the results of this paper freely.

Let $\mathcal{H}$ be a Hilbert space. The Banach space $\gamma(\mathcal{H}, X)$ is defined as the completion of $\mathcal{H} \otimes X$ with respect to the norm

\[ \left\| \sum_{n=1}^{N} h_n \otimes x_n \right\|_{\gamma(\mathcal{H}, X)}^2 := \mathbb{E} \left\| \sum_{n=1}^{N} \gamma_n \otimes x_n \right\|^2. \]

Here we assume that $(h_n)_{n=1}^{N}$ is an orthonormal sequence in $\mathcal{H}$, $(x_n)_{n=1}^{N}$ is a sequence in $X$, and $(\gamma_n)_{n=1}^{N}$ is a standard Gaussian sequence on some probability space. The space $\gamma(H, X)$ embeds continuously into $\mathcal{L}(\mathcal{H}, X)$ and it elements are referred to as the $\gamma$-radonifying operators from $\mathcal{H}$ to $X$. In the special cases where $\mathcal{H} = L^2(S)$ and $\mathcal{H} = L^2(S; H)$ we write

\[ \gamma(L^2(S); X) = \gamma(S; X), \quad \gamma(L^2(S; H), X) = \gamma(S; H, X). \]

If $X$ is a space with type 2, then we have a continuous embedding

\[ L^2(S, \gamma(H, X)) \hookrightarrow \gamma(S; H, X) \]

with norm depending only on the type 2 constant of $X$.

We shall frequently need the so-called ideal property. It states that if $\mathcal{H}_1$ and $\mathcal{H}_2$ are Hilbert spaces and $X_1$ and $X_2$ are Banach spaces, then for all $V \in \mathcal{L}(\mathcal{H}_2, \mathcal{H}_1)$, $T \in \gamma(\mathcal{H}_1, X_1)$, and $U \in \mathcal{L}(X_1, X_2)$ we have $UTV \in \gamma(\mathcal{H}_2, X_2)$ and

\[ \| UTV \|_{\gamma(\mathcal{H}_2, X_2)} \leq \| U \| \| V \|_{\gamma(\mathcal{H}_1, X_1)} \| R \|. \]

By [29, Proposition 2.6], the mapping $J : L^p(R; \gamma(\mathcal{H}, X)) \to \mathcal{L}(\mathcal{H}, L^p(R; X))$, where $p \in [1, \infty)$, defined by

\[ (Jf)h(r) := f(r)h, \quad r \in R, \quad h \in \mathcal{H}, \]

defines an isomorphism of Banach spaces

\[ L^p(R; \gamma(\mathcal{H}, X)) \simeq \gamma(\mathcal{H}, L^p(R, X)). \]
2.1. Stochastic integration. An \( H \)-cylindrical Brownian motion with respect to a filtration \( (\mathcal{F}_t)_{t \in [0,T]} \) is a linear mapping \( W_H : L^2(0,T; H) \rightarrow L^2(\Omega) \) with the following properties:

1. for all \( f \in L^2(0,T; H) \), \( W_H(f) \) is Gaussian;
2. for all \( f_1, f_2 \in L^2(0,T; H) \) we have \( \mathbb{E}(W_H(f_1)W_H(f_2)) = [f_1, f_2] \);
3. for all \( h \in H \) and \( t \in [0,T] \), \( W_H(1_{(0,t]} \otimes h) \) is \( \mathcal{F}_t \)-measurable;
4. for all \( h \in H \) and \( 0 \leq s \leq t < \infty \), \( W_H(1_{(s,t]} \otimes h) \) is independent of \( \mathcal{F}_s \).

For all \( f_1, \ldots, f_n \in L^2(0,T; H) \) the random variables \( W_H(f_1), \ldots, W_H(f_n) \) are jointly Gaussian. As a consequence, these random variables are independent if and only if \( f_1, \ldots, f_n \) are orthogonal in \( L^2(0,T; H) \). For further details on cylindrical Brownian motions see [27, Section 3].

The stochastic integral with respect to an \( H \)-cylindrical Brownian motion \( W_H \) of a finite rank adapted step process \( \Phi : (0,T) \times \Omega \rightarrow H \otimes X \) of the form

\[
\Phi(t, \omega) = \sum_{n=1}^{N} \sum_{m=1}^{M} 1_{(t_{n-1},t_n]}(t) \sum_{k=1}^{K} h_k \otimes x_{nmk},
\]

where \( 0 \leq t_0 < t_1 < \ldots < t_N < T \), \( A_{nm} \in \mathcal{F}_{t_{n-1}} \), \( x_{nmk} \in X \), and the vectors \( h_k \) are orthonormal in \( H \), is defined by

\[
\int_0^T \Phi \, dW_H := \sum_{n=1}^{N} \sum_{m=1}^{M} 1_{A_{nm}}(\omega) \sum_{k=1}^{K} W_H(1_{(t_{n-1},t_n]} \otimes h_k) \otimes x_{nmk}.
\]

In the above, for \( h \in H \), \( \phi \in L^2(\Omega) \), and \( x \in X \), we write \( h \otimes x \) for the rank one operator from \( H \) to \( X \) given by \( h' \mapsto [h, h']x \) and \( \phi \otimes x \) for the random variable \( \omega \mapsto \phi(\omega)x \).

Theorem 2.1 (Burkholder-Davis-Gundy estimates ([29] for \( p \in (1, \infty) \), [8] for \( p \in (0, \infty) \)). Let \( X \) be a UMD Banach space and let \( p \in (1, \infty) \) be fixed. For every finite rank adapted step process \( \Phi : (0,T) \times \Omega \rightarrow H \otimes X \) we have

\[
\mathbb{E} \sup_{0 \leq t \leq T} \norm{\int_0^t \Phi \, dW_H}_X^p \approx_p \mathbb{E} \norm{\Phi}_{p(0,T; H, X)}^p,
\]

the implied constants being independent of \( \Phi \).

In what follows we denote by

\[
L^p_\mathcal{F}(\Omega; \gamma(0,T; H, X))
\]

the closure in \( L^p(\Omega; \gamma(0,T; H, X)) \) of the adapted finite rank step processes. Due to estimate (2.3), an adapted measurable process \( \Phi : [0,T] \times \Omega \rightarrow X \) is \( L^p \)-stochastically integrable with respect to \( W_H \) if and only if it defines an element of \( L^p_\mathcal{F}(\Omega; \gamma(0,T; H, X)) \).
Replacing the role of the Gaussian sequence by a Rademacher sequence we arrive

\[ X \]

the converse holds if

\[ q, r \]

we want to emphasize the domain and range spaces we shall write

\[ \gamma \]

where the constant of the embedding depends on \( \Phi \in \gamma \) and more information

\[ C \]

of operators

\[ \text{Randomized boundedness for analytic semigroups.} \]

\[ 2.3. \]

\[ \mathcal{B}(2.5) \]

\[ \text{embedding} \]

\[ \text{with, for } h \in \mathbb{R}, \]

\[ T'_h f(s) = \begin{cases} f(s + h); & s + h \in I, \\ 0; & s + h \notin I. \end{cases} \]

Observe that if \( I' \subseteq I \) are nested intervals, then we have a natural contractive restriction mapping from \( B^s_{q,r}(I; X) \) into \( B^s_{q,r}(I'; X) \).

If (and only if) a Banach space \( X \) has type \( \tau \in [1, 2] \), by [29] we have a continuous embedding

\[ B^{\frac{1}{2} - \frac{1}{2}}_{\tau, \tau}(I; \gamma(H, X)) \hookrightarrow \gamma(I; H, X), \]

where the constant of the embedding depends on \( |I| \) and the type \( \tau \) constant of \( X \).

\[ 2.3. \text{Randomized boundedness for analytic semigroups.} \]

Let \( (\gamma_k)_{k \geq 1} \) denote a sequence of real-valued independent standard Gaussian random variables. A family of operators \( \mathcal{R} \subseteq \mathcal{L}(X_1, X_2) \) is called \( \gamma \)-\textit{bounded} if there exists a finite constant \( C \geq 0 \) such that for all finite choices \( R_1, \ldots, R_n \in \mathcal{R} \) and vectors \( x_1, \ldots, x_n \in X_1 \) we have

\[ \mathbb{E} \left\| \sum_{k=1}^{n} \gamma_k R_k x_k \right\|_{X_2}^2 \leq C^2 \mathbb{E} \left\| \sum_{n=1}^{n} \gamma_k x_k \right\|_{X_1}^2. \]

The least admissible constant \( C \) is called the \( \gamma \)-\textit{bound} of \( \mathcal{R} \), notation \( \gamma(\mathcal{R}) \). When we want to emphasize the domain and range spaces we shall write \( \gamma_{[X_1, X_2]}(\mathcal{R}) \).

Replacing the role of the Gaussian sequence by a Rademacher sequence we arrive at the related notion of \( R \)-\textit{boundedness}. Every \( R \)-bounded set is \( \gamma \)-bounded, and the converse holds if \( X_1 \) has finite cotype. We refer to [6, 10, 23, 36] for examples and more information \( \gamma \)-boundedness and \( R \)-boundedness.
The following $\gamma$-multiplier result, due to Kalton and Weis [21] (see also [27]), establishes a relation between stochastic integrability and $\gamma$-boundedness.

**Theorem 2.2 ($\gamma$-Multiplier theorem).** Suppose $X_1$ does not contain a closed subspace isomorphic to $c_0$. Suppose $M : (0, T) \to \mathcal{L}(X_1, X_2)$ is a strongly measurable function with $\gamma$-bounded range $\mathcal{M} = \{ M(t) : t \in (0, T) \}$. If $\Phi \in \gamma(0, T; H, X_1)$ then $M\Phi \in \gamma(0, T; H, X_2)$ and:

$$\| M\Phi \|_{\gamma(0, T; H, X_2)} \leq \gamma(X_1, X_2) \| \Phi \|_{\gamma(0, T; H, X_1)}.$$

Due to Theorem 2.1, the theorem above implies that if $\Phi \in L_p^p(\Omega; \gamma(0, T; H, X_1))$ for some $p \in (1, \infty)$, then the function $M\Phi : (0, T) \times \Omega \to \gamma(H, X_2)$ is $L^p$-stochastically integrable and

$$\left\| \int_0^T M\Phi \, dW_H \right\|_{L^p(\Omega; H, X_2)} \lesssim \gamma(X_1, X_2) \left\| \int_0^T \Phi \, dW_H \right\|_{L^p(\Omega; H, X_1)}.$$

The $\gamma$-multiplier theorem will frequently by applied in conjunction with the following basic result due to Kaiser and Weis [20, Corollary 3.6]:

**Theorem 2.3.** Let $X$ be a Banach space with finite cotype. Define, for every $h \in H$, the operator $U_h : X \to \gamma(H, X)$ by

$$U_h x := h \otimes x, \quad x \in X.$$

Then the family $\{ U_h : \|h\| \leq 1 \}$ is $\gamma$-bounded.

We proceed with a useful $\gamma$-boundedness results which will be used frequently below. It is a minor variation of [30, Lemma 4.1], which is a key ingredient in the proof of the existence and uniqueness result for stochastic evolution equations proved there. The proof is entirely analogous and is left to the reader; it is based on the fact that if $A$ generates an analytic $C_0$-semigroup on $X$, then for all $\alpha > 0$ and any $T > 0$ there exists a constant $C$ such that

$$\| S(t) \|_{\mathcal{L}(X, X_\alpha)} \leq Ct^{-\alpha}; \quad \text{for all } t \in (0, T];$$

and

$$\| S(t) - I \|_{\mathcal{L}(X_\alpha, X)} \leq Ct^{\alpha \wedge 1}; \quad \text{for all } t \in (0, T].$$

In the typical application of the lemma, it is combined with the Kalton-Weis $\gamma$-multiplier theorem to estimate $\gamma$-radonifying norms of certain vector-valued or operator-valued functions.

**Lemma 2.4.** Let $A$ generate an analytic $C_0$-semigroup $S$ on a Banach space $X$.

1. For all $0 \leq \alpha < \beta$ and $t \in (0, T]$ the set $\mathcal{S}_{\beta, t} = \{ s^\beta S(s) : s \in [0, t] \}$ is $\gamma$-bounded in $\mathcal{L}(X, X_\alpha)$ and we have

$$\gamma(X, X_\alpha)(\mathcal{S}_{\beta, t}) \lesssim t^{\beta - \alpha},$$

with implied constant independent of $t \in (0, T]$.

2. For all $0 < \alpha \leq 1$ and $t \in (0, T]$ the set $\mathcal{S}_t = \mathcal{S}_{0, t} = \{ S(s) : s \in [0, t] \}$ is $\gamma$-bounded in $\mathcal{L}(X_\alpha, X)$ and we have

$$\gamma(X_\alpha, X)(\mathcal{S}_t) \lesssim t^{\alpha},$$

with implied constant independent of $t \in (0, T]$. 
(3) For all $0 < \alpha \leq 1$ and $t \in (0, T]$ the set $\mathcal{R}_t = \{S(s) - I : s \in [0, t]\}$ is $\gamma$-bounded in $\mathcal{L}(\mathcal{C}^\gamma X, X)$ and we have

$$\gamma|\mathcal{C}^\gamma X(\mathcal{R}_t)| \lesssim t^\alpha,$$

with implied constant independent of $t \in (0, T]$.

Let us emphasize that the constants in Lemma 2.4 may depend on the final time $T$: all we are asserting is that, given $T$, the constants are independent of $t \in [0, T]$.

2.4. Stochastic evolution equations.

2.4.1. The equation. We are interested in the convergence rate of various numerical schemes associated with stochastic evolution equations of the form:

$$\begin{align*}
\text{(SEE)} & \quad \begin{cases}
dU(t) = AU(t) \, dt + F(t, U(t)) \, dt + G(t, U(t)) \, dW_H(t); & t \in [0, T], \\
U(0) = x_0.
\end{cases}
\end{align*}$$

Here, $0 < T < \infty$ is fixed and $W_H$ is an $H$-cylindrical $(\mathcal{F}_t)_{t \in [0, T]}$-Brownian motion on a probability space $(\Omega, \mathcal{F}, \mathbb{P})$. We make the following standing assumptions on the Banach space $X$, the operator $A$, and the functions $F$ and $G$:

(A) $A$ generates an analytic $C_0$-semigroup on a UMD Banach space $X$.

(F) For some $\theta_F > -1 + \left(\frac{1}{2} - \frac{1}{2}\right)$, where $\tau$ is the type of $X$, the function $F : [0, T] \times X \to X_{\theta_F}$ is measurable in the sense that for all $x \in X$ the mapping $F(\cdot, x) : [0, T] \to X_{\theta_F}$ is strongly measurable. Moreover, $F$ is uniformly Lipschitz continuous and uniformly of linear growth in its second variable. That is to say, there exist constants $C_0$ and $C_1$ such that for all $t \in [0, T]$ and all $x, x_1, x_2 \in X$:

$$\begin{align*}
&\|F(t, x_1) - F(t, x_2)\|_{X_{\theta_F}} \leq C_0\|x_1 - x_2\|_X, \\
&\|F(t, x)\|_{X_{\theta_F}} \leq C_1(1 + \|x\|_X).
\end{align*}$$

The least constant $C_0$ such that the above holds is denoted by $\text{Lip}(F)$, and the least constant $C_1$ such that the above holds is denoted by $M(F)$.

(G) For some $\theta_G > -\frac{1}{2}$, the function $G : [0, T] \times X \to \mathcal{L}(H, X_{\theta_G})$ is measurable in the sense that for all $h \in H$ and $x \in X$ the mapping $G(\cdot, x)h : [0, T] \to X_{\theta_G}$ is strongly measurable. Moreover, $G$ is uniformly $L^2(0, t; H)$-Lipschitz continuous and uniformly of linear growth in its second variable. That is to say, there exist constants $C_0$ and $C_1$ such that for all $\alpha \in [0, \frac{1}{2})$, all $t \in [0, T]$, and all simple functions $\phi_1, \phi_2, \phi : [0, T] \to X$ one has:

$$\begin{align*}
&\|s \mapsto (t - s)^{-\alpha}[G(s, \phi_1(s)) - G(s, \phi_2(s))]\|_{\gamma(0, t; H, X_{\theta_G})} \\
&\quad \leq C_0\|s \mapsto (t - s)^{-\alpha}[\phi_1 - \phi_2]\|_{L^2(0, t; X) \cap \gamma(0, t; X)}; \\
&\|s \mapsto (t - s)^{-\alpha}G(s, \phi(s))\|_{\gamma(0, t; H, X_{\theta_G})} \\
&\quad \leq C_1(1 + \|s \mapsto (t - s)^{-\alpha}\phi(s)\|_{L^2(0, t; X) \cap \gamma(0, t; X)}).
\end{align*}$$

The least constant $C_0$ such that the above holds is denoted by $\text{Lip}_\gamma(G)$, and the least constant $C_1$ such that the above holds is denoted by $M_\gamma(G)$.

Our definition of $L^2_\gamma$-Lipschitz continuity is a slight adaptation of the definition given in [30]. Examples of $L^2_\gamma$-Lipschitz continuous operators can be found in that article. In particular:
• If \( G \) is defined by an Nemytskii map on \([0, T] \times L^p(R)\), where \( p \in [1, \infty) \) and \((R, \mathcal{R}, \mu)\) a \( \sigma \)-finite measure space, then \( G \) is \( L^2 \)-Lipschitz continuous (see [30, Example 5.5]).

• if \( G : [0, T] \times X_1 \rightarrow \gamma(H, X_2) \) is Lipschitz continuous, uniformly in \([0, T]\), and \( X_2 \) is a type 2 space, then \( G \) is \( L^2 \)-Lipschitz continuous (see [30, Lemma 5.2]).

2.4.2. Existence and uniqueness. In [30, Theorem 6.2] an existence and uniqueness result is presented for solutions to (SEE) under the conditions (A), (F), (G). In this section we give a variation to these results (see Theorem 2.7 and Remark 2.8 below). We shall prove existence and uniqueness of a solution to (SEE) in a class of spaces which turns out to be the most suitable for proving convergence of the various numerical schemes under consideration.

Definition 2.5. For \( \alpha \geq 0 \) and \( 0 \leq a < b < \infty \) we define \( \mathcal{Y}_\alpha^p([a, b] \times \Omega; X) \) to be the space containing all adapted processes \( \Phi \in L_p^\alpha(\Omega; \gamma(a, b; X)) \) for which the following norm is finite:

\[
\|\Phi\|_{\mathcal{Y}_\alpha^p([a, b] \times \Omega; X)} = \|\Phi\|_{L^\infty(a, b; L_p(\Omega; X))} + \sup_{a \leq t \leq b} \|s \mapsto (t-s)^{-\alpha}\Phi(s)\|_{L_p(\Omega; \gamma(a, t; X))}.
\]

For \( 0 \leq \beta \leq \alpha < \frac{1}{2} \) the \( \gamma \)-multiplier theorem (Theorem 2.2) implies:

\[
(2.8) \quad \|\Phi\|_{\mathcal{Y}_\alpha^p([a, b] \times \Omega; X)} \leq (b-a)^{\beta-\alpha}\|\Phi\|_{\mathcal{Y}_{\alpha}^p([a, b] \times \Omega; X)}.
\]

One also checks that for \( 0 \leq a \leq b \leq T \):

\[
(2.9) \quad \|\Phi\|_{\mathcal{Y}_\alpha^p([a, b] \times \Omega; X)} = \|\Phi\|_{\mathcal{Y}_\alpha^p([a, b] \times \Omega; X)}.
\]

Finally, if \( G : [0, T] \times X \rightarrow \mathcal{L}(H, X_{\theta G}) \) satisfies (G) and \( \Phi_1, \Phi_2 \in \mathcal{Y}_\alpha^p([0, T] \times \Omega; X) \) for some \( p \geq 2 \), then:

\[
(2.10) \quad \sup_{0 \leq t \leq T} \|s \mapsto (t-s)^{-\alpha}[G(s, \Phi_1(s)) - G(s, \Phi_2(s))]\|_{L_p(\Omega; \gamma(0, t; X_{\theta G}))}
\leq\ \text{Lip}_\gamma(G) \sup_{0 \leq t \leq T} \|s \mapsto (t-s)^{-\alpha}[\Phi_1(s) - \Phi_2(s)]\|_{L_p(\Omega; L^2(0, t; X))}.
\]

\[
(2.11) \quad \sup_{0 \leq t \leq T} \|s \mapsto (t-s)^{-\alpha}G(s, \Phi(s))\|_{L_p(\Omega; \gamma(0, t; X_{\theta G}))} \leq (1 + T^{\frac{1}{2}-\alpha})\text{Lip}_\gamma(G) \left(1 + \|\Phi\|_{\mathcal{Y}_\alpha^p([0, T] \times \Omega; X)}\right).
\]

Definition 2.6. An adapted, strongly measurable process \( U : [0, T] \times \Omega \rightarrow X \) is called a mild solution of (SEE) if, for all \( t \in [0, T] \),

1. \( s \mapsto S(t-s)F(s, U(s)) \in L^0(\Omega, L^1(0, T; X)) \),
2. \( s \mapsto S(t-s)G(s, U(s)) \) is \( H \)-strongly measurable, adapted and almost surely in \( \gamma(0, t; H, X) \),

and moreover \( U \) satisfies:

\[
(2.12) \quad U(t) = S(t)x_0 + \int_0^t S(t-s)F(s, U(s)) \, ds + \int_0^t S(t-s)G(s, U(s)) \, dW_H(s).
\]
almost surely for all \( t \in [0, T] \).

A rigorous definition of the stochastic integral in (2.12) can be given if condition (2) in the definition above is satisfied, but this is beyond the theory discussed in 2.1. We refer to [30] for the details. With exception of Section 7, throughout this article the process \( s \mapsto S(t-s)G(s, U(S))1_{[0,t]} \) will always be \( L^p \)-stochastically integrable for some \( p \in (1, \infty) \).

The proof of the following theorem, which is entirely analogous to the proof given for [30, Theorem 6.2], is presented in Appendix C.

We set
\[
\eta_{\text{max}} := \min\{1 - (\frac{1}{p} - \frac{1}{2}) + \theta_F, \frac{1}{2} + \theta_G\}.
\]

**Theorem 2.7.** Let \( 0 \leq \eta < \eta_{\text{max}} \) and \( p \in [2, \infty) \). For all initial values \( x_0 \in L^p(\Omega; \mathcal{F}_0; X_\eta) \) and all \( \alpha \in [0, \frac{1}{2}) \), the problem (SEE) has a unique mild solution \( U \) in \( \mathcal{Y}^\alpha_{\infty,p}(0, T \times \Omega; X_\eta) \). It satisfies

\[
||U||_{\mathcal{Y}^\alpha_{\infty,p}(0, T \times \Omega; X_\eta)} \lesssim 1 + ||x_0||_{L^p(\Omega; X_\eta)}.
\]

**Remark 2.8.** In [30] the authors prove existence in the space \( V^\alpha_{\infty,p}(0, T \times \Omega; X_\eta) \) of all continuous adapted processes \( \Phi \in L^p(\Omega; \gamma(0, T; X)) \) for which the norm

\[
||\Phi||_{V^\alpha_{\infty,p}(0, T \times \Omega; X_\eta)} := ||\Phi||_{L^p(\Omega; \gamma(0, T; X_\eta))} + \sup_{0 \leq t \leq T} ||s \mapsto (t-s)^{-\alpha}\Phi(s)||_{L^p(\Omega; \gamma(0, t; X_\eta))}
\]

is finite, under the assumption that \( \frac{1}{p} < \frac{1}{2} + (\theta_G \cap 0) \) and \( 0 \leq \eta < \min\{1 - (\frac{1}{p} - \frac{1}{2}) + (\theta_F \cap 0), \frac{1}{2} + (\theta_G \cap 0) - \frac{1}{2}\} \). The extra factor \( \frac{1}{p} \) arises from a Kolmogorov-type estimate on the stochastic integral.

Note that the approximations obtained by the splitting scheme and the Euler scheme are not continuous. Accordingly we first prove convergence of the various schemes in \( \mathcal{Y}^\alpha_{\infty,p}(0, T \times \Omega; X) \) for arbitrarily large \( p \in (2, \infty) \). Pathwise convergence results (in the grid points) can then be obtained by a Kolmogorov argument. However, if we were to use the existence and uniqueness results of [30], we would lose a factor \( \frac{1}{p^2} \) twice. Instead, we shall use Theorem 2.7.

3. **Convergence of the splitting-up method**

We consider the stochastic differential equation (SEE) under the assumptions (A), (F), (G) and with initial value \( x_0 \in L^p(\Omega; \mathcal{F}_0; X_\eta) \) with \( 0 \leq \eta < \eta_{\text{max}} \).

In order to define a scheme, which we shall call the modified splitting scheme (for reasons to be explained shortly), we fix an initial value \( y_0 \in L^p(\Omega; \mathcal{F}_0; X) \), possibly different from \( x_0 \), and fix an integer \( n \in \mathbb{N} \). For \( j = 1, \ldots, n \) we define the process

\[
U_j^{(n)} : [t_{j-1}^{(n)}, t_j^{(n)}] \times \Omega \to X \text{ as the mild solution to the problem}
\]

\[
\begin{cases}
\frac{dU_j^{(n)}(t)}{dt} = S(\frac{t}{n})[F(t, U_j^{(n)}(t)) dt + G(t, U_j^{(n)}(t)) dW_H(t)], \quad t \in [t_{j-1}^{(n)}, t_j^{(n)}]; \\
U_j^{(n)}(t_{j-1}^{(n)}) = S(\frac{t}{n})U_j^{(n)}(t_{j-1}^{(n)}),
\end{cases}
\]

where we set \( U_0^{(n)}(0) := y_0 \); recall that \( t_j^{(n)} := \frac{jT}{n} \).

The existence of a unique mild solution to (3.3) in \( \mathcal{Y}^p_{\infty,p}([t_{j-1}^{(n)}, t_j^{(n)}] \times \Omega; X) \), for \( \alpha \in [0, \frac{1}{2}) \) and \( p \in [2, \infty) \), is guaranteed by Theorem 2.7. Here we use that \( S(\frac{t}{n})F : [0, T] \times X \to X \) satisfies (F) and that \( S(\frac{t}{n})G : [0, T] \times X \to \gamma(H, X) \) satisfies (G).
For $j = 1, \ldots, n$ we define $I_j^{(n)} := [t_{j-1}^{(n)}, t_j^{(n)})$. Observe that the adapted process $U^{(n)} : [0, T) \times \Omega \to X$ defined by

\begin{equation}
U^{(n)} := \sum_{j=1}^{n} 1_{I_j^{(n)}}(t) U_j^{(n)}(t), \quad t \in [0, T),
\end{equation}

defines an element of $\mathcal{V}_{\infty}^{\alpha,p}([0, T \times \Omega; X)$. In the next subsection we prove convergence of $U^{(n)}$ against $U$ in this space.

There is a subtle difference between the modified splitting scheme and the classical splitting scheme, which is defined by

\begin{equation}
\begin{cases}
\frac{d\tilde{U}_j^{(n)}(t)}{dt} = F(t, \tilde{U}_j^{(n)}(t)) \, dt + G(t, \tilde{U}_j^{(n)}(t)) \, dW_H(t), & t \in [t_{j-1}^{(n)}, t_j^{(n)}]; \\
\tilde{U}_j^{(n)}(t_{j-1}^{(n)}) = S(\frac{T}{n})\tilde{U}_{j-1}^{(n)}(t_{j-1}^{(n)})
\end{cases}
\end{equation}

with $\tilde{U}_0^{(n)}(0) := y_0$. The existence of a unique mild solution $\tilde{U}_j^{(n)}$ to (3.3) in $\mathcal{V}_{\infty}^{\alpha,p}([t_{j-1}^{(n)}, t_j^{(n)}] \times \Omega; X)$, for every $\alpha \in [0, \frac{1}{2})$ and $p \in [2, \infty)$, is again guaranteed by Theorem 2.7 provided that $\theta_F \geq 0$ and $\theta_G \geq 0$. However, if $\theta_F < 0$ or $\theta_G < 0$, then we have no means to define a solution to (3.3) in $X$, since we cannot guarantee that the integrals corresponding to $F$ and $G$ in the definition of a mild solution take values in $X$. In the modified splitting scheme, this problem is overcome by extra operator $S(\frac{T}{n})$ with provides the required additional smoothing.

Once convergence of the modified splitting scheme has been established, convergence of the classical splitting scheme is derived from it under the additional assumptions $\theta_F \geq 0$ and $\theta_G \geq 0$ (Theorem 3.3).

### 3.1. Convergence of the modified splitting scheme.

For $t \in I_j^{(n)}$ we define

\begin{equation}
\underline{t} := t_{j-1}^{(n)}, \quad \overline{t} := t_j^{(n)}.
\end{equation}

In particular, $\overline{t}_{j-1}^{(n)} = \overline{t}_j^{(n)}$. It should be kept in mind that in the notation $\overline{t}$ and $\underline{t}$ we suppress the dependence on $n$ and $T$.

The key idea of our approach is the following observation.

**Claim.** Let $U^{(n)}$ be defined by (3.2). Almost surely, for all $t \in [0, T)$ we have:

\begin{equation}
U^{(n)}(t) = S(\overline{t})y_0 + \int_0^t S(\overline{t} - \zeta)F(s, U^{(n)}(s)) \, ds \\
+ \int_0^t S(\overline{t} - \zeta)G(s, U^{(n)}(s)) \, dW_H(s).
\end{equation}
Proof of Claim. It suffices to prove that for any \( j \in \{1, \ldots, n\} \), almost surely the following identity holds for all \( t \in I_j^{(n)} \):

\[
U_j^{(n)}(t) = S(t^{(n)})y_0 + \int_{t^{(n)}_{j-1}}^{t^{(n)}} S(\frac{t}{n})F(s, U_j^{(n)}(s)) \, ds
+ \int_{t^{(n)}_{j-1}}^{t^{(n)}} S(\frac{t}{n})G(s, U_j^{(n)}(s)) \, dW_H(s)
+ \sum_{k=1}^{j-1} \int_{I_k^{(n)}} S(t^{(n)}_{j-k+1})F(s, U_k^{(n)}(s)) \, ds
+ \sum_{k=1}^{j-1} \int_{I_k^{(n)}} S(t^{(n)}_{j-k+1})G(s, U_k^{(n)}(s)) \, dW_H(s).
\]

(3.5)

By definition, the process \( U_j^{(n)} \), being a mild solution to (3.1), satisfies:

\[
U_j^{(n)}(t) = S(\frac{t}{n})U_j^{(n)}(t^{(n)}_{j-1}) + \int_{t^{(n)}_{j-1}}^{t^{(n)}} S(\frac{t}{n})F(s, U_j^{(n)}(s)) \, ds
+ \int_{t^{(n)}_{j-1}}^{t^{(n)}} S(\frac{t}{n})G(s, U_j^{(n)}(s)) \, dW_H(s)
\]

(3.6)

almost surely for all \( t \in I_j^{(n)} \). For \( j = 1 \) (3.5) follows directly from (3.6), and for \( j \in \{2, \ldots, n\} \) it follows by induction. \( \square \)

As always we assume that \((A)\), \((F)\), \((G)\) hold, and we denote by \( U \) the mild solution of the problem (SEE) with initial value \( x_0 \). We define \( U^{(n)} \) as above with initial value \( y_0 \). The proof of the following theorem uses the strong resemblance between identity (3.4) for \( U^{(n)} \) and the identity (2.12) satisfied by \( U \).

**Theorem 3.1.** Let \( 0 \leq \eta \leq 1 \) satisfy \( \eta < \eta_{\text{max}} \), let \( p \in [2, \infty] \), and assume that \( x_0 \in L^p(\mathcal{F}_0, X) \) and \( y_0 \in L^p(\mathcal{F}_0, X) \). Then for all \( \alpha \in (0, \frac{1}{2}) \) one has:

\[
\|U - U^{(n)}\|_{\mathcal{X}_{p,\infty}^{n,p}(0,T;\Omega;X)} \lesssim \|x_0 - y_0\|_{L^p(\Omega;X)} + n^{-\eta}(1 + \|x_0\|_{L^p(\Omega;X)})
\]

(3.7)

with implied constants independent of \( n \), \( x_0 \) and \( y_0 \).

**Proof.** Let \( \varepsilon > 0 \) be such that

\[
\varepsilon < \min\{\frac{1}{2}, 1 - 2\alpha, \eta_{\text{max}} - \eta\}.
\]

In particular we have \( \varepsilon < \frac{1}{2} + \theta_G \) and thus, by replacing \( \alpha \in [0, \frac{1}{2}) \) by some larger value if necessary, we may assume that

\[
\max\{1 - \frac{1}{2}\varepsilon, \varepsilon - 2\theta_G\} < 2\alpha < 1 - \varepsilon.
\]

We split the proof of (3.7) into several parts. In each part, constants will be allowed to depend on the final time \( T \). Thus, the statement ‘\( A(t) \lesssim B \) with a constant independent of \( t \in [0,T] \)’ is to be interpreted as short-hand for ‘there is a constant \( C \), possibly depending on \( T \), such that \( \sup_{t \in [0,T]} A(t) \leq CB \).’
**Part 1.** Fix $n \in \mathbb{N}$. Let $x, y \in L^p(\Omega; X_n)$. By the identities (2.12) and (3.4), for all $s \in [0, T]$ we have:

$$U(s) - U^{(n)}(s) = (S(s) - S(\bar{s}))x_0 + S(\bar{s})(x_0 - y_0) + \int_0^s [S(s - u) - S(\bar{s} - u)]F(u, U(u))\,du$$

$$+ \int_0^s (S(s - u) - S(\bar{s} - u))F(u, U(u))\,du$$

$$+ \int_0^s [S(s - u) - S(\bar{s} - u)]G(u, U(u))\,dW_H(u)$$

(3.8)

$$+ \int_0^s [S(s - u) - S(\bar{s} - u)]G(u, U(u))\,dW_H(u).$$

We shall estimate the $Y^{\alpha,p}_\infty([0, T_0] \times \Omega; X)$-norm of each of the six terms separately for arbitrary $T_0 \in [0, T]$. In the fourth and sixth term (Part 1d and 1f below) it will be necessary to keep track of the dependence on $T_0$.

**Part 1a.** We start with the first term in (3.8). Fix an arbitrary $\beta \in (0, \frac{1}{2})$. Writing $S(s) - S(\bar{s}) = (I - S(\bar{s} - s))S(s)$ and $S(s) = s^{-\beta}s^\beta S(s)$, from Lemma 2.4 (1) and (3) and Theorem 2.2 we obtain, almost surely for all $t \in [0, T]$:

$$\|s \mapsto (t - s)^{-\alpha}(S(s) - S(\bar{s}))x_0\|_{Y^{\alpha}_\infty(0,t; X)} \lesssim n^{-\eta}\|s \mapsto (t - s)^{-\alpha}S(s)x_0\|_{Y^{\alpha}_\infty(0,t; X)}$$

$$\lesssim n^{-\eta}\|s \mapsto (t - s)^{-\alpha}s^{-\beta}x_0\|_{Y^{\alpha}_\infty(0,t; X)}$$

$$= n^{-\eta}\|s \mapsto (t - s)^{-\alpha}s^{-\beta}\|L^2(0,t)\|x_0\|X$$

$$\lesssim n^{-\eta}\|x_0\|X,$$

with implied constants independent of $n$, $t$, and $x_0$. Also, by (2.7),

$$\|s \mapsto (S(s) - S(\bar{s}))x_0\|_{L^\infty(0,T; X)} \leq \sup_{s \in [0,T_0]} \|S(s)\|_{L^\infty(0,T_0)} \|I - S(\bar{s} - s)\|_{L^\infty(0,T_0; X)} \|x_0\|_{X_n} \lesssim n^{-\eta}\|x_0\|_{X_n}.$$

By taking $p^{th}$ moments it follows that that for every $T_0 \in [0, T]$ we have:

(3.9) $$\|s \mapsto (S(s) - S(\bar{s}))x_0\|_{Y^{\alpha}_\infty([0,T_0] \times \Omega; X)} \lesssim n^{-\eta}\|x_0\|_{L^p(\Omega; X_n)}$$

with implied constant independent of $n$, $T_0$ and $x_0$.

**Part 1b.** Concerning the second term on the right-hand side in (3.8) we note that, almost surely:

$$\|s \mapsto S(\bar{s})(x_0 - y_0)\|_{L^\infty(0,T; X)} \lesssim \|x_0 - y_0\|X,$$

with implied constant independent of $n$, $x_0$ and $y_0$. Also, by Lemma 2.4 (1) and Theorem 2.2, almost surely we have, for all $t \in [0, T]$:

$$\|s \mapsto (t - s)^{-\alpha}S(\bar{s})(x_0 - y_0)\|_{Y^{\alpha}_\infty(0,t; X)}$$

$$\lesssim \|s \mapsto (t - s)^{-\alpha}(\bar{s})^{-\epsilon}(x_0 - y_0)\|_{Y^{\alpha}_\infty(0,t; X)}$$

$$= \|s \mapsto (t - s)^{-\alpha}(\bar{s})^{-\epsilon}\|L^2(0,t)\|x_0 - y_0\|X$$

$$\lesssim \|x_0 - y_0\|X,$$
with implied constants are independent of \( n, t, x_0 \) and \( y_0 \).

Combining these estimates we obtain, for all \( T_0 \in [0, T] \):
\[
\| s \mapsto S(\sigma)(x_0 - y_0) \|_{\mathcal{Y}_{\infty}^\infty([0, T_0] \times \Omega; X)} \lesssim \| x_0 - y_0 \|_{L^p(\Omega; X)}
\]
with implied constants independent of \( n, T_0, x_0 \) and \( y_0 \).

**Part 1c.** Concerning the third term on the right-hand side in (3.8) we observe that:
\[
S(s - u) - S(\overline{\sigma} - \overline{u}) = (I - S(\overline{\sigma} - s))S(s - u) + S(\overline{\sigma} - s)S(s - u)(I - S(u - \overline{u}))
\]
and hence
\[
\int_0^s [S(s - u) - S(\overline{\sigma} - \overline{u})]F(u, U(u)) \, du
\]
\[
\quad = (I - S(\overline{\sigma} - s)) \int_0^s S(s - u)F(u, U(u)) \, du
\]
\[
\quad + S(\overline{\sigma} - s) \int_0^s S(s - u)(I - S(u - \overline{u}))F(u, U(u)) \, du.
\]

Let \( T_0 \in [0, T] \). It follows from Lemma 2.4, part (2) (with exponent \( \frac{1}{2} \varepsilon \)) and part (3) (with exponent \( \frac{3}{2} + \theta F - \frac{1}{2} - \varepsilon \)) and the \( \gamma \)-multiplier theorem (Theorem 2.2) that:
\[
\| s \mapsto \int_0^s [S(s - u) - S(\overline{\sigma} - \overline{u})]F(u, U(u)) \, du \|_{\mathcal{Y}_{\infty}^\infty([0, T_0] \times \Omega; X)}
\]
\[
\lesssim n^{-\min\left(\frac{3}{2} + \theta F - \frac{1}{2} - \varepsilon, 1\right)} \| s \mapsto \int_0^s S(s - u)F(u, U(u)) \, du \|_{\mathcal{Y}_{\infty}^\infty([0, T_0] \times \Omega; X^{\frac{3}{4} + \theta F - \frac{1}{4} - \varepsilon})}
\]
\[
\quad + \| s \mapsto \int_0^s S(s - u)(I - S(u - \overline{u}))F(u, U(u)) \, du \|_{\mathcal{Y}_{\infty}^\infty([0, T_0] \times \Omega; X^{\frac{3}{2} + \theta F - \frac{1}{2} - \varepsilon})},
\]
with implied constants independent of \( n \) and \( T_0 \). We shall estimate the two terms on the right-hand side of (3.12) separately.

We begin with the first term. Recall that \( U \in \mathcal{Y}_{\infty}^\infty([0, T_0] \times \Omega; X) \) and therefore, by (F), we have \( F(\cdot, U(\cdot)) \in L^\infty(0, T_0; L^p(\Omega; X_{\theta F})) \). By Lemma B.1 (applied with \( Y = X_{\frac{3}{2} + \theta F - \frac{1}{4} - \varepsilon} \), \( \Phi(u) = F(u, U(u)) \), and \( \delta = -\frac{3}{2} + \frac{1}{2} + \varepsilon \)) we obtain, for all \( t \in [0, T_0] \):
\[
\| s \mapsto \int_0^s S(s - u)F(u, U(u)) \, du \|_{\mathcal{Y}_{\infty}^\infty([0, T_0] \times \Omega; X^{\frac{3}{2} + \theta F - \frac{1}{4} - \varepsilon})}
\]
\[
\lesssim \| u \mapsto F(u, U(u)) \|_{L^\infty(0, T_0; L^p(\Omega; X_{\theta F}))}
\]
\[
\lesssim (1 + \| U \|_{L^\infty(0, T_0; L^p(\Omega; X))}),
\]
with implied constants independent of \( n, x_0 \) and \( T_0 \).

For the second term in the right-hand side of (3.12) we apply Lemma B.1 (with \( Y = X_{\frac{3}{2} + \varepsilon} \), \( \delta = -\frac{3}{2} + \frac{1}{2} + \frac{1}{2} \varepsilon \) and \( \Phi(u) = (I - S(u - \overline{u}))F(u, U(u)) \)). Note that \( \Phi \in L^\infty(0, T; L^p(\Omega; X_{\frac{3}{2} + \frac{1}{4} + \varepsilon})) \) by the boundedness of \( u \mapsto (I - S(u - \overline{u})) \) in \( \mathcal{L}(X_{\theta F}, X_{\frac{3}{4} + \frac{1}{4} + \varepsilon}) \), the linear growth condition in (F) and the fact that \( U \in \mathcal{L}^\infty(0, T; L^p(\Omega; X_{\theta F})) \).
\( V_\infty^{\alpha,p}(0, T_0] \times \Omega; X) \). We obtain:

\[
\begin{align*}
\left\| s \mapsto \int_0^s S(s-u)(I-S(u-u))F(u, U(u)) \, du \right\|_{V_\infty^{\alpha,p}(0, T_0] \times \Omega; X_{\frac{1}{2}}} \\
\lesssim \left\| u \mapsto (I-S(u-u))F(u, U(u)) \right\|_{L^\infty(0, T_0; L^p(\Omega; X_{\frac{1}{2}}))} \\
\lesssim n^{-\min\left(\frac{1}{2}+\theta_F - \frac{1}{2} - \varepsilon, 1\right)} \left\| u \mapsto F(u, U(u)) \right\|_{L^\infty(0, T_0; L^p(\Omega; X_{\theta_F}))} \\
\lesssim n^{-\min\left(\frac{1}{2}+\theta_F - \frac{1}{2} - \varepsilon, 1\right)} (1 + \|U\|_{L^\infty(0, T_0; L^p(\Omega; X))}),
\end{align*}
\]

with implied constants independent of \( n, x_0 \) and \( T_0 \). For the penultimate estimate we used (2.7).

Combining these estimates, applying (2.13), and recalling the assumptions \( \eta \leq 1 \) and \( \eta < \frac{1}{2} - \frac{1}{2} - \varepsilon + \theta_F \), we obtain:

\[
\begin{align*}
\left\| s \mapsto \int_0^s [S(s-u) - S(u-u)]F(u, U(u)) \, du \right\|_{V_\infty^{\alpha,p}(0, T_0] \times \Omega; X} \\
\lesssim n^{-\min\left(\frac{1}{2}+\theta_F - \frac{1}{2} - \varepsilon, 1\right)} (1 + \|U\|_{L^\infty(0, T_0; L^p(\Omega; X))}) \\
\lesssim n^{-\eta}(1 + \|x_0\|_{L^p(\Omega; X)}),
\end{align*}
\]

with implied constants independent of \( n, x_0 \) and \( T_0 \).

**Part 1d.** Concerning the fourth term on the right-hand side in (3.8) we first apply Theorem 2.2 and Lemma 2.4 (2) (with exponent \( \frac{1}{2} \varepsilon \)) and then apply Lemma B.1 (with \( Y = X_{\frac{1}{2}}, \delta = \theta_F - \varepsilon \) and \( \Phi(u) = S(u-u)(F(u, U(u)) - F(u, U^{(n)}(u))) \)). Observe that \( \Phi \in L^\infty(0, T; L^p(\Omega; X)) \) by the fact that both \( U \) and \( U^{(n)} \) belong to \( V_\infty^{\alpha,p}(0, T] \times \Omega; X \), (E), and the uniform boundedness of \( u \mapsto S(u-u) \) in \( \mathcal{L}(X_{\theta_F}, X_{\theta_F - \frac{1}{2} \varepsilon}) \). We obtain:

\[
\begin{align*}
\left\| s \mapsto S(s-s) \int_0^s S(s-u)[F(u, U(u)) - F(u, U^{(n)}(u))] \, du \right\|_{V_\infty^{\alpha,p}(0, T_0] \times \Omega; X} \\
\lesssim \left\| s \mapsto \int_0^s S(s-u)[F(u, U(u)) - F(u, U^{(n)}(u))] \, du \right\|_{V_\infty^{\alpha,p}(0, T_0] \times \Omega; X_{\frac{1}{2}}} \\
\lesssim (T_0^{-1-(\theta_F - \varepsilon)} + T_0^{\frac{1}{2} - \alpha}) \\
\times \|u \mapsto S(u-u)[F(u, U(u)) - F(u, U^{(n)}(u))]\|_{L^\infty(0, T_0; L^p(\Omega; X_{\theta_F - \frac{1}{2} \varepsilon}))} \\
\lesssim (T_0^{-1-(\theta_F - \varepsilon)} + T_0^{\frac{1}{2} - \alpha}) \|U - U^{(n)}\|_{L^\infty(0, T_0; L^p(\Omega; X))},
\end{align*}
\]

with implied constants independent of \( n \) and \( T_0 \).

**Part 1e.** For the fifth term on the right-hand side in (3.8) we proceed as in Part 1c. Using (3.11), Lemma 2.4, part (2) (with exponent \( \frac{1}{2} \varepsilon \)) and part (3) (with exponent
\( \frac{1}{2} + \theta_G - \frac{2}{3} \varepsilon \), and Theorem 2.2, we obtain:

\[
\| s \mapsto \int_0^s [S(s-u) - S(\pi - u)] G(u, U(u)) \, dW_H(u) \|_{\mathcal{Y}_\infty^\alpha([0,T_0] \times \Omega; X)} \lesssim n^{-\min\left(\frac{1}{2} + \theta_G - \frac{2}{3} \varepsilon, 1\right)} \|

\]

\[
\| s \mapsto \int_0^s S(s-u) G(u, U(u)) \, dW_H(u) \|_{\mathcal{Y}_\infty^\alpha([0,T_0] \times \Omega; X_{\frac{1}{2} + \theta_G - \frac{2}{3} \varepsilon})} + \| s \mapsto \int_0^s S(s-u)(I - S(u-u)) G(u, U(u)) \, dW_H(u) \|_{\mathcal{Y}_\infty^\alpha([0,T_0] \times \Omega; X_{\frac{1}{2} + \theta_G - \frac{2}{3} \varepsilon})}.
\]

Now we apply Lemma B.2 to the two terms on the right-hand side of (3.15). For the first term we apply Lemma B.2 with \( Y = X_{\frac{1}{2} + \theta_G - \frac{2}{3} \varepsilon} \), \( \delta = -\frac{1}{2} + \frac{2}{3} \varepsilon \) and \( \Phi(u) = G(u, U(u)) \), noting that \( \alpha > \frac{1}{2} - \frac{2}{3} \varepsilon = -\delta \). Assumption (B.1) is satisfied due to (2.11) and the fact that \( U \in \mathcal{Y}_\infty^\alpha([0,T] \times \Omega; X) \). By Lemma B.2 and (2.11) we obtain:

\[
\| s \mapsto \int_0^s S(s-u) G(u, U(u)) \, dW_H(u) \|_{\mathcal{Y}_\infty^\alpha([0,T_0] \times \Omega; X_{\frac{1}{2} + \theta_G - \frac{2}{3} \varepsilon})} \lesssim \sup_{s \in [0,T_0]} \| u \mapsto (s-u)^-\alpha G(u, U(u)) \|_{L_p(\Omega; \gamma(0,s;X_{\theta_G}))} \lesssim 1 + \| U \|_{\mathcal{Y}_\infty^\alpha([0,T_0] \times \Omega; X)},
\]

with implied constants independent of \( n \), \( x_0 \) and \( T_0 \).

For the second term we take \( Y = X_{\frac{1}{2} + \theta_G - \frac{2}{3} \varepsilon} \), \( \delta = -\frac{1}{2} + \frac{2}{3} \varepsilon \) and \( \Phi(u) = (I - S(u-u)) G(u, U(u)) \) in Lemma B.2, noting that \( \alpha > \frac{1}{2} - \frac{2}{3} \varepsilon = -\delta \); assumption (B.1) is satisfied because of \( U \in \mathcal{Y}_\infty^\alpha([0,T_0] \times \Omega; X) \), (2.11), and the fact that the operators \( I - S(u-u) \) are \( \gamma \)-bounded from \( X_{\theta_G} \) to \( X_{\frac{1}{2} + \varepsilon} \) by Lemma 2.4 (3).

By Lemma B.2, Lemma 2.4 (3) (applied with exponent \( \frac{1}{2} + \theta_G - \varepsilon \)) and (2.11) we obtain:

\[
\| s \mapsto \int_0^s S(s-u)(I - S(u-u)) G(u, U(u)) \, dW_H(u) \|_{\mathcal{Y}_\infty^\alpha([0,T_0] \times \Omega; X_{\frac{1}{2} + \theta_G - \frac{2}{3} \varepsilon})} \lesssim \sup_{s \in [0,T_0]} \| u \mapsto (s-u)^-\alpha (I - S(u-u)) G(u, U(u)) \|_{L_p(\Omega; \gamma(0,s;X_{\theta_G}))} \lesssim n^{-\min\left(\frac{1}{2} + \theta_G - \frac{2}{3} \varepsilon, 1\right)} \sup_{s \in [0,T_0]} \| u \mapsto (s-u)^-\alpha G(u, U(u)) \|_{L_p(\Omega; \gamma(0,s;X_{\theta_G}))} \lesssim n^{-\eta}(1 + \| U \|_{\mathcal{Y}_\infty^\alpha([0,T_0] \times \Omega; X)})),
\]

with implied constants independent of \( n \), \( x_0 \) and \( T_0 \).

Combining these estimates and applying (2.13) we obtain:

\[
\| s \mapsto \int_0^s [S(s-u) - S(\pi - u)] G(u, U(u)) \, dW_H(u) \|_{\mathcal{Y}_\infty^\alpha([0,T_0] \times \Omega; X)} \lesssim n^{-\eta}(1 + \| U \|_{\mathcal{Y}_\infty^\alpha([0,T_0] \times \Omega; X)}) \lesssim n^{-\eta}(1 + \| x_0 \|_{L_p(\Omega; X)}),
\]

with implied constants independent of \( n \), \( x_0 \) and \( T_0 \).

**Part 1f.** For the final term in (3.8) we proceed as in Part 1d. First we apply Theorem 2.2 in combination with Lemma 2.4 (2) (with exponent \( \frac{1}{4} \varepsilon \) to get rid of the term \( S(\pi - s) \). Then we apply Lemma B.2 (with \( Y = X_{\frac{1}{4} \varepsilon}, \delta = \theta_G - \frac{1}{2} \varepsilon \), and
\(\Phi = S(u - y)G(u, U(u)) - G(u, U^{(n)}(u))\). Note that \(\alpha > \frac{1}{2}\epsilon - \theta_G = -\delta\). Assumption (B.1) is satisfied because \(U\) and \(U^{(n)}\) are in \(Y_{\infty, p}([0, T_0] \times \Omega; X)\), condition (G) holds, and the operators \(S(u - y)\) are \(\gamma\)-bounded from \(X_{\theta_G}\) to \(X_{\theta_G - \frac{1}{2}\epsilon}\). Finally, we apply Theorem 2.2 again in combination with Lemma 2.4 (2) (with exponent \(\frac{1}{4}\)) to get rid of the term \(S(u - y)\). We obtain that there exists an \(\epsilon > 0\), independent of \(T_0 \in [0, T]\), such that:

\[
\left\| s \mapsto \int_0^s S(\pi - y)[G(u, U(u)) - G(u, U^{(n)}(u))] \, dW_H(u) \right\|_{Y_{\infty, p}([0, T_0] \times \Omega; X)} \leq T_0^\epsilon \sup_{0 \leq s \leq T_0} \| u \mapsto (s - u)^{-\alpha}S(u - y) \times [G(u, U(u)) - G(u, U^{(n)}(u))] \|_{L^p(\Omega; \gamma(0, s; H, X_{\theta_G - \frac{1}{4}}))}
\]

\[
\leq T_0^\epsilon \sup_{0 \leq s \leq T_0} \| s \mapsto (s - u)^{-\alpha}[G(u, U(u)) - G(u, U^{(n)}(u))] \|_{L^p(\Omega; \gamma(0, s; H, X_{\theta_G}))}
\]

\[
\leq T_0^\epsilon \| U - U^{(n)} \|_{Y_{\infty, p}([0, T_0] \times \Omega; X)},
\]

where the last step used (2.10); the implied constants are independent of \(n\) and \(T_0\).

**Part 2.** Substituting (3.9), (3.10), (3.13), (3.14), (3.16), (3.17) into (3.8) we obtain that there exists an exponent \(\epsilon_0 > 0\) and a constant \(C > 0\), both of which are independent of \(n\), \(x_0\), and \(y_0\), such that for all \(T_0 \in [0, T]\) we have:

\[
\| U - U^{(n)} \|_{Y_{\infty, p}([0, T_0] \times \Omega; X)} \leq C\left(\| x_0 - y_0 \|_{L^p(\Omega; X)} + n^{-\eta}(1 + \| x_0 \|_{L^p(\Omega; X_n)})\right)
\]

\[
+ C T_0^\epsilon \| U - U^{(n)} \|_{Y_{\infty, p}([0, T_0] \times \Omega; X)}.
\]

From now on we fix \(T_0 := \min\{(2C)^{1/\epsilon_0}, T\}\). Note that \(T_0\) is independent of \(n\), \(x_0\), and \(y_0\), and we have:

\[
\| U - U^{(n)} \|_{Y_{\infty, p}([0, T_0] \times \Omega; X)} \leq 2C\| x_0 - y_0 \|_{L^p(\Omega; X)} + 2C n^{-\eta}(1 + \| x_0 \|_{L^p(\Omega; X_n)}).
\]

**Part 3.** Let us fix \(n \in \mathbb{N}\) and pick \(t_0 \in \{t_j^{(n)} : j = 0, 1, \ldots, n\}\). For \(x \in L^p(\Omega, F_{t_0}; X)\) we denote by \(U(x, t_0, \cdot)\) the process in \(Y_{\infty, p}([t_0, t_0 + T] \times \Omega; X)\) satisfying, almost surely for all \(s \in [t_0, t_0 + T]\):

\[
U(x, t_0, s) = S(t - t_0)x + \int_{t_0}^s S(t - t_0 - s)F(s, U(x, t_0, s)) \, ds
\]

\[
+ \int_{t_0}^s S(t - t_0 - s)G(s, U(x, t_0, s)) \, dW_H(s).
\]

By \(U^{(n)}(x, t_0, \cdot)\) we denote the process obtained from the modified splitting scheme initiated in \(t_0\) with initial value \(x \in L^p(\Omega, F_{t_0}; X)\). Thus, almost surely for \(t \in [t_0, t_0 + T]\):

\[
U^{(n)}(x, t_0, t) = S(t - t_0)x + \int_{t_0}^t S(t - t_0 - s)F(s, U^{(n)}(x, t_0, s)) \, ds
\]
\[ + \int_{t_0}^t S(\bar t - t_0)G(s, U^n(x, t_0, s)) \, dW_H(s). \]

From the proof of (3.18) it follows that for any \( x \in L^p(\Omega, \mathcal{F}_t; X) \) and \( y \in L^p(\Omega, \mathcal{F}_t; X) \) we have:

\[
\|U(x, t_0, \cdot) - U^n(y, t_0, \cdot)\|_{L^p([t_0, t_0 + T_0] \times \Omega; X)} \leq 2C\|x - y\|_{L^p(\Omega; X)} + 2Cn^{-\eta}(1 + \|x\|_{L^p(\Omega; X)}),
\]

with \( C \) as in (3.18).

**Part 4.** Let \( T_0 \) be as in Part 2 and fix \( N \in \mathbb{N} \) large enough such that \( \frac{T}{N} \leq T_0 \). Let \( M = \lceil 2T/T_0 \rceil \). Then \( M \geq 2T \leq MT_0 \leq 2T + T_0 \leq 3T \).

Let us now fix \( n \geq N \). Then \( \frac{1}{2}T_0 \leq \max\{T_0 - \frac{T}{n}, \frac{T}{n}\} \) and therefore \( T_0 \geq \frac{1}{2}T_0 \). Hence, \( T \leq MT_0 \leq 3T \).

From now on we fix an integer \( n \geq N \). By the uniqueness of the mild solution to (1.1) and by the definition of \( U^n \) we have, for any \( s_0, t_0 \in \{t_j^n : j = 0, 1, \ldots, M\} \), any \( x \in L^p(\Omega, \mathcal{F}_s; X) \) and any \( t \in [t_0, t_0 + T_0] \) that:

\[
U(x, s_0, t) = U(U(x, s_0, t_0), t_0, t);
\]

\[
U^n(x, s_0, t) = U^n(U^n(x, s_0, t_0), t_0, t).
\]

For \( j \in \{1, \ldots, M\} \), from (3.19) (with \( x = U(x_0, (j - 1)T_0) \) and \( y = U^n(y_0, (j - 1)T_0) \)) we obtain:

\[
\|U(x_0, 0, jT_0) - U^n(y_0, 0, jT_0)\|_{L^p(\Omega; X)} = \|U(U(x_0, 0, (j - 1)T_0), (j - 1)T_0) - U^n(U^n(x_0, 0, (j - 1)T_0), (j - 1)T_0)\|_{L^p(\Omega; X)}
\]

\[
\leq \|U(x_0, 0, (j - 1)T_0) - U^n(y_0, 0, (j - 1)T_0)\|_{L^p(\Omega; X)} + n^{-\eta}(1 + \|U^n(x_0, 0, (j - 1)T_0)\|_{L^p(\Omega; X)}),
\]

with implied constants independent of \( j, n, x_0, y_0 \).

By (2.13) we have

\[
\sup_{1 \leq j \leq M} \|U(x_0, 0, jT_0)\|_{L^p(\Omega; X)} \leq \sup_{s \in [0, 3T]} \|U(x_0, 0, s)\|_{L^p(\Omega; X)} \leq 1 + \|x_0\|_{L^p(\Omega; X)},
\]

and therefore, by (3.20):

\[
\|U(x_0, 0, jT_0) - U^n(y_0, 0, jT_0)\|_{L^p(\Omega; X)} \leq \|U(x_0, 0, (j - 1)T_0) - U^n(y_0, 0, (j - 1)T_0)\|_{L^p(\Omega; X)} + n^{-\eta}(1 + \|x_0\|_{L^p(\Omega; X)}),
\]

with implied constants independent of \( j \) and \( n \). By induction we obtain:

\[
\sup_{1 \leq j \leq M} \|U(x_0, 0, jT_0) - U^n(y_0, 0, jT_0)\|_{L^p(\Omega; X)} \leq \|x_0 - y_0\|_{L^p(\Omega; X)} + n^{-\eta}(1 + \|x_0\|_{L^p(\Omega; X)}),
\]

with implied constants independent of \( j \) and \( n \). As \( M \) is independent of \( n \).
The estimate (3.22) is precisely what we need to extend (3.18) to the interval [0, T]. To do so, we once again fix \( j \in \{1, \ldots, M\} \). Set
\[
x = U(x_0, 0, (j - 1)\mathbb{T}_0) \quad \text{and} \quad y = U^{(n)}(y_0, 0, (j - 1)\mathbb{T}_0)
\]
in (3.19) to obtain, using (3.21) and (3.22):
\[
\|U(x_0, 0, \cdot) - U^{(n)}(y_0, 0, \cdot)\|_{\mathcal{F}^{\infty, p}([0, 1] \times \Omega; X)}
= \|U(U(x_0, 0, (j - 1)\mathbb{T}_0), (j - 1)\mathbb{T}_0, \cdot)
- U^{(n)}(y_0, 0, (j - 1)\mathbb{T}_0), (j - 1)\mathbb{T}_0, \cdot)\|_{\mathcal{F}^{\infty, p}([0, 1] \times \Omega; X)}
\leq \|U(x_0, 0, (j - 1)\mathbb{T}_0) - U^{(n)}(y_0, 0, (j - 1)\mathbb{T}_0)\|_{L^p(\Omega; X)}
+ n^{-\eta}(1 + \|U(x_0, 0, (j - 1)\mathbb{T}_0)\|_{L^p(\Omega; X)})
\leq \|x_0 - y_0\|_{L^p(\Omega; X)} + n^{-\eta}(1 + \|x_0\|_{L^p(\Omega; X)})
\]
with implied constants independent of \( j \) and \( n \).

Due to inequality (2.9) we thus obtain:
\[
\|U - U^{(n)}\|_{\mathcal{F}^{\infty, p}([0, T] \times \Omega; X)}
\leq \sum_{j=1}^{M} \|U(U(x_0, 0, (j - 1)\mathbb{T}_0), (j - 1)\mathbb{T}_0, \cdot)
- U^{(n)}(y_0, 0, (j - 1)\mathbb{T}_0), (j - 1)\mathbb{T}_0, \cdot)\|_{\mathcal{F}^{\infty, p}([0, 1] \times \Omega; X)}
\leq \sum_{j=1}^{M} \|x_0 - y_0\|_{L^p(\Omega; X)} + n^{-\eta}(1 + \|x_0\|_{L^p(\Omega; X)})
\leq \|x_0 - y_0\|_{L^p(\Omega; X)} + n^{-\eta}(1 + \|x_0\|_{L^p(\Omega; X)})
\]
since \( M \) is independent of \( n \). This proves estimate (3.7).

In the next subsection we shall need the following corollary of Theorem 3.1:

**Corollary 3.2.** Let the setting be as in Theorem 3.1. Let \( 0 < \delta < \eta_{\max} \) and \( p \in [2, \infty) \), and assume that \( y_0 \in L^p(\Omega; X_\delta) \). Then for all \( \alpha \in [0, \frac{1}{2}) \) one has:
\[
\sup_{n \in \mathbb{N}} \|U^{(n)}\|_{\mathcal{F}^{\infty, p}([0, T] \times \Omega; X_\delta)} \lesssim 1 + \|y_0\|_{L^p(\Omega; X_\delta)}.
\]

**Proof.** By assumption one can pick \( \varepsilon > 0 \) such that \( \delta + \varepsilon < \eta_{\max} \). Because \( \theta_F > \delta - 1 + (\frac{1}{2} - \frac{1}{2}) + \varepsilon \), the restriction of \( F : [0, T] \times X \to X_{\theta_F} \) to \([0, T] \times X_\delta\) induces a mapping \( F : [0, T] \times X_\delta \to X_{\delta - \frac{1}{2} + \varepsilon} \) which satisfies (F) with \( \tilde{\theta}_F = -1 + (\frac{1}{2} - \frac{1}{2}) + \varepsilon \). Similarly, from \( \theta_G > \delta - \frac{1}{2} + \varepsilon \) we obtain a mapping \( G : [0, T] \times X_\delta \to X_{\delta - \frac{1}{2} + \varepsilon} \) which satisfies (G) with \( \tilde{\theta}_G = -\frac{1}{2} + \varepsilon \). The desired result is now obtained by combining Theorem 3.1 (with state space \( X_\delta \), initial conditions \( x_0 = y_0 \), and exponent \( \eta = 0 \)) and Theorem 2.7.

### 3.2. Convergence of the classical splitting scheme

We consider the stochastic differential equation (SDE) under the assumptions (A), (F), (G), with initial value \( x_0 \), under the additional assumption that \( \theta_F, \theta_G \geq 0 \).
For $n \in \mathbb{N}$ let $(\tilde{U}_j^{(n)})_{j=0}^n$ be defined by (3.3) and set

$$\tilde{U}^{(n)}(t) := \sum_{j=1}^n 1_{j^{(n)}}(t) \tilde{U}_j^{(n)}(t), \quad t \in [0, T).$$

As before, $U$ denotes the mild solution of (SEE) with initial value $x_0$.

**Theorem 3.3.** Let $0 \leq \eta < \eta_{\text{max}}$, and suppose $x_0 \in L^p(\mathcal{F}_0, X_\eta)$ and $\tilde{y}_0 \in L^p(\mathcal{F}_0, X)$ for some $p \in [2, \infty)$. Then for all $\alpha \in \left[0, \frac{1}{2}\right)$ we have:

$$\|U - \tilde{U}^{(n)}\|_{\mathcal{H}^{\alpha, p}([0, T] \times \Omega; X)} \lesssim \|x_0 - \tilde{y}_0\|_{L^p(\Omega; X)} + n^{-\eta}(1 + \|x_0\|_{L^p(\Omega; X)}),$$

with implied constants independent of $n$, $x_0$ and $\tilde{y}_0$.

**Proof.** Fix $T > 0$, $n \in \mathbb{N}$. For $\tilde{U}^{(n)}$ the following relation holds (see also (3.4)):

$$\tilde{U}^{(n)}(s) = S(\sigma)\tilde{y}_0 + \int_0^s S(\sigma - u)F(u, \tilde{U}^{(n)}(u)) \, du$$

$$+ \int_0^s S(\sigma - u)G(u, \tilde{U}^{(n)}(u)) \, dW_H(u).$$

At first sight the processes $\tilde{U}^{(n)}$ and $U^{(n)}$ are very similar, and one would expect the proof of Theorem 3.3 to be entirely analogous to the proof of Theorem 3.1. However, there is a subtle difficulty when considering $\tilde{U}^{(n)}$: for the proof of Theorem 3.1 we make use of the fact that $\sigma - u \geq s - u$ for all $0 \leq u \leq s$, $s \in [0, T]$. This allows us to write

$$S(\sigma - u) = S(\sigma - s)S(s - u)S(u - u)$$

and (see (3.11)):

$$S(s - u) - S(\sigma - u) = (I - S(\sigma - s))S(s - u) + S(\sigma - s)S(s - u)(I - S(u - u)),\$$

As a result, we can interpret the (deterministic and stochastic) integral terms in (3.8) as (stochastic) convolutions and use Lemmas B.1 and B.2 to obtain estimates for these terms.

For $\tilde{U}^{(n)}$ one of the difficulties lies in the fact that for $s \in [0, T] \setminus \{t_j^{(n)} : j = 1, \ldots, n\}$ fixed we have $\sigma - u > s - u$ for some values of $u \in [0, s]$, but $\sigma - u < s - u$ for other values of $u \in [0, s]$. Instead of (3.24) we have, for $s - u \geq \frac{\sigma - u}{n}$:

$$S(\sigma - u) = S(\sigma - s)S(s - u)S(u - u).$$

Roughly speaking, this allows one to apply Lemmas B.1 and B.2 on the interval $[0, (s - \frac{\sigma - u}{n})]$. However, an extra argument is needed for the remainder of the interval.

Another difficulty in dealing with $\tilde{U}^{(n)}$ is that for $s \in [0, T] \setminus \{t_j^{(n)} : j = 1, \ldots, n\}$ and $u \in [\sigma, s]$ we have $S(\sigma - u) = I$, and thus we cannot use the smoothing property of the semigroup there. Note that this occurs precisely in the aforementioned remainder.

**Part 1.** It is easier to deal with the remainder if we compare $\tilde{U}^{(n)}$ with $U^{(n)}$ instead of comparing $\tilde{U}^{(n)}$ with $U$: by Theorem 3.1 suffices to prove that

$$\|U^{(n)} - \tilde{U}^{(n)}\|_{\mathcal{H}^{\alpha, p}([0, T] \times \Omega; X)} \lesssim \|y_0 - \tilde{y}_0\|_{L^p(\Omega; X)} + n^{-\eta}(1 + \|y_0\|_{L^p(\Omega; X)}),$$

where $\eta_{\text{max}}$ is defined in (3.9) and $F$ and $G$ are the functions defined in (3.2) and (3.3), respectively.
with implied constants independent of $n$, $y_0$ and $\tilde{y}_0$, where $U^{(n)}$ denotes the process obtained by applying the modified splitting scheme to (1.1) with initial value $y_0 \in L^p(\Omega, \mathcal{F}_0; X_\eta)$.

By (3.4) and (3.23) we have:

$$U^{(n)}(s) - \tilde{U}^{(n)}(s) = S(\eta)(y_0 - \tilde{y}_0)$$

$$\quad + (S(\frac{T}{n}) - I) \int_0^s S(\tilde{x} - u) F(u, U^{(n)}(u)) \, du$$

$$\quad + \int_0^s S(\tilde{x} - u) [F(u, U^{(n)}(u)) - F(u, \tilde{U}^{(n)}(u))] \, du$$

$$\quad + (S(\frac{T}{n}) - I) \int_0^s S(\tilde{x} - u) G(u, U^{(n)}(u)) \, dW_H(u)$$

$$\quad + \int_0^s S(\tilde{x} - u) [G(u, U^{(n)}(u)) - G(u, \tilde{U}^{(n)}(u))] \, dW_H(u).$$

As mentioned above, we can rewrite each of the (deterministic and stochastic) integrals above as a (deterministic or stochastic) convolution and a remainder term. Below, we will demonstrate this for the first deterministic integral term in (3.26). The convolutions can be dealt with in the same manner as in the proof of Theorem 3.1, and in Part 2 of this proof we will demonstrate how to deal with the remainder.

For the first deterministic integral in (3.26) we have, by (3.25):

$$(S(\frac{T}{n}) - I) \int_0^s S(\tilde{x} - u) F(u, U^{(n)}(u)) \, du$$

$$= (S(\frac{T}{n}) - I) S(\eta - s) \int_0^{(s - \frac{T}{n})^+} S((s - \frac{T}{n})^+ - u) S(u - y) F(u, U^{(n)}(u)) \, du$$

$$\quad + (S(\frac{T}{n}) - I) \int_{(s - \frac{T}{n})^+}^s S(\tilde{x} - u) F(u, U^{(n)}(u)) \, du$$

Note that the first term on the right-hand side above involves the convolution of the process

$$u \mapsto S(u - y) F(u, U^{(n)}(u))$$

with the semigroup $S$, evaluated in $(s - \frac{T}{n})^+$. By arguments analogous to Part 1c in the proof of Theorem 3.1 we can estimate this term, using Corollary 3.2 where in Part 1c the estimate of Theorem 2.7 is applied:

$$\left\| s \mapsto (S(\frac{T}{n}) - I) S(\eta - s) \right\|_{\mathcal{F}_\infty^\infty; [0, T_0] \times \Omega; X_\eta}$$

$$\leq n^{-\eta}(1 + \|y_0\|_{L^p(\Omega; X_\eta)}).$$

For the remainder term we apply, for the time being, only the ideal property for $\gamma$-radonifying operators (2.1) to get rid of the term $S(\frac{T}{n}) - I$. We thus obtain, for all $T_0 \in [0, T]$:

$$\left\| s \mapsto (S(\frac{T}{n}) - I) \int_0^s S(\tilde{x} - u) F(u, U^{(n)}(u)) \, du \right\|_{\mathcal{F}_\infty^\infty; [0, T_0] \times \Omega; X_\eta}$$

$$\leq n^{-\eta}(1 + \|y_0\|_{L^p(\Omega; X_\eta)})$$
\[ + n^{-\theta_F}(1) \left\| s \mapsto \int_{(s-t_0)^+} S(s-u)F(u, U^n(u)) \, du \right\|_{\mathcal{Y}^{\alpha,p}([0,T_0] \times \Omega; X_{s_0})}. \]

By applying similar arguments to the other three integral terms in (3.26) and by applying the argument of Part 1b in the proof of Theorem 3.1 to the first term in (3.26), one obtains that there exists an \( \varepsilon_0 \in (0, \frac{1}{2}) \) such that for \( T_0 \in [0,T] \) and \( \alpha \) sufficiently large we have, setting \( I_s := [(s-t_0)^+, s] \):

\[ \left\| \| U(n) - \bar{U}^n(\cdot) \|_{\mathcal{Y}^{\alpha,p}([0,T] \times \Omega; X)} \right\| \lesssim \left\| y_0 - \bar{y}_0 \right\|_{L^p(\Omega; X)} + n^{-\eta}(1 + \left\| y_0 \right\|_{L^p(\Omega; X_\delta)}) + T_0^n \left\| U(n) - \bar{U}^n(\cdot) \right\|_{\mathcal{Y}^{\alpha,p}([0,T_0] \times \Omega; X)} \]

(i) \[ + n^{-\theta_F}(1) \left\| s \mapsto \int_{I_s} S(s-u)F(u, U^n(u)) \, du \right\|_{\mathcal{Y}^{\alpha,p}([0,T_0] \times \Omega; X_{s_0})} \]

(ii) \[ + \left\| s \mapsto \int_{I_s} S(s-u) \left[ F(u, U^n(u)) - F(u, \bar{U}^n(u)) \right] \, du \right\|_{\mathcal{Y}^{\alpha,p}([0,T_0] \times \Omega; X)} \]

(iii) \[ + n^{-\theta_G}(1) \left\| s \mapsto \int_{I_s} S(s-u)G(u, U^n(u)) \, du \right\|_{\mathcal{Y}^{\alpha,p}([0,T_0] \times \Omega; X_{s_0})} \]

(iv) \[ + \left\| s \mapsto \int_{I_s} S(s-u) \left[ G(u, U^n(u)) - G(u, \bar{U}^n(u)) \right] \, du \right\|_{\mathcal{Y}^{\alpha,p}([0,T_0] \times \Omega; X)} \]

Part 2. We now demonstrate how (i) - (iv) can be estimated using the following two claims. The proofs of the claims are postponed to Parts 3 and 4.

Claim 1. Let \( \delta \in \mathbb{R}, \alpha \in [0, \frac{1}{2}) \) and \( \Phi \in \mathcal{Y}^{\alpha,p}([0,T] \times \Omega; X_\delta) \). Then for all \( \varepsilon > 0 \) and all \( T_0 \in [0,T] \) we have:

\[ \left\| \Phi \right\|_{\mathcal{Y}^{\alpha,p}([0,T_0] \times \Omega; X_\delta)} \lesssim \left( \frac{T}{T_0} \wedge T_0 \right)^{\frac{3}{2} - \frac{1}{2} - \varepsilon} \left\| \Phi \right\|_{L^\infty([0,T_0]; L^p(\Omega, X_\delta))}, \]

with implied constant independent of \( n \) and \( T_0 \).

Claim 2. Let \( \delta \in \mathbb{R}, \alpha \in [0, \frac{1}{2}) \), and \( \Phi \in \mathcal{Y}^{\alpha,p}([0,T] \times \Omega; X_\delta) \). Then for all \( T_0 \in [0,T] \) we have:

\[ \left\| \Phi \right\|_{\mathcal{Y}^{\alpha,p}([0,T_0] \times \Omega; X_\delta)} \lesssim \left( \frac{T}{T_0} \wedge T_0 \right)^{\alpha} \sup_{0 \leq t \leq T_0} \left\| (t-s)^{-\alpha} \Phi(s) \right\|_{L^\infty([0,t]; L^p(\Omega, X_\delta))}, \]

with implied constant independent of \( n \) and \( T_0 \).

Pick \( \varepsilon > 0 \) such that

\[ \varepsilon < \frac{3}{2} - \frac{1}{2} + \theta_F - \eta. \]

We shall apply Claim 1 with this choice of \( \varepsilon \). To be precise, for (i) we apply Claim 1 with \( \Phi = F(\cdot, U^n(\cdot)) \) and \( \delta = \theta_F \). For (ii) we apply Claim 1 with \( \Phi = F(\cdot, U^n(\cdot)) - F(\cdot, \bar{U}^n(\cdot)) \) and \( \delta = \theta_F \). For (iii) we apply Claim 1 with \( \Phi = G(\cdot, U^n(\cdot)) - G(\cdot, \bar{U}^n(\cdot)) \) and \( \delta = \theta_G \). For (iv) we apply Claim 1 with \( \Phi = G(\cdot, U^n(\cdot)) - G(\cdot, \bar{U}^n(\cdot)) \) and \( \delta = \theta_G \).
Replacing $\alpha \in [0, \frac{1}{2})$ by a larger value if necessary, we may assume $\eta - \theta_G < \alpha < \frac{1}{2}$. For (iii) we apply Claim 2 with $\Phi = G(\cdot, U^{(n)}(\cdot))$ and $\delta = \theta_G$. Finally, for (iv) we apply Claim 2 with $\Phi = G(\cdot, U^{(n)}(\cdot)) - G(\cdot, \tilde{U}^{(n)}(\cdot))$ and $\delta = 0$. This gives:

\[
\|U^{(n)} - \tilde{U}^{(n)}\|_{\gamma^\alpha_p([0,T_0] \times \Omega; X)} \lesssim \|y_0 - \tilde{y}_0\|_{L^p(\Omega; X)} + n^{-\eta}(1 + \|y_0\|_{L^p(\Omega; X_M)}) + T_0^{\delta_0} \|U^{(n)} - \tilde{U}^{(n)}\|_{\gamma^\alpha_p([0,T_0] \times \Omega; X)} \\
+ n^{-\eta}\sup_{0 \leq t \leq T_0} \|s \mapsto (t - s)^{-\alpha}G(u, U^{(n)}(u))\|_{L^p(\Omega, \gamma(0,t; X_M))} \\
+ T_0^{\delta_0 - \frac{1}{2} - \varepsilon} \sup_{0 \leq t \leq T_0} \|s \mapsto (t - s)^{-\alpha}[G(u, U^{(n)}(u)) - G(u, \tilde{U}^{(n)}(u))]\|_{L^p(\Omega, \gamma(0,t; X))}.
\]

Note that, as $\theta_F, \theta_G \geq 0$, we have continuous inclusions $X_{\theta_F} \hookrightarrow X$ and $X_{\theta_G} \hookrightarrow X$, so that the norms in $L^p(\Omega; X)$ and $\gamma(0,t; X)$ may be estimated by the norms in $L^p(\Omega; X_{\theta_F})$ and $\gamma(0,t; X_{\theta_G})$ in the third and fifth line, respectively. Applying assumption (F) and the estimates (2.11) and (2.10), then Corollary 3.2 (with $\delta = 0$), we obtain that there exists an $\bar{\varepsilon}_0 > 0$ such that:

\[
\|U^{(n)} - \tilde{U}^{(n)}\|_{\gamma^\alpha_p([0,T_0] \times \Omega; X)} \lesssim \|y_0 - \tilde{y}_0\|_{L^p(\Omega; X)} + n^{-\eta}(1 + \|y_0\|_{L^p(\Omega; X_M)}) + T_0^{\delta_0} \|U^{(n)} - \tilde{U}^{(n)}\|_{\gamma^\alpha_p([0,T_0] \times \Omega; X)}.
\]

The remainder of the proof is entirely analogous to Parts 3 and 4 in the proof of Theorem 3.1.

**Part 3: Proof of Claim 1.** Fix $\varepsilon > 0$. Recall that $I_s = [(s - \frac{\varepsilon}{n})^+, s]$. Observe that for $s \in [0, T_0]$ we have:

\[
\int_{I_s} S(s - u)\Phi(u) du = S(\frac{T}{n}) \int_{(s - \frac{\varepsilon}{n})^+}^s \Phi(u) du + \int_{\frac{T}{n}}^s \Phi(u) du.
\]

Let $a, b : [0, T] \to \mathbb{R}$ be measurable and satisfy $a \leq b$. We shall prove that:

\[
\|s \mapsto \int_{I_s} 1_{\{a(s) \leq u \leq b(s)\}} \Phi(u) du\|_{\gamma^\alpha_p([0,T_0] \times \Omega; X_s)} \lesssim \left(\frac{T}{n} \wedge T_0\right)^{\frac{1}{2} - \frac{1}{2} - \varepsilon} \|\Phi\|_{L^\infty(0,T_0; L^p(\Omega; X_s))}.
\]

The claim follows by applying the above estimate with $a(s) = (s - \frac{T}{n})^+$; $b(s) = s$ to the first term in (3.27), and with $a(s) = \frac{T}{n}$; $b(s) = s$ to the second term. (The term $S(\frac{T}{n})$ can be estimated away by (2.1).)

For $s \in [0, T_0]$ we have $|I_s| = s - (s - \frac{T}{n})^+ \leq \frac{T}{n} \wedge T_0$, and thus:

\[
\|s \mapsto \int_{I_s} 1_{\{a(s) \leq u \leq b(s)\}} \Phi(u) du\|_{L^\infty(0,T_0; L^p(\Omega; X_s))} \lesssim \left(\frac{T}{n} \wedge T_0\right) \|\Phi\|_{L^\infty(0,T_0; L^p(\Omega; X_s))}.
\]

For the estimate in the $\gamma$-radonifying norm we shall use the Besov embedding of Section 2.2 and Lemma A.3 in the Appendix. We begin with a simple observation. If $\Psi \in L^\infty(0,T_0; Y)$ for some Banach space $Y$ and $\alpha \in (0,1]$, then:

\[
\|s \mapsto \int_{I_s} \Psi(u) du\|_{C^\alpha(0,T_0; Y)}
\]
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There exists an ε0 > 0 such that if

\[ \text{If } t-s \geq \frac{T}{n}, \text{ then:} \]

\[ (t-s)^{-\alpha} \left\| \int_{I_s} \Psi(u) du - \int_{I_t} \Psi(u) du \right\|_{Y} \leq 2(t-s)^{-\alpha} \sup_{s \in [0,T_0]} \left| I_s \right| \left\| \Psi \right\|_{L^\infty(0,T_0;Y)} \leq 2\left( \frac{T}{n} \wedge T_0 \right)^{1-\alpha} \left\| \Psi \right\|_{L^\infty(0,T_0;Y)}. \]

On the other hand, if \( t-s \leq \frac{T}{n} \), then:

\[ (t-s)^{-\alpha} \left( \left\| \int_{I_t} \Psi(u) du - \int_{I_s} \Psi(u) du \right\|_{Y} \right) \leq (t-s)^{-\alpha} \left( \left\| \int_{s}^{t} \Psi(u) du \right\|_{Y} + \left\| \int_{(s-\frac{T}{n})^+}^{(t-\frac{T}{n})^+} \Psi(u) du \right\|_{Y} \right) \leq 2(t-s)^{-\alpha} \left\| \Psi \right\|_{L^\infty(0,T_0;Y)} \leq 2\left( \frac{T}{n} \wedge T_0 \right)^{1-\alpha} \left\| \Psi \right\|_{L^\infty(0,T_0;Y)}. \]

It follows that:

(3.28) \[ \left\| s \mapsto \int_{I_s} \Psi(u) du \right\|_{C^0(0,T_0;Y)} \leq 3\left( \frac{T}{n} \wedge T_0 \right)^{1-\alpha} \left\| \Psi \right\|_{L^\infty(0,T_0;Y)}. \]

Note that as \( p \geq 2 \) the type of \( L^p(\Omega, X) \) is the same as the type \( \tau \) of \( X \). Without loss of generality we may assume that \( \tau < 2 \). Fix \( q \geq 2 \) such that \( \frac{1}{q} < \frac{1}{\tau} - \alpha \). By isomorphism (2.2), the Besov embedding (2.5), and Lemma A.3 there exists an \( \varepsilon_0 > 0 \) such that we have:

\[ \sup_{t \in [0,T_0]} \left\| s \mapsto (t-s)^{-\alpha} \int_{I_s} 1_{\{a(s) \leq u \leq b(s)\}} \Phi(u) du \right\|_{L^p(\Omega; C(\gamma(t,t_s)))} \approx \sup_{t \in [0,T_0]} \left\| s \mapsto (t-s)^{-\alpha} \int_{I_s} 1_{\{a(s) \leq u \leq b(s)\}} \Phi(u) du \right\|_{L^p(\Omega; \gamma(t,t_s))} \leq \sup_{t \in [0,T_0]} \left\| s \mapsto (t-s)^{-\alpha} \int_{I_s} 1_{\{a(s) \leq u \leq b(s)\}} \Phi(u) du \right\|_{B_{p,\alpha}^{\frac{1}{2}-\frac{1}{\tau}}(0,T_0;L^p(\Omega;X_s))} \approx T_0^\alpha \left\| s \mapsto \int_{I_s} 1_{\{a(s) \leq u \leq b(s)\}} \Phi(u) du \right\|_{L^\infty(0,T_0;L^p(\Omega;X_s))} + T_0^\alpha \left\| s \mapsto \int_{I_s} 1_{\{a(s) \leq u \leq b(s)\}} \Phi(u) du \right\|_{B_{p,\alpha}^{\frac{1}{2}-\frac{1}{\tau}}(0,T_0;L^p(\Omega;X_s))} \lesssim T_0^\alpha \left\| s \mapsto \int_{I_s} 1_{\{a(s) \leq u \leq b(s)\}} \Phi(u) du \right\|_{C^{\frac{1}{2}-\frac{1}{\tau}}(0,T_0;L^p(\Omega;X_s))} \lesssim \left( \frac{T}{n} \wedge T_0 \right)^{2-\frac{2+\tau}{\tau}} \left\| \Phi \right\|_{L^\infty(0,T_0;X_s)}, \]

where in the final estimate we used (3.28).
Part 4: Proof of Claim 2. As before let \(a, b : [0, T] \to \mathbb{R}\) be measurable and satisfy \(a \leq b\). It suffices to prove that:

\[
\begin{align*}
\|s \mapsto \int_{I_s} \Phi(u) dW_H(u)\|_{L^p([0, T] \times \Omega; X_\delta)} \\
&\leq (\mathcal{T}_n \wedge T_0)^\alpha \|\Phi\|_{L^p([0, T] \times \Omega; X_\delta)}.
\end{align*}
\]

Note that for any \(s \in [0, T_0]\) we have:

\[
\begin{align*}
\|\int_{I_s} \Phi(u) dW_H(u)\|_{L^p([0, T] \times \Omega; X_\delta)} \\
&= \left\| \int_0^s \Phi(u) dW_H(u) - \int_0^{(s - \frac{T}{n})^+} \Phi(u) dW_H(u) \right\|_{L^p([0, T] \times \Omega; X_\delta)} \\
&\leq (s - (s - \frac{T}{n})^+)^\alpha \|s \mapsto \int_0^s \Phi(u) dW_H(u)\|_{C^0([0, T] \times \Omega; L^p([0, T] \times \Omega; X_\delta))}.
\end{align*}
\]

Thus by (2.4) we have:

\[
\begin{align*}
\|s \mapsto \int_{I_s} \Phi(u) dW_H(u)\|_{L^p([0, T] \times \Omega; X_\delta)} \\
&\leq \sup_{0 \leq r \leq T_0} |I_r| \|s \mapsto \int_0^s \Phi(u) dW_H(u)\|_{C^0([0, T] \times \Omega; L^p([0, T] \times \Omega; X_\delta))} \\
&\leq (\mathcal{T}_n \wedge T_0)^\alpha \sup_{0 \leq t \leq T_0} \|s \mapsto (t - s)^{-\alpha} \Phi(s)\|_{L^p([0, T] \times \Omega; H, X_\delta)}.
\end{align*}
\]

Let \(t \in [0, T_0]\). We wish to apply Lemma A.2 with

\[f(r, u)(s) = (t - s)^{-\alpha}(t - u)^\alpha 1_{\{(s - \frac{T}{n})^+ \leq u \leq s\}} 1_{\{a \leq u \leq b\}},\]

\(R = [0, 1]\), \(S = [0, t]\) (both with the Lebesgue measure), \(X_1 = X_2 = X_\delta\), \(\Phi_2 \equiv I\) and \(\Phi_1(u) = (t - u)^{-\alpha} \Phi(u)\). Note that:

\[
\sup_{(r, u) \in [0, 1] \times [0, t]} \|f(r, u)\|_{L^2([0, t])} \leq \sup_{u \in [0, t]} (t - u)^\alpha \|s \mapsto (t - s)^{-\alpha} 1_{\{(s - \frac{T}{n})^+ \leq u \leq s\}}\|_{L^2([0, t])}^2
\leq \sup_{u \in [0, t]} (t - u)^{2\alpha} \int_u^{(u + \frac{T}{n})^+} (t - s)^{-2\alpha} ds.
\]

If \(t - u \geq \frac{2T}{n}\), then \(t - (u + \frac{T}{n}) \geq \frac{1}{2}(t - u)\) and

\[
\int_u^{(u + \frac{T}{n})^+} (t - s)^{-2\alpha} ds \leq \frac{T}{n}(t - (u + \frac{T}{n}))^{-2\alpha} \leq 2^{2\alpha} T_n(t - u)^{-2\alpha},
\]

while if \(t - u \leq \frac{2T}{n}\), then

\[
\int_u^{(u + \frac{T}{n})^+} (t - s)^{-2\alpha} ds \leq \int_u^t (t - s)^{-2\alpha} ds = \frac{1}{1 - 2\alpha} (t - u)^{1 - 2\alpha} \leq \frac{1}{1 - 2\alpha} 2T_n(t - u)^{-2\alpha}.
\]

In both cases, we also have the estimate

\[
\int_u^{(u + \frac{T}{n})^+} (t - s)^{-2\alpha} ds \leq \int_u^t (t - s)^{-2\alpha} ds \approx (t - u)^{1 - 2\alpha} \leq T_0(t - u)^{-2\alpha}.
\]

Combining this with the previous estimates we find:

\[
\sup_{(r, u) \in [0, 1] \times [0, t]} \|f(r, u)\|_{L^2([0, t])} \lesssim (\frac{T}{n} \wedge T_0)^{\frac{\alpha}{2}}.
\]
Thus Lemma A.2 gives:
\[
\left\| s \mapsto (t - s)^{-\alpha} \int_0^s 1_{\{a(s) \leq u \leq b(s)\}} \Phi(u) dW_H(u) \right\|_{L^p(\Omega; (0,T; X))} \\
\lesssim \left( \frac{t}{n} \wedge T_0 \right)^\alpha \| u \mapsto (t - u)^{-\alpha} \Phi(u) \|_{L^p(\Omega; (0,T; H; X))}.
\]
Taking the supremum over \( t \in [0,T_0] \) above and combining the result with (3.30) we obtain (3.29). This completes the proof of the claim. \( \square \)

4. Approximating semigroup operators

In this section we prove a \( \gamma \)-boundedness result for families of operators defined in terms of the so-called Hille-Phillips functional calculus of an operator \( A \) that generates an analytic \( C_0 \)-semigroup on a Banach space \( X \). This result will be used in the next section, where we prove an abstract convergence result for time discretization schemes of (SEE).

Let \((\mu_n)_{n \geq N}\) be a sequence of non-negative finite measures on \([0,\infty)\) and let \( R \geq 0 \) be given. For \( j \in \mathbb{N} \) let \( \mu_n^* = \mu_n \ast \cdots \ast \mu_n \) denote the \( j \)-fold convolution product of \( \mu_n \) with itself. Consider the following properties:

(M1) For all \( n \geq 1 \) we have:
\[
\int_0^\infty t \, d\mu_n(t) = \frac{1}{n}.
\]

(M2) There exists an \( N \geq 1 \) such that for all \( n \geq N \), all \( j = 1, \ldots, n \), and every \( \alpha \in (-1,1] \) we have:
\[
\int_0^\infty t^\alpha e^{Rt} \, d\mu_n^*(t) < \infty;
\]

(M3) For every \( \alpha \in (-1,1] \) we have:
\[
\sup_{n \geq N} \sup_{1 \leq j \leq n} \left| j \int_0^\infty \left[ 1 - \left( \frac{tn}{j} \right)^\alpha \right] e^{Rt} \, d\mu_n^*(t) \right| < \infty.
\]

Let \( A \) be the generator of an analytic \( C_0 \)-semigroup \( S \) on \( X \) and let \( \omega \geq 0 \) be such that \( (e^{-tS(t)})_{t \geq 0} \) is uniformly bounded. Fix \( T > 0 \). Let \((\mu_n)_{n \geq N}\) be a sequence of non-negative \( \sigma \)-finite measures on \([0,\infty)\) that satisfy (M1), (M2), (M3) for \( R = \omega T \). Let \( N \) be as in (M2). For \( n \geq N \) define \( E(h_n) \in \mathcal{L}(X) \) by

\[
E\left( \frac{T}{n} \right) x := \int_0^\infty S(t) x \, d\mu_n(t/T), \quad x \in X,
\]

where for \( n \in \mathbb{N} \), \( j \in \mathbb{N} \) we define:
\[
\int_0^\infty f(t) \, d\mu_n^*(t/T) := \int_0^\infty f(t) \, d\mu_n^*(t).
\]

By (M2), the right-hand side of (4.1) is well defined as a Bochner integral in \( X \). It is an easy consequence of the semigroup property that, for all \( j \geq 1 \),

\[
E((h_n)^j) x := [E(h_n)]^j x = \int_0^\infty S(t) x \, d\mu_n^*(t/T), \quad x \in X.
\]

We supplement these definitions by putting \( E(0) := I \).
Example 4.1. In Section 4.1 below we will demonstrate that the family of measures 
\[ d\mu_n(t) = ne^{-nt} \, dt \]
satisfy (M1), (M2), (M3). For these measures we have 
\[ E(t_j^{(n)})x = (I - \frac{T}{n}A)^{-j}x, \]
which means that \( E(t_j^{(n)}) \) is the \( j \)th Euler approximation of \( S(t_j^{(n)}) \).

The following proposition and corollary give the \( \gamma \)-boundedness estimates for the differences \( E(t_j^{(n)}) - S(t_j^{(n)}) \) that play the same role in the proof of Theorem 5.2 as the estimates of Lemma 2.4 played in the proof of Theorem 3.1.

**Proposition 4.2.** Let the setting be as described above.
(1) For all \( \delta \in (-1, 1] \) there exists a constant \( C \) such that for all \( n \geq N \):
\[ \sup_{j=1, \ldots, n} \left\| (t_j^{(n)})^{1-\delta}[E(t_j^{(n)}) - S(t_j^{(n)})] \right\|_{\mathcal{L}(X;X)} \leq Cn^{-1}. \]
(2) For all \( \delta \in (-1, 1] \), \( 0 \leq \beta \leq 1 - \delta \), and \( \epsilon > 0 \) there exists a constant \( C \) such that for all \( n \geq N \):
\[ \gamma_{\mathcal{L}(X;X)} \{ (t_j^{(n)})^{\beta}[E(t_j^{(n)}) - S(t_j^{(n)})] : j = 1, \ldots, n \} \leq Cn^{-\beta-\delta+\epsilon}. \]

**Remark 4.3.** Stronger \( \gamma \)-boundedness estimates can be obtained by imposing stronger conditions on the measures \( \mu_n \). Such conditions would correspond to using higher-order numerical approximation schemes. However, this will not improve the overall convergence rates as proven in Theorem 5.2 because the bottleneck for convergence rate is the noise approximation.

**Remark 4.4.** The first part of Proposition 4.2, concerning the uniform boundedness, has been known since the 1970’s for the case that \( \delta = 0 \) and \( E(t_j^{(n)}) \) is the Euler approximation. Generally such results are proven by functional calculus methods. Our proof may be read as an extension of the approach taken by Bentkus and Paulauskas [3], which is of more probabilistic nature and seems the most suitable for our needs.

Before turning to the proof of Proposition 4.2, we state a simple corollary.

**Corollary 4.5.** Let the setting be as described above.
(1) For all \( \delta \in (-1, 0] \) there exists a constant \( C \) such that for all \( n \geq N \):
\[ \sup_{j=1, \ldots, n} \left\| (t_j^{(n)})^{-\delta}E(t_j^{(n)}) \right\|_{\mathcal{L}(X;X)} \leq C. \]
(2) For all \( \delta \in (-1, 0] \), \( -\delta < \beta \leq 1 - \delta \), and \( 0 < \epsilon < \beta + \delta \) there exists a constant \( C \) such that for all \( n \geq N \) and all \( k = 1, \ldots, n \) we have
\[ \gamma_{\mathcal{L}(X;X)} \{ (t_j^{(n)})^{\beta}E(t_j^{(n)}) : j = 1, \ldots, k \} \leq C(t_k^{(n)})^{\beta+\delta-\epsilon}. \]

**Proof.** By the first part of Proposition 4.2, for all \( 1 \leq j \leq n \)
\[ \left\| (t_j^{(n)})^{-\delta}(E(t_j^{(n)}) - S(t_j^{(n)})) \right\|_{\mathcal{L}(X;X)} \leq Cn^{-1}(t_j^{(n)})^{-1} \lesssim 1. \]
Moreover, by the analyticity of the semigroup \( S \) (i.e., estimate (2.6)), and the fact that \( \delta \leq 0 \),
\[ \sup_{0 \leq j \leq n} (t_j^{(n)})^{-\delta}S(t_j^{(n)}) \lesssim 1. \]
This proves the first part of the corollary.

By Part (1) of Lemma 2.4 and Part (2) of Proposition 4.2, observing that \( \beta > -\delta \geq 0 \), we have:

\[
\gamma_{[x, X]} \{ (t_j^{(n)})^\beta E(t_j^{(n)}) : j = 1, \ldots, k \} \\
\leq \gamma_{[x, X]} \{ (t_j^{(n)})^\beta S(t_j^{(n)}) : j = 1, \ldots, k \} \\
+ \gamma_{[x, X]} \{ (t_j^{(n)})^\beta (E(t_j^{(n)}) - S(t_j^{(n)})) : j = 1, \ldots, k \} \\
\leq (t_j^{(n)})^{\beta + \delta} + n^{-\beta - \delta + \epsilon} \\
\leq (t_j^{(n)})^{\beta + \delta - \epsilon},
\]

with implied constants independent of \( k \) and \( n \) (although they may depend on \( T \)).

In order to prove Proposition 4.2 we shall make use the following simple observation. Suppose \( \mu \) is a probability measure on \([0, \infty)\), \( t = \int_0^\infty s \, d\mu(s) \), and \( f : [0, \infty) \to X \) is twice continuously differentiable. By integration by parts one has:

\[
(4.3) \quad \int_0^\infty f(s) \, d\mu(s) - f(t) = \int_0^\infty \int_t^s (s-r) f''(r) \, dr \, d\mu(s).
\]

We substitute \( f(s) = S(s)x, x \in X \), and \( \mu = \mu_n^x \) for \( n > N \) and \( j \in \{1, \ldots, n \} \) in the above. From (M1) we have that \( \int_0^\infty s \, d\mu_n^x(s/T) = t_j^{(n)} \). Thus by setting \( t = t_j^{(n)} \) in (4.3) we obtain, for \( x \in X \):

\[
(4.4) \quad E(t_j^{(n)})x - S(t_j^{(n)})x = \int_0^\infty S(s)x \, d\mu_n^x(s/T) - S(t_j^{(n)})x \\
= \int_0^\infty \int_{t_j^{(n)}}^s (s-r) A^2 S(r)x \, dr \, d\mu_n^x(s/T).
\]

Proof of Proposition 4.2. We first prove the statement concerning \( \gamma \)-boundedness. Let \( N \) be as in assumption (M2). Let \( n > N \). Without loss of generality we may assume \( \delta - \epsilon \neq 0 \) and \( \delta - \epsilon \neq -1 \). For \( j = 1, 2, \ldots, n \) define \( \phi_j : [0, \infty) \times [0, \infty) \to \mathbb{R} \) by

\[
\phi_j(s, r) := (t_j^{(n)})^\beta r^{-2+\delta-\epsilon}(s-r)e^{\omega r}(1_{\{t_j^{(n)} \leq r \leq s\}} - 1_{s \leq r \leq t_j^{(n)}}).
\]

By equality (4.4) we have, for \( x \in X \):

\[
(4.5) \quad (t_j^{(n)})^\beta [E(t_j^{(n)})x - S(t_j^{(n)})x] \\
= \int_0^\infty \int_0^\infty \phi_j(s, r)r^{2-\delta+\epsilon} e^{-\omega r} A^2 S(r)x \, dr \, d\mu_n^x(s/T),
\]

\( j = 1, 2, \ldots, n \).

In a similar fashion as used for Lemma 2.4 in Section 2.3 one may prove that for \( \delta \leq 2 \) the set

\[
\{ r^{2-\delta+\epsilon} e^{-\omega r} A^2 S(r) : r \in [0, \infty) \}
\]
is $\gamma$-bounded in $\mathcal{L}(X_\delta; X)$. By [36, Proposition 2.5] and equation (4.5) it follows that
\[
\gamma_{n,\delta, X} \left\{ (t^{(n)}_j)^{\beta} E(t^{(n)}_j) - S(t^{(n)}_j) \right\} : j = 1, \ldots, n \right\} \lesssim \sup_{1 \leq j \leq n} \| \phi_j \|_{L^1([0, \infty) \times [0, \infty), \mu^{\ast}_n(\cdot/T) \times \lambda)},
\]
with implied constant independent of $n$, where $\lambda$ is the Lebesgue measure on $[0, \infty)$.

Observe that for all $j, \delta \geq 0$, where $\omega$ is independent of $n$ and $\epsilon > 0$.
\[
\| \phi_j \|_{L^1([0, \infty) \times [0, \infty), \mu^{\ast}_n(\cdot/T) \times \lambda)} = (t^{(n)}_j)^{\beta} \int_0^\infty \int_{s \in [0, \infty)} r^{-2+\delta-\epsilon} (s-r) e^{\omega r} dr d\mu^{\ast}_n(s/T)
\]
\[
\lesssim (t^{(n)}_j)^{\beta} \int_0^\infty e^{(\omega^\ast T) s} \int_{s \in [0, \infty)} r^{-2+\delta-\epsilon} (s-r) dr d\mu^{\ast}_n(s/T).
\]

As $\delta - \epsilon \neq 0$ and $\delta - \epsilon \neq -1$, basic calculus gives:
\[
(4.7) \int_{s \in [0, \infty)} r^{-2+\delta-\epsilon} (s-r) dr = \frac{(t^{(n)}_j)^{\beta-\epsilon}}{1 - \delta + \epsilon} \left[ \frac{1}{\delta - \epsilon} \left( 1 - \left( \frac{s}{t^{(n)}_j} \right)^{\delta-\epsilon} \right) + \frac{s}{t^{(n)}_j} - 1 \right].
\]

In the final estimate below we apply assumption (M3) (recall that we have $R = \omega(T)$). Due to that assumption there exists a constant $C$ independent of $n$ and $j \in \{1, \ldots, n\}$ such that:
\[
(4.8) \| \phi_j \|_{L^1([0, \infty) \times [0, \infty), \mu^{\ast}_n(\cdot/T) \times \lambda)} \lesssim \frac{C}{t^{(n)}_j} \int_0^\infty e^{(\omega^\ast T) s} \int_{s \in [0, \infty)} r^{-2+\delta-\epsilon} (s-r) dr d\mu^{\ast}_n(s/T)
\]
\[
\lesssim (t^{(n)}_j)^{\beta-\delta+\epsilon-j} - 1 \lesssim n^{-\beta-\delta+\epsilon}.
\]

This in combination with estimate (4.6) completes the proof, as $\beta + \delta - \epsilon < 1$.

As for the statement concerning uniform boundedness, by (4.4) and analyticity (estimate (2.6)) we have, for $\delta 
eq 0$:
\[
(\hat{t}^{(n)}_j)^{1-\delta} \| E(\hat{t}^{(n)}_j) - S(\hat{t}^{(n)}_j) \|_{\mathcal{L}(X_\delta; X)} \lesssim (\hat{t}^{(n)}_j)^{1-\delta} \int_0^\infty \int_{s \in [0, \infty)} r^{-2+\delta} (s-r) A^{2-\delta} S(r) dr d\mu^{\ast}_n(s/T)
\]
\[
\lesssim (\hat{t}^{(n)}_j)^{1-\delta} \int_0^\infty \int_{s \in [0, \infty)} r^{-2+\delta} (s-r) e^{\omega r} dr d\mu^{\ast}_n(s/T)
\]
\[
\lesssim n^{-1},
\]
where the final estimate follows by similar arguments as used to obtain (4.8). In the case that $\delta = 0$ or $\delta = 1$ the evaluation of the integral in (4.7) will contain a logarithmic term, which can be estimated in a suitable manner by observing that $\ln x \leq x - 1$ for all $x > 0$. We leave the details to the reader.
4.1. Examples. We have two main examples in mind, which lead to a splitting scheme with discretized noise and the implicit Euler scheme, respectively.

Example 4.6 (Splitting with discretized noise). The simplest example obtained by taking
\[ \mu_n := \delta_n, \]
which correspond to the trivial choice
\[ E(T_n) := S(T_n). \]
The conditions (M1), (M2), (M3) are trivially fulfilled for any \( R \geq 0. \)

Example 4.7 (Implicit Euler). We will show that the measures
\[ d\mu_n(t) = ne^{-nt} dt \]
fulfill assumptions (M1), (M2), (M3) for any \( R \geq 0. \) These measures give rise to the operators
\[ E(T_n) = (I - T_n A)^{-1}. \]
To start the proof, first note that by induction,
\[ d\mu^*_j n(t) = \left( \frac{nt}{j-1} \right) e^{-nt} dt, \]
and thus, for all \( \alpha > -j \) and all \( n > \omega T \), one has:
\[ \int_0^\infty t^\alpha e^{\omega T t} d\mu^* j n(t) = \frac{n^j}{(j-1)!} \int_0^\infty u^{j-1} e^{-u} du = \frac{n^j}{(n - \omega T)^{j+\alpha}} \frac{\Gamma(j + \alpha)}{\Gamma(j)}. \]
This proves that (M1) and (M2) are satisfied with \( N > \omega T. \)

As for (M3), by (4.9) we have for \( \alpha \in (-1, 1] \) and \( n \geq N \geq 2\omega T: \)
\[ \int_0^\infty \left[ 1 - \left( \frac{n}{n - \omega T} \right)^\alpha \right] e^{\omega T t} d\mu^* j n(t) \]
\[ = \left( \frac{n}{n - \omega T} \right)^j \left( 1 - \left( \frac{n}{n - \omega T} \right)^\alpha \right) + \left( \frac{n}{n - \omega T} \right)^{j+\alpha} \left[ 1 - \frac{\Gamma(j + \alpha)}{j^\alpha \Gamma(j)} \right]. \]
As \( \left( \frac{n}{n - \omega T} \right)^{n+1} \to e^{\omega T} \) as \( n \to \infty, \) there exists a constant \( M \) such that
\[ \sup_{n \in \mathbb{N}} \sup_{s \in [0, n+1]} \left| \frac{n}{n - \omega T} \right|^s \leq M. \]
Moreover, for \( n \geq 2\omega T \) we have:
\[ \left| 1 - \left( \frac{n}{n - \omega T} \right)^\alpha \right| = |\alpha| \int_0^1 \frac{n^{s-1}}{n - \omega T} s^{\alpha-1} \, ds \leq |\alpha| \frac{\omega T}{n - \omega T} \leq 2|\alpha| \frac{\omega T}{n}. \]
From (4.10) and the above estimates we thus obtain:
\[ \int_0^\infty \left| 1 - \left( \frac{n}{n - \omega T} \right)^\alpha \right| e^{\omega T t} d\mu^* j n(t) \leq 2M|\alpha| \frac{\omega T}{n} + M \left[ 1 - \frac{\Gamma(j + \alpha)}{j^\alpha \Gamma(j)} \right]. \]
For \( j \geq 2 \) define \( g_j : [-1, 1] \to \mathbb{R} \) by:
\[ g_j(x) = \begin{cases} \frac{1}{x} \left( 1 - \frac{\Gamma(j + x)}{j^x \Gamma(j)} \right); & x \neq 0, \\ \ln j - \Psi(j); & x = 0, \end{cases} \]
where $\Psi = (\ln(\Gamma))^\prime$ is the di-gamma function.

Assumption (M3) follows from (4.11) for $N \geq 2\omega T$ once the following claim is established:

Claim. For $j \geq 2$ we have $g_j(x) \in [0, \frac{1}{j-1}]$ for all $x \in [-1, 1]$.

Proof of Claim. As $g_j(-1) = \frac{1}{j-1}$ and $g_j(1) = 0$ for all $j \geq 2$, it suffices to show that $g_j$ is non-increasing on $[-1, 1]$.

For $j \geq 2$ define the function $h_j : [-1, 1] \to \mathbb{R}$ by $h_j(x) := 1 - x^2 \frac{\Gamma(j+x)}{\Gamma(j)}$. For $x \in [-1, 1]$ and $j \geq 2$ we have:

$$h_j'(x) = \frac{\Gamma(j+x)}{x^2 \Gamma(j)} [\ln j - \Psi(j+x)] = (1 - h_j(x))(\ln j - \Psi(j+x));$$

$$h_j''(x) = \frac{-\Gamma(j+x)}{x^2 \Gamma(j)} [(\Psi(j+x) - \ln j)^2 + \Psi'(j+x)].$$

As the $\Gamma$-function is log-convex on $(0, \infty)$, we have that $\Psi'$ is positive on that interval. As $j \geq 2$ and $x \in [-1, 1]$ we have that $j + x > 0$ and thus $h_j''(x) \leq 0$ for $x \in [-1, 1]$.

One may check that $g_j$ is continuously differentiable and

$$g_j'(x) = \begin{cases} \frac{1}{x^2} (xh_j'(x) - h_j(x)); & x \neq 0, \\ -\frac{1}{2} (\Psi(j) - \ln j)^2 + \Psi'(j); & x = 0. \end{cases}$$

To prove that $g_j$ is non-increasing on $[-1, 1]$ it suffices to prove that

$$xh_j'(x) - h_j(x) \leq 0, \text{ for all } x \in [-1, 1].$$

Observe that $g_j'(0) \leq 0$, hence it suffices to prove that $x \mapsto xh_j'(x) - h_j(x)$ is non-decreasing on $[-1, 0]$ and non-increasing on $[0, 1]$. This follows from the fact that

$$\frac{d}{dx}[xh_j'(x) - h_j(x)] = xh_j''(x) \text{ and } h_j''(x) \leq 0 \text{ on } [-1, 1].$$

5. **AN ABSTRACT TIME DISCRETIZATION**

In this section we prove a convergence result for a general class of approximation schemes for (SEE) involving the operators $E(t_j^{(h)})$ as defined in (4.1) and discretized noise. In particular, the convergence result contains the implicit Euler scheme as the special case that $E(t_j^{(h)}) = (I - \frac{T}{n} A)^{-j}$ (see Example 4.7).

Throughout this section we consider the problem (SEE) under the assumptions (A), (F), (G). On the part of $X$ we shall assume that it is an UMD space with Pisier’s property ($\alpha$) introduced in [33]. For an extensive discussion of this property and its use in the theory of stochastic evolution equations we refer to [21, 31]. Examples of Banach spaces with property ($\alpha$) are the Hilbert spaces and the spaces $L^p(\mu)$ with $1 \leq p < \infty$ and $\mu$ $\sigma$-finite. Here we need the fact (see [21, 31] for the proof and generalizations) that if $X$ has property ($\alpha$), then for any two measure spaces $(R_1, \mathcal{A}, \mu_1)$ and $(R_2, \mathcal{A}_2, \mu_2)$ and any Hilbert space $H$ we have a natural isomorphism

$$(5.1) \quad \gamma(R_1; \gamma(R_2; H, X)) \simeq \gamma(R_1 \times R_2; H, X),$$

which is given by the mapping $f_1 \otimes ((f_2 \otimes h) \otimes x) \mapsto ((f_1 \otimes f_2) \otimes h) \otimes x$.

Set:

$$(5.2) \quad \zeta_{\text{max}} := \min\{1 - \left(\frac{1}{r} - \frac{1}{2}\right) + (\theta_F \wedge 0), \frac{1}{2} + (\theta_G \wedge 0)\},$$
where $\tau \in (1,2]$ is the type of $X$. In addition to the assumptions (F), (G) we shall assume:

(F') There exists a constant $C$ such that for all $x \in X$ and $s, t \in [0, T]$ we have:

$$\|F(t, x) - F(s, x)\|_{X_{\theta_F \wedge 0}} \leq C|t - s|^{\zeta_{\max}}(1 + \|x\|_X).$$

(G') There exists a constant $C$ such that for all $x \in X$ and $s, t \in [0, T]$ we have:

$$\|G(t, x) - G(s, x)\|_{\gamma(H,X_{\theta_G \wedge 0})} \leq C|t - s|^{\zeta_{\max} + \frac{1}{2} - \frac{1}{2}}(1 + \|x\|_X).$$

Moreover, for every $t \in [0, T]$ there exists a constant $C_t$ such that:

$$\|G(t, x) - G(t, y)\|_{\gamma(H,X_{\theta_G \wedge 0})} \leq C_t\|x - y\|_X.$$ 

Remark 5.1. Clearly, condition (F') is automatically satisfied if $F$ is not time-dependent and satisfies (F).

Condition (G') is also automatically satisfied if $G$ is not time-dependent and satisfies (G), provided we assume $G(0) \in \gamma(H,X_{\theta_G \wedge 0})$. Indeed, in that case, from [30, Lemma 5.3] it follows that $G$ takes values in $\gamma(H,X_{\theta_G \wedge 0})$ and the conditions of (G') concerning Hölder continuity in the first variable and Lipschitz continuity the second are satisfied.

The reader will have noticed that the above assumptions are phrased in terms of $\theta_F \wedge 0$ and $\theta_G \wedge 0$. The reason for this is explained in Remark 5.3 below. Because of this, for the rest of this section, without loss of generality we shall assume that $\theta_F, \theta_G \geq 0$. The other assumptions on $\theta_F$ and $\theta_G$ remain in force. Explicitly, we assume

$$-1 + \left(\frac{1}{2} - \frac{1}{2}\right) < \theta_F \leq 0, \quad -\frac{1}{2} < \theta_G \leq 0.$$ 

Once this convention is in force, of course one has $\zeta_{\max} = \eta_{\max}$. In order to remind the reader of the convention, we shall continue the use of $\zeta_{\max}$.

Let us now introduce the discrete-time approximation scheme that will be studied in this section. Fix $T > 0$ and let $(\mu_n)_{n=1}^{\infty}$ be a family of measures satisfying (M1), (M2), (M3) for $R = \omega T$, where $\omega \geq 0$ is such that $e^{-\omega t}S(t)$ is uniformly bounded in $t \in [0, \infty)$. Let $E(t_j^{(n)})$ be defined by (4.2). We fix $p > 2$ and let $U$ be the mild solution to (SEE) with initial value $x_0 \in L^p(\Omega,\mathcal{F}_0;X)$. We fix another initial value $y_0 \in L^p(\Omega,\mathcal{F}_0;X)$ (in the applications below, the typical situation is that $y_0$ is a close approximation to $x_0$). Let $n \geq N$, where $N$ is as in (M2). Set $V_0^{(n)} := y_0$ and define $V_j^{(n)}$, $j = 1, \ldots, n$, inductively as follows:

$$V_j^{(n)} := E(\frac{T}{n})[V_{j-1}^{(n)} + \frac{T}{n}F(t_{j-1}^{(n)}, V_{j-1}^{(n)}) + G(t_{j-1}^{(n)}, V_{j-1}^{(n)})\Delta W_j^{(n)}].$$

Here,

$$\Delta W_j^{(n)} := W_H(t_j^{(n)}) - W_H(t_{j-1}^{(n)}).$$

The rigorous interpretation of the term $G(t_{j-1}^{(n)}, V_{j-1}^{(n)})\Delta W_j^{(n)}$ proceeds in three steps.

Step 1: Let us first fix an operator $R \in \gamma(H,X_{\theta_G})$. By standard results on $\gamma$-radonifying operators (see, e.g., [27]) may write

$$R \equiv \sum_{k=1}^{\infty} h_k \otimes x_k$$
for some orthonormal sequence \((h_k)_{k=1}^\infty\) in \(H\) and a sequence \((x_k)_{k=1}^\infty\) in \(X_{\theta_G}\) (the convergence of the sum being in \(\gamma(H, X_{\theta_G})\)). For sets \(B \in \mathcal{F}_{j-1}^{(n)} := \mathcal{F}_{j-1}^{(n)}\), we now define

\[
(1_B \otimes R) \Delta W_j^{(n)} := 1_B \sum_{k=1}^\infty W_H(h_k \otimes 1_{(t_j^{(n)}, t_j^{(n)})}) \otimes x_k.
\]

The sum on the right-hand side above converges in \(L^p(\Omega, X_{\theta_G})\) since \(W_H\) extends to a bounded operator from \(\gamma(L^2(0, T; H); X_{\theta_G})\) into \(L^p(\Omega, X_{\theta_G})\) (see [27]). By the independence of \(W_H(h_k \otimes 1_{(t_j^{(n)}, t_j^{(n)})})\) and \(\mathcal{F}_{j-1}^{(n)}\), the product of \(1_B\) and this sum converges in \(L^p(\Omega, X_{\theta_G})\) as well. Moreover, by the Kahane-Khintchine inequality,

\[
\|1_B \otimes R) \Delta W_j^{(n)}\|_{L^p(\Omega, X_{\theta_G})} \
\lesssim \left(\mathbb{E}(1_B)\right)^{\frac{1}{p}} \left\| \sum_{k=1}^\infty (h_k \otimes 1_{(t_j^{(n)}, t_j^{(n)})}) \otimes x_k \right\|_{\gamma(L^2(0, T; H); X_{\theta_G})} \
= (t_j^{(n)} - t_j^{(n-1)})^{\frac{1}{2}} \left(\mathbb{E}(1_B)\right)^{\frac{1}{p}} \left\| \sum_{k=1}^\infty h_k \otimes x_k \right\|_{\gamma(H; X_{\theta_G})} \
= \left(\frac{2}{n}\right)^{\frac{1}{2}} \left(\mathbb{E}(1_B)\right)^{\frac{1}{p}} \|R\|_{\gamma(H; X_{\theta_G})}
\]

with implied constants depending on \(p\) only.

**Step 2:** Now fix a simple random variable \(\phi \in L^p(\Omega, \mathcal{F}_{j-1}^{(n)}; \gamma(H, X_{\theta_G}))\), say \(\phi = \sum_{j=1}^k 1_{B_j} \otimes R_j\) with the sets \(B_j \in \mathcal{F}_{j-1}^{(n)}\) disjoint. By the above,

\[
\|\phi \Delta W_j^{(n)}\|_{L^p(\Omega, X_{\theta_G})} \lesssim \left(\frac{2}{n}\right)^{\frac{1}{2}} \sum_{j=1}^k \left(\mathbb{E}(1_{B_j})\right)^{\frac{1}{p}} \|R_j\|_{\gamma(H; X_{\theta_G})} \
= \left(\frac{2}{n}\right)^{\frac{1}{2}} \|\phi\|_{L^p(\Omega; \gamma(H, X_{\theta_G}))}.
\]

**Step 3:** Let \((v_m)_{m \in \mathbb{N}}\) be a sequence of simple \(X\)-valued random variables approximating \(V_j^{(n)}\) in \(L^p(\Omega, \mathcal{F}_{j-1}^{(n)}; X)\). By the norm estimate (5.4) and the Lipschitz condition in \((G')\) it follows that the sequence \((G(t_j^{(n)}, v_m) \Delta W_j^{(n)})_{m \in \mathbb{N}}\) is Cauchy in \(L^p(\Omega; X_{\theta_G})\). Now we define

\[
G(t_j^{(n)}, V_j^{(n)}) \Delta W_j^{(n)} := \lim_{m \to \infty} G(t_j^{(n)}, v_m) \Delta W_j^{(n)}.
\]

This completes the construction.

Returning to the abstract scheme (5.3), we have the following explicit expression for \(V_j^{(n)}\):

\[
V_j^{(n)} = E(t_j^{(n)} y_0 + \frac{2}{n} \sum_{k=1}^j E(t_j^{(n)} - t_{j-k+1}) F(t_k^{(n)}, V_{k-1}) \\
+ \sum_{k=0}^{j-1} E(t_j^{(n)} - t_{j-k+1}) G(t_k^{(n)}, V_k^{(n)}) \Delta W_k^{(n)}, \quad j = 0, \ldots, n.
\]
Unlike the case in Theorem 5.2, we begin by observing that, due to the assumption 2 < p < \frac{1}{2}, the Hölder conditions of (F) and (G) can be weakened: in order to obtain convergence rate \eta in Theorem 5.2 it suffices that the Hölder exponent in (F') is \eta instead of \zeta_{\text{max}}, and that the exponent in (G') is \eta + \frac{1}{\tau} - \frac{1}{2} instead of \zeta_{\text{max}} + \frac{1}{\tau} - \frac{1}{2}.

Proof of Theorem 5.2. We begin by observing that, due to the assumption 2 < p < \infty, the spaces X and L^p(\Omega; X) has the same type \tau.

Part 1. The main issue is to prove that there exists \tau_0 \in (0, T] and a constant C such that for all \tau \in \mathbb{N} and \tau \in \{0, \ldots, n\} we have:

\begin{align*}
\|U - V^{(n)}(\tau_0)\|_{\mathcal{F}_{\tau_0}^\alpha(\Omega; X)} & \leq C\|U_0 - V_0^{(n)}\|_{L^p(\Omega; X)} + Cn^{-\eta}(1 + \|U(t_j^{(n)})\|_{L^p(\Omega; X)})
\end{align*}

This statement is entirely analogous to the result obtained in Part 3 of the proof of Theorem 3.1. Once it has been established, the extension to the interval [0, T] can be obtained in precisely the same way as in Part 4 of Theorem 3.1.

Until further notice we fix \tau \geq N and \tau_0 \in [0, T].
Let \((U_j^{(n)})_{j=1}^n\) be the modified splitting scheme as defined by (3.3) in Section 3, with initial datum \(x_0\). Let \(U^{(n)}\) be the corresponding process as defined by (3.2) in Section 3. By Theorem 3.1 we have, for all \(j\):

\[
\|U - V^{(n)}\|_{\mathcal{F}_{\alpha,p}([t_j^{(n)}, t_j^{(n)} + \underline{t}_0] \times \Omega; X)} \\
\leq \|U - U_j^{(n)}\|_{\mathcal{F}_{\alpha,p}([t_j^{(n)}, t_j^{(n)} + \underline{t}_0] \times \Omega; X)} + \|U^{(n)} - V^{(n)}\|_{\mathcal{F}_{\alpha,p}([t_j^{(n)}, t_j^{(n)} + \underline{t}_0] \times \Omega; X)} \\
\lesssim n^{-\eta}(1 + \|U(t_j^{(n)})\|_{L^p(\Omega; X_n)}) + ||U^{(n)} - V^{(n)}\|_{\mathcal{F}_{\alpha,p}([t_j^{(n)}, t_j^{(n)} + \underline{t}_0] \times \Omega; X)},
\]

with implied constants independent of \(n\) and \(j\). Thus it suffices to show that

\[
(5.6) \quad \|U^{(n)} - V^{(n)}\|_{\mathcal{F}_{\alpha,p}([t_j^{(n)}, t_j^{(n)} + \underline{t}_0] \times \Omega; X)} \\
\lesssim C\|U(t_j^{(n)}) - V_j^{(n)}\|_{L^p(\Omega; X)} + C\eta^{-\eta}(1 + \|U(t_j^{(n)})\|_{L^p(\Omega; X)}).
\]

**Part 2.** For simplicity we shall prove this for \(j = 0\) (careful examination of the proof reveals that the other \(t_j^{(n)}\) do not generate extra difficulties). In that case we have \(U(t_j^{(n)}) = U(0) = x_0\) and \(V_j^{(n)} = V_0^{(n)} = y_0\).

From the integral representations (3.4) and (5.5) we have, for \(n \geq N\):

\[
(5.7) \quad U^{(n)}(t) - V^{(n)}(t) = S(t) x_0 - E(t) y_0 \\
+ \int_0^t S(t - s) F(s, U^{(n)}(s)) \, ds - \int_0^t E(t - s) F(\underline{u}, V^{(n)}(s)) \, ds \\
+ \int_0^t S(t - s) G(s, U^{(n)}(s)) \, dW_H(s) - \int_0^t E(t - s) G(\underline{u}, V^{(n)}(s)) \, dW_H(s) \\
= (S(t) - E(t)) x_0 + E(t)(x_0 - y_0) + \int_0^t [S(t - s) - E(t - s)] F(s, U^{(n)}(s)) \, ds \\
+ \int_0^t E(t - s) [F(s, U^{(n)}(s)) - F(s, V^{(n)}(s))] \, ds \\
+ \int_0^t E(t - s) [F(s, V^{(n)}(s)) - F(s, V^{(n)}(s))] \, ds \\
+ \int_0^t S(t - s) F(s, U^{(n)}(s)) \, ds \\
+ \int_0^t [S(t - s) - E(t - s)] G(s, U^{(n)}(s)) \, dW_H(s) \\
+ \int_0^t E(t - s) [G(s, U^{(n)}(s)) - G(s, V^{(n)}(s))] \, dW_H(s) \\
+ \int_0^t E(t - s) [G(s, V^{(n)}(s)) - G(s, V^{(n)}(s))] \, dW_H(s) \\
+ \int_0^t S(t - s) G(s, U^{(n)}(s)) \, dW_H(s).
\]
We shall estimate each of the ten terms on the right-hand side above separately. The implied constants in these estimates may depend on $T$, although this will not be stated explicitly. However, for the fourth, fifth, eighth and ninth term (Part 2d and 2g below) it will be necessary to keep track of the dependence upon $T_0$.

Without loss of generality we may assume that $\tau \in (1, 2)$. We fix $0 < \varepsilon < \frac{1}{2}$ such that
\begin{equation}
(5.8) \quad \varepsilon < \max \{ \zeta_{\text{max}} - \eta, 1 - 2\alpha \},
\end{equation}
where $\zeta_{\text{max}}$ is defined as in (5.2). As $\mathcal{Y}_{\infty}^{\alpha, p} \hookrightarrow \mathcal{Y}_{\infty}^{\beta, p}$ for $\alpha > \beta$, we may also assume that
\begin{equation}
\frac{1}{2} - \frac{2}{3}\varepsilon < \alpha < \frac{1}{2} - \frac{1}{2}\varepsilon.
\end{equation}

**Part 2a.** For the first term on the right-hand side of (5.7) we have, by the uniform boundedness estimate of Proposition 4.2 with $\delta = \eta$, pointwise in $\omega \in \Omega$,
\begin{equation}
(5.9) \quad \| s \mapsto (S(s) - E(s))x_0 \|_{L^\infty(0, T_0; X)} \lesssim n^{-1} \sup_{1 \leq j \leq n} \| t_j^{(n)} \|^{1 - \eta} \| x_0 \|_{X_n} \lesssim n^{-\eta} \| x_0 \|_{X_n}.
\end{equation}

Let $t \in [0, T_0]$. By the $\gamma$-boundedness result of Proposition 4.2 with $\beta = \epsilon = \frac{1}{2}\varepsilon$ and $\delta = \eta$, the $\gamma$-multiplier theorem (Theorem 2.2), and (2.2) we have, pointwise in $\omega \in \Omega$:
\begin{equation}
\| s \mapsto (t - s)^{-\alpha}(S(s) - E(s))x_0 \|_{\gamma(0, t; X)} \lesssim n^{-\eta} \| s \mapsto (t - s)^{-\alpha} x_0 \|_{\gamma(0, t; X_n)} \\
= \| s \mapsto (t - s)^{-\alpha} x_0 \|_{L^2(0, t)} \| x_0 \|_{X_n} \lesssim n^{-\eta} \| x_0 \|_{X_n},
\end{equation}
with implied constants independent of $n$, $T_0$ and $x_0$. As $\frac{1}{2} - \alpha - \frac{1}{2}\varepsilon > 0$, we have $t^{\frac{1}{2} - \alpha - \frac{1}{2}\varepsilon} \leq T^{\frac{1}{2} - \alpha - \frac{1}{2}\varepsilon}$. By taking the supremum over $t \in [0, T_0]$ in the above, combining the result with (5.9), and then taking $p$th moments, one obtains:
\begin{equation}
(5.10) \quad \| s \mapsto (S(s) - E(s))x_0 \|_{L^{\infty, p}(0, T_0) \times \Omega; X)} \lesssim n^{-\eta} \| x_0 \|_{L^p(\Omega; X_n)},
\end{equation}
with implied constants independent of $n$, $T_0$, and $x_0$.

**Part 2b.** Concerning the second term on the right-hand side of (5.7) recall that as $A$ generates an analytic $C_0$-semigroup, there exists a constant $M$ such that
\begin{equation}
\sup_{n \geq N} \sup_{k \in \{1, \ldots, n\}} \| E(t_k^{(n)}) \|_{\mathcal{L}(X)} \leq M.
\end{equation}
Thus pointwise in $\omega \in \Omega$ we have:
\begin{equation}
\| s \mapsto E(s)(x_0 - y_0) \|_{L^\infty(0, T_0; X)} \lesssim \| x_0 - y_0 \| X.
\end{equation}

Let $t \in [0, T_0]$. We apply the second part of Corollary 4.5 with $\beta = \varepsilon$, $\delta = 0$, $\epsilon = \frac{1}{2}\varepsilon$, $k = n$. Arguing as in the previous estimate we obtain:
\begin{equation}
\| s \mapsto (t - s)^{-\alpha} E(s)(x_0 - y_0) \|_{\gamma(0, t; X)} \lesssim \| s \mapsto (t - s)^{-\theta} x_0 \|_{L^2(0, t)} \| x_0 - y_0 \| X \lesssim \| x_0 - y_0 \| X,
\end{equation}
where
with implied constants independent of $n$, $T_0$, $x_0$ and $y_0$.

As $t \in [0, T_0]$ was arbitrary, by taking $p^\text{th}$ moments it follows that:
\begin{equation}
\|s \mapsto E(\mathcal{F})(x_0 - y_0)\|_{L^p(\Omega; X)} \lesssim \|x_0 - y_0\|_{L^p(\Omega; X)}
\end{equation}
with implied constants independent of $n$, $T_0$, $x_0$, and $y_0$.

**Part 2c.** By the uniform boundedness estimate of Proposition 4.2 with $\delta = \theta_F$ one has, for $s \in [0, T_0]$ fixed:
\begin{align*}
\left\| \int_0^s \left[ S(\mathcal{F} - \mathcal{S}) - E(\mathcal{F} - \mathcal{S}) \right] F(u, U^{(n)}(u)) \, du \right\|_{L^p(\Omega; X)} &
\leq \int_0^s \| \left[ S(\mathcal{F} - \mathcal{S}) - E(\mathcal{F} - \mathcal{S}) \right] F(u, U^{(n)}(u)) \|_{L^p(\Omega; X)} \, du \\
&\lesssim \int_0^s n^{-1} \| \mathcal{F} - \mathcal{S} \|_{L^\infty(0, T_0; L^p(\Omega; X_{\theta_F}))} \| \mathcal{F} - \mathcal{S} \|_{L^\infty(0, T_0; L^p(\Omega; X_{\theta_F}))} \| F(\cdot, U^{(n)}) \|_{L^\infty(0, T_0; L^p(\Omega; X))} \\
&\leq \frac{1}{n} \sum_{j=1}^n (T/n) \cdot (jT/n)^{-1+\theta_F} \| F(\cdot, U^{(n)}) \|_{L^\infty(0, T_0; L^p(\Omega; X))} \\
&\lesssim n^{-1-\theta_F} \sum_{j=1}^n j^{-1+\theta_F} (1 + \| U^{(n)} \|_{L^\infty(0, T_0; L^p(\Omega; X))}) \\
&\lesssim n^{-1-\theta_F + \frac{3}{4} \varepsilon} (1 + \| x_0 \|_{L^p(\Omega; X)}),
\end{align*}
where we used the linear growth condition in (F) and that $\sum_{j=1}^n j^{-1+\theta_F} \lesssim 1$ if $\theta_F < 0$ and $\sum_{j=1}^n j^{-1+\theta_F} \lesssim \ln n \lesssim n^{\frac{3}{4} \varepsilon}$ if $\theta_F = 0$. In the last step we used Corollary 3.2 with $\delta = 0$. The implied constants are independent of $n$, $T_0$, and $x_0$.

By taking the supremum over $s \in [0, T_0]$ we obtain:
\begin{equation}
\left\| \int_0^s \left[ S(\mathcal{F} - \mathcal{S}) - E(\mathcal{F} - \mathcal{S}) \right] F(u, U^{(n)}(u)) \, du \right\|_{L^\infty(0, T_0; L^p(\Omega; X))} \lesssim n^{-1-\theta_F + \frac{3}{4} \varepsilon} (1 + \| x_0 \|_{L^p(\Omega; X)}).
\end{equation}

For the estimate in the weighted $\gamma$-space we shall use Lemma A.3. Define $\Psi : [0, T_0] \to L^p(\Omega; X)$ by
\begin{equation}
\Psi(s) = \int_0^s \left[ S(\mathcal{F} - \mathcal{S}) - E(\mathcal{F} - \mathcal{S}) \right] F(u, U^{(n)}(u)) \, du.
\end{equation}

Let $t \in [0, T_0]$ and let $q = \left( \frac{1}{p} - \frac{1}{2} + \frac{3}{4} \varepsilon \right)^{-1}$ (so $\frac{1}{p} - \frac{1}{2} < \frac{1}{q} < \frac{1}{p} - \alpha$). By Lemma A.3 we have:
\begin{equation}
\sup_{t \in [0, T_0]} \| s \mapsto (t - s)^{-\alpha} \Psi(s) \|_{\gamma(0, t; L^p(\Omega; X))} \lesssim \| \Psi \|_{L^p(\Omega; X)} \| \| \|_{L^\infty(0, T_0; L^p(\Omega; X))},
\end{equation}
with implied constant independent of $T_0$.

Let $\rho \in [0, 1]$ and let $0 < |h| < \rho$. We have, with $I = [0, T_0]$,
\begin{align*}
\| T_h^I \Psi(s) - \Psi(s) \|_{L^p(\Omega; X)} &\leq \left\{ \begin{array}{ll}
0, & s + h = \pi, s + h \in [0, T_0]; \\
2 \| \Psi \|_{L^\infty(0, T_0; L^p(\Omega; X))}, & s + h \neq \pi \text{ or } s + h \notin [0, T_0].
\end{array} \right.
\end{align*}
Suppose $|h| < \frac{T}{n}$. Define $I_h = \{s \in [0, T_0]: s + nh \neq \pi\}$ and observe that $|I_h| \leq n|h|$. Thus by the definition of $q$ and by (5.12):
\[
\|T^n_h \Psi - \Psi\|_{L^p(0,T_0;L^p(\Omega;X))} \lesssim (n|h|)^{\frac{1}{2}} \|\Psi\|_{L^\infty(0,T_0;L^p(\Omega;X))} \\
\lesssim |h|^{\frac{1}{2} - \frac{1}{p} + \frac{1}{2}n - \frac{3}{2} + \frac{1}{2} - \eta - \varepsilon} (1 + \|x_0\|_{L^p(\Omega;X)}).
\]

On the other hand, if $|h| \geq \frac{T}{n}$ then we have:
\[
\|T^n_h \Psi - \Psi\|_{L^p(0,T_0;L^p(\Omega;X))} \lesssim \|\Psi\|_{L^\infty(0,T_0;L^p(\Omega;X))} \\
\lesssim n^{-\eta - \varepsilon} (1 + \|x_0\|_{L^p(\Omega;X)}) \\
= |h|^{\frac{1}{2} - \frac{1}{p} + \frac{1}{2}n - \frac{3}{2} + \frac{1}{2} - \eta} (1 + \|x_0\|_{L^p(\Omega;X)}).
\]

Combining the two cases and recalling that $\eta < \frac{3}{2} - \frac{1}{p} + \theta_F - \varepsilon$ by (5.8) we obtain:
\[
\sup_{0 < |h| \leq \rho} \|T^n_h \Psi(s) - \Psi(s)\|_{L^p(0,T_0;L^p(\Omega;X))} \lesssim \rho^\frac{1}{2} - \frac{1}{p} + \frac{1}{2}n - \eta(1 + \|x_0\|_{L^p(\Omega;X)}).
\]

By the definition of $B_{q,p}^{\frac{1}{2},\frac{1}{2}}$ and equation (5.12), this gives:
\[
\|\Psi\|_{B_{q,p}^{\frac{1}{2},\frac{1}{2}}(0,T_0;L^p(\Omega;X))} \\
\lesssim \|\Psi\|_{L^p(0,T_0;L^p(\Omega;X))} + n^{-\eta} \int_0^1 \rho^\frac{1}{2} \varepsilon^{-1} \rho(1 + \|x_0\|_{L^p(\Omega;X)})) \\
\lesssim \|\Psi\|_{L^\infty(0,T_0;L^p(\Omega;X))} + n^{-\eta}(1 + \|x_0\|_{L^p(\Omega;X)}) \\
\lesssim n^{-\eta}(1 + \|x_0\|_{L^p(\Omega;X)}),
\]

with implied constant independent of $n$, $T_0$, and $x_0$.

Inserting the above and (5.12) into (5.13) we obtain:
\[
(5.14) \sup_{t \in [0,T_0]} \|t \mapsto (t-s)^{-\alpha} \Psi(s)\|_{L^\alpha(0,T;L^p(\Omega;X))} \lesssim n^{-\eta}(1 + \|x_0\|_{L^p(\Omega;X)}).
\]

Finally, by combining (5.12) and (5.14) we obtain that
\[
(5.15) \|\Psi\|_{Y^{\infty,p}(0,T_0) \times \Omega;X} \lesssim n^{-\eta}(1 + \|x_0\|_{L^p(\Omega;X)}),
\]

with implied constants independent of $n$, $T_0$, and $x_0$.

**Part 2d.** In this part we provide estimates for the fourth and fifth term in (5.7). In order to do so, we shall prove that there exists an $\varepsilon_1 > 0$ such that for any $\Phi \in L^\infty(0,T;L^p(\Omega;X_{\theta_F}))$ we have:
\[
(5.16) \left\|s \mapsto \int_0^s E(\tau - u) \Phi(u) du \right\|_{Y^{\infty,p}(0,T_0) \times \Omega;X} \lesssim T_0^{-1} \Phi\|_{L^\infty(0,T_0;L^p(\Omega;X_{\theta_F}))},
\]

with implied constants independent of $n$, $T_0$, and $\Phi$.

Once (5.16) is obtained, we immediately get:
\[
(5.17) \left\|s \mapsto \int_0^s E(\tau - u)|F(u, U^{(n)}(u)) - F(u, V^{(n)}(u))| du \right\|_{Y^{\infty,p}(0,T_0) \times \Omega;X} \\
\lesssim T_0^{-1} \|V^{(n)} - U^{(n)}\|_{L^\infty(0,T_0;L^p(\Omega;X))},
\]
by \((\text{F})\). Moreover, by \((\text{F}')\) we get:

\[
\left\| s \mapsto \int_{0}^{s} E(\bar{s} - u) [F(u, V^{(n)}(u)) - F(u, V^{(n)}(u)))]
\right\|_{L^{p}(0, T_{0}; \mathcal{F}(\Omega; X))} 
\]

(5.18) \[
\lesssim T_{0}^{-1} n^{-\zeta_{\text{max}}} (1 + \|V^{(n)}\|_{L^{\infty}(0, T_{0}; L^{p}(\Omega; X))}) 
\]

\[
\lesssim T_{0}^{-1} \left[ \|V^{(n)} - U^{(n)}\|_{L^{\infty}(0, T_{0}; L^{p}(\Omega; X))} + n^{-n}(1 + \|U^{(n)}\|_{L^{\infty}(0, T_{0}; L^{p}(\Omega; X))}) \right] 
\]

\[
\lesssim T_{0}^{-1} \left[ \|V^{(n)} - U^{(n)}\|_{L^{\infty}(0, T_{0}; L^{p}(\Omega; X))} + n^{-n}(1 + \|x_{0}\|_{L^{p}(\Omega; X))} \right],
\]

the last step being again a consequence of Corollary 3.2 with \(\delta = 0\).

It remains to prove (5.16). We fix \(\Phi \in L^{\infty}(0, T; L^{p}(\Omega; X_{q,p}))\). By the uniform boundedness estimate of Corollary 4.5 with \(\delta = \theta_{F}\) we obtain:

\[
\left\| s \mapsto \int_{0}^{s} E(\bar{s} - u) \Phi(u) du \right\|_{L^{\infty}(0, T_{0}; L^{p}(\Omega; X))} 
\]

(5.19) \[
\leq \sup_{0 \leq s \leq T_{0}} \int_{0}^{s} \|E(\bar{s} - u) \Phi(u)\|_{L^{p}(\Omega; X)} du 
\]

\[
\lesssim \sup_{0 \leq s \leq T_{0}} \int_{0}^{s} (\bar{s} - u)^{\theta_{F}} du \|\Phi\|_{L^{\infty}(0, T_{0}; L^{p}(\Omega; X_{q,p}))} 
\]

\[
\lesssim T_{0}^{-1 + \theta_{F}} \|\Phi\|_{L^{\infty}(0, T_{0}; L^{p}(\Omega; X_{q,p}))},
\]

where the last step follows from a similar calculation as in (5.12), the difference being that now we consider the terms ‘up to \(T_{0}\)’. The implied constants are independent of \(n, T_{0}\), and \(\Phi\).

For the estimate in the weighted \(\gamma\)-space we shall again use Lemma A.3. Define \(\Psi : [0, T_{0}] \rightarrow L^{p}(\Omega; X)\) by

\[
\Psi(s) := \int_{0}^{s} E(\bar{s} - u) \Phi(u) du.
\]

(5.20) Let \(t \in [0, T_{0}]\) and let \(q = (\frac{1}{2} - \frac{1}{\gamma} + \frac{1}{2}\zeta) \in (1, 1] \leq \frac{1}{2} < \frac{1}{\gamma} < 1 - \alpha\). Combining Lemma A.3 and (5.19) we obtain, for some \(\varepsilon_{0} > 0\):

\[
\sup_{t \in [0, T_{0}]} \|s \mapsto (t - s)^{-\alpha} \Psi(s)\|_{L^{p}(0, t; L^{p}(\Omega; X))} 
\]

(5.21) \[
\lesssim T_{0}^{-\varepsilon_{0}} \left( \|\Psi\|_{B_{q,1}^{-\frac{1}{2}}([0, T_{0}; L^{p}(\Omega; X))}) + \|\Psi\|_{L^{\infty}(0, T_{0}; L^{p}(\Omega; X))} \right) 
\]

\[
\lesssim T_{0}^{-\varepsilon_{0}} \left( \|\Psi\|_{B_{q,1}^{-\frac{1}{2}}([0, T_{0}; L^{p}(\Omega; X))}) + T_{0}^{-1 + \theta_{F}} \|\Phi\|_{L^{\infty}(0, T_{0}; L^{p}(\Omega; X_{q,p}))} \right),
\]

with implied constant independent of \(T_{0}\) and \(\Psi\).

In order to estimate the Besov norm in the right-hand side, let us first fix \(s \in [0, T_{0}]\) and \(k \in \{1, \ldots, n\}\) such that \(s + t_{k}^{(n)} \leq T_{0}\). We have, using (4.2),

\[
\Psi(s + t_{k}^{(n)}) - \Psi(s) = \int_{0}^{s} (E(t_{k}^{(n)})) (E(\bar{s} - u)) du 
\]

(5.22) \[
+ \int_{s}^{s + t_{k}^{(n)}} E(\bar{s} + t_{k}^{(n)} - u) \Phi(u) du.
\]
By Lemma 2.4 (3) and the uniform boundedness result of Proposition 4.2 with \( \delta = 1 + \theta_F - \frac{1}{2} \varepsilon \) (note that \( \delta > 0 \) by (5.8)) we have:

\[
\| E(t_k^{(n)}) - I \|_{\mathcal{L}^p(X_{t_k^{(n)}}; X)} \lesssim (t_k^{(n)})^{1 + \theta_F - \frac{1}{2} \varepsilon}.
\]

Moreover, by the uniform boundedness result in Corollary 4.5 with \( \delta = -1 + \frac{1}{2} \varepsilon \) we have:

\[
\| E(t_k^{(n)}) \|_{\mathcal{L}^p(X_{\theta F}; X)} \lesssim (t_k^{(n)})^{-1 + \frac{1}{2} \varepsilon},
\]

and with \( \delta = \theta_F \):

\[
\| E(t_k^{(n)}) \|_{\mathcal{L}^p(X_{\theta F}; X)} \lesssim (t_k^{(n)})^{\theta_F}.
\]

Thus for the first term in (5.22) we have:

\[
\left\| \int_0^{\infty} (E(t_k^{(n)}) - I) E(\bar{\sigma} - u) \Phi(u) \, du \right\|_{L^p(\Omega; X)}
\]

\[
\leq \int_0^{\infty} \| E(t_k^{(n)}) - I \|_{\mathcal{L}^p(X_{t_k^{(n)}}; X)} \| E(\bar{\sigma} - u) \|_{\mathcal{L}^p(X_{\theta F}; X)} \| \Phi \|_{L^\infty(0,T_0;L^p(\Omega;X_{\theta F}))} \, du
\]

\[
\lesssim (t_k^{(n)})^{1 + \theta_F - \frac{1}{2} \varepsilon} \int_0^{\infty} (\bar{\sigma} - u)^{-1 + \frac{1}{2} \varepsilon} \, du \| \Phi \|_{L^\infty(0,T_0;L^p(\Omega;X_{\theta F}))}
\]

\[
\lesssim (t_k^{(n)})^{1 + \theta_F - \frac{1}{2} \varepsilon} \| \Phi \|_{L^\infty(0,T_0;L^p(\Omega;X_{\theta F}))}.
\]

For the second term in (5.22) we have:

\[
\left\| \int_0^{\infty} E(\bar{\sigma} + t_k^{(n)} - u) \Phi(u) \, du \right\|_{L^p(\Omega; X)}
\]

\[
\lesssim \int_0^{\infty} (t_k^{(n)} - u)^{\theta_F} \, du \| \Phi \|_{L^\infty(0,T_0;L^p(\Omega;X_{\theta F}))}
\]

\[
\lesssim (t_k^{(n)})^{1 + \theta_F} \| \Phi \|_{L^\infty(0,T_0;L^p(\Omega;X_{\theta F}))}.
\]

Combining the two estimates above we obtain:

\[
(5.23) \quad \| \Psi(s + t_k^{(n)}) - \Psi(s) \|_{L^p(\Omega; X)} \lesssim (t_k^{(n)})^{1 + \theta_F} \| \Phi \|_{L^\infty(0,T_0;L^p(\Omega;X_{\theta F}))}.
\]

This enables us to find the right estimate for the Besov norm in (5.21). Fix \( \rho \in [0,1) \) and \( 0 < |h| < \rho \). Set \( I = [0,T_0] \). Suppose first that \( |h| \leq \frac{T_0}{n} \). In that case we have, by (5.23):

\[
\| T_k^I \Psi(s) - \Psi(s) \|_{L^p(\Omega; X)}
\]

\[
\leq \begin{cases}
0, & \text{if } h \neq \bar{\sigma} \text{ and } s + h \in [0,T_0]; \\
\| \Psi(s + \frac{T_0}{n}) - \Psi(s) \|_{L^p(\Omega; X)}, & \text{if } s + h \notin [0,T_0]; \\
\| \Psi(s) \|_{L^p(\Omega; X)}, & \text{if } s + h \notin [0,T_0]; \\
\| \Phi \|_{L^\infty(0,T_0;L^p(\Omega;X_{\theta F}))}, & \text{if } s + h = \bar{\sigma} \text{ and } s + h \notin [0,T_0].
\end{cases}
\]

In the above we used \( \| \Psi(s) \|_{L^p(\Omega; X)} \leq \| \Psi \|_{L^\infty(0,T_0;L^p(\Omega; X))} \) and (5.19).
Define \( I_h = \{ s \in [0, T_0] : s + h \neq \bar{s} \} \) and observe that \( |I_h| \leq n|h| \). Moreover \( |\{ s \in [0, T_0] : s + h \notin [0, T_0] \}| \leq |h| \). Thus by the definition of \( q \) and by (5.23),

\[
\| T^t_h \Psi - \Psi \|_{L^q([0, T_0); L^p(\Omega; X))} \\
\leq \left[ \left( n|h| \right)^\frac{1}{n} n^{1-\theta_p} + |h|^{\frac{1}{n}} \right] \| \Phi \|_{L^\infty(0, T_0; L^p(\Omega; X_{q_p}))} \\
\leq |h|^{\frac{1}{n} - \frac{1}{2} + \frac{1}{2}} \| \Phi \|_{L^\infty(0, T_0; L^p(\Omega; X_{q_p}))}.
\]

Next let \( |h| > \frac{T}{n} \). Then, \( |h|/2 < |h| \leq |h| \) and \( |h| < |\bar{h}| \leq 2|h| \). Let us deal with the case \( h > \frac{T}{n} \); the case \( -h > \frac{T}{n} \) is dealt with entirely analogously. It follows from the definition of \( \Phi \) in (5.20) that for each \( s \in [0, T_0] \) either we have \( \Psi(s + h) = \Psi(\bar{s} + \bar{h}) \) or \( \Psi(s + h) = \Psi(\bar{s} - \bar{h}) \). Hence, by (5.23):

\[
\| T^t_h \Psi - \Psi \|_{L^p([0, T_0); L^p(\Omega; X))} \\
\leq \| 1_{\{s + \rho \notin [0, T_0] \}} \|_{L^q([0, T_0); L^p(\Omega; X))} \\
+ \| T^t_h \Psi - \Psi \|_{L^p([0, T_0); L^p(\Omega; X))} \\
\leq \left( h \right)^\frac{1}{n} \| \Phi \|_{L^\infty(0, T_0; L^p(\Omega; X_{q_p}))} \\
\leq \| h \|^{\frac{1}{n} - \frac{1}{2} + \frac{1}{2}} \| \Phi \|_{L^\infty(0, T_0; L^p(\Omega; X_{q_p}))},
\]

where we use that \( 1 + \theta_p \geq \frac{1}{2} - \frac{1}{2} + \frac{1}{2} \varepsilon \) (by (5.8)).

Thus we have:

\[
\sup_{|h| \leq \rho} \| T^t_h \Psi - \Psi \|_{L^q([0, T_0); L^p(\Omega; X))} \leq \rho \left( h \right)^\frac{1}{n} \| \Phi \|_{L^\infty(0, T_0; L^p(\Omega; X_{q_p}))}.
\]

With (5.19) it follows that

\[
\| \Psi \|_{B_{\frac{1}{n}, \frac{1}{p}}(0, T_0; L^p(\Omega; X))} \leq \| \Psi \|_{L^q([0, T_0); L^p(\Omega; X))} + \| \Phi \|_{L^\infty(0, T_0; L^p(\Omega; X_{q_p}))} \\
\leq \| \Phi \|_{L^\infty(0, T_0; L^p(\Omega; X))} + \| \Phi \|_{L^\infty(0, T_0; L^p(\Omega; X_{q_p}))} \\
\leq \| \Phi \|_{L^\infty(0, T_0; L^p(\Omega; X_{q_p}))},
\]

and thus, by (5.21),

\[
\sup_{t \in [0, T_0]} \| s \mapsto (t - s)^{-\alpha} \| \Psi(s) \|_{\gamma([0, T_0); L^p(\Omega; X))} \leq T_0^{-\alpha} \| \Phi \|_{L^\infty(0, T_0; L^p(\Omega; X_{q_p}))}.
\]

This completes the proof of (5.16).

**Part 2e.** For the sixth term we again use Besov embeddings. First of all observe that by (2.6), the linear growth condition on \( F \) in (F) and Corollary 3.2, for all \( s \in [0, T_0] \) we have:

\[
\| \int_s^\bar{X} S(t/\bar{X}) F(u, U^{(n)}(u)) \|_{L^p(\Omega; X)} \leq \left( \frac{\bar{X}}{X} \right)^{\theta_p} \int_s^\bar{X} \| F(u, U^{(n)}(u)) \|_{L^p(\Omega; X_{q_p})} \ du \\
\leq \left( \frac{\bar{X}}{X} \right)^{\theta_p} \int_s^\bar{X} \| U^{(n)}(u) \|_{L^p(\Omega; X)} \ du \\
\leq (\bar{X} - s) n^{-\theta_p} (1 + \| x_0 \|_{L^p(\Omega; X)}) \\
\leq n^{-1-\theta_p} (1 + \| x_0 \|_{L^p(\Omega; X)})
\]
and therefore
\begin{equation}
\| s \mapsto \int_{s}^{\tau} S(\frac{\tau}{n}) F(u, U^{(n)}(u)) \, du \|_{L^\infty(0,T_0; L^p(\Omega; X))} \lesssim n^{-1-\theta_F} (1 + \| x_0 \|_{L^p(\Omega; X)}).
\end{equation}

Similarly, for \( h \in (0, \frac{T}{n}] \):
\begin{equation}
\| s \mapsto \int_{s}^{s+h} S(\frac{\tau}{n}) F(u, U^{(n)}(u)) \, du \|_{L^\infty(0,T_0; L^p(\Omega; X))} \\
\lesssim h n^{-\theta_F} (1 + \| x_0 \|_{L^p(\Omega; X)}) \\
\lesssim h^{\frac{1}{2} + \frac{\epsilon}{2}} n^{-\theta_F + \frac{1}{2}(1 + \| x_0 \|_{L^p(\Omega; X)}).
\end{equation}

Define \( \Psi : [0, T_0] \to L^p(\Omega; X) \) by
\[
\Psi(s) := \int_{s}^{\tau} S(\frac{\tau}{n}) F(u, U^{(n)}(u)) \, du.
\]

Fix \( \rho \in [0, 1] \) and let \( 0 \leq h < \rho \) (the case that \( -\rho < h \leq 0 \) is entirely analogous).
Suppose first that \( h < \frac{T}{n} \). Then, for \( s \in I := [0, T_0] \):
\[
\| T_h^s \Psi(s) - \Psi(s) \|_{L^p(\Omega; X)} \\
\lesssim \| s \mapsto \int_{s}^{s+h} S(\frac{\tau}{n}) F(u, U^{(n)}(u)) \, du \|_{L^p(\Omega; X)}, \quad s + h \in [0, T_0]; \\
2\| \Psi \|_{L^\infty(0,T_0; L^p(\Omega; X))}, \quad \text{otherwise}.
\]

Recall that \( |\{ s \in [0, T_0] : s \neq s + h \} | \leq nh \) and \( |\{ s \in [0, T_0] : s + h \notin [0, T_0] \} | \leq h \).
Let \( q = \left( \frac{1}{r} - \frac{1}{2} + \frac{\epsilon}{2} \right)^{-1} \). By (5.24) and (5.25) we have:
\[
\| T_h^s \Psi - \Psi \|_{L^p(\Omega; L^p(\Omega; X))} \\
\lesssim (h^{\frac{1}{2} + \frac{\epsilon}{2}} n^{-\frac{1}{2} + \frac{\theta_F}{2} + \frac{1}{2} + \frac{1}{q}} + ((n+1)h)^{\frac{1}{q}} n^{-1-\theta_F})(1 + \| x_0 \|_{L^p(\Omega; X)}) \\
\lesssim h^{\frac{1}{2} + \frac{\epsilon}{2}} n^{-\theta_F + \frac{1}{2} + \frac{1}{q}} (1 + \| x_0 \|_{L^p(\Omega; X)}).
\]

On the other hand, if \( h > \frac{T}{n} \), then by (5.24):
\[
\| T_h^s \Psi - \Psi \|_{L^p(\Omega; L^p(\Omega; X))} \lesssim 2\| \Psi \|_{L^\infty(0,T_0; L^p(\Omega; X))} \\
\lesssim n^{-\theta_F} (1 + \| x_0 \|_{L^p(\Omega; X)}) \\
\lesssim h^{\frac{1}{2} + \frac{\epsilon}{2}} n^{-1-\theta_F + \frac{1}{2} + \frac{1}{q}} (1 + \| x_0 \|_{L^p(\Omega; X)}).
\]

As in Part 2c, using that by (5.8) we have \( \eta + \epsilon < \frac{3}{2} - \frac{1}{r} + \theta_F < 1 + \theta_F - \frac{1}{q} + \epsilon \),
this implies
\[
\| \Psi \|_{B^{\frac{1}{2} + \frac{\epsilon}{2}}_{r, \infty}(0,T_0; L^p(\Omega; X))} \lesssim n^{-\eta}(1 + \| x_0 \|_{L^p(\Omega; X)}),
\]
with implied constants independent of \( n, x_0 \) and \( T_0 \). By Lemma A.3 it now follows that
\[
\sup_{t \in [0,T_0]} \| s \mapsto (t-s)^{-\alpha} \int_{s}^{\tau} S(\frac{\tau}{n}) F(u, U^{(n)}(u)) \, du \|_{\gamma(0,t; L^p(\Omega; X))} \\
\lesssim n^{-\eta}(1 + \| x_0 \|_{L^p(\Omega; X)}),
\]
with implied constants independent of $n$, $x_0$ and $T_0$. Combining this with (5.24) we obtain:

\[(5.26) \left\| s \mapsto \int_{s}^{T} S\left(\frac{z}{n}\right) F(u, U^{(n)}(u)) \, du \right\|_{L_{\infty}^{\infty}([0, T_{0}] \times \Omega; X)} \lesssim n^{-\eta}(1 + \|x_0\|_{L_{\infty}^{p}(\Omega; X)}),\]

with implied constants independent of $n$, $x_0$ and $T_0$.

Part 2f: By Theorem 2.1 we have, for any $s \in [0, T_{0}]$:

\[
\left\| \int_{0}^{T} [S(\bar{s} - \bar{u}) - E(\bar{s} - \bar{u})] G(u, U^{(n)}(u)) \, dW_{H}(u) \right\|_{L_{p}(\Omega; X)}
\leq \left\| u \mapsto [S(\bar{s} - \bar{u}) - E(\bar{s} - \bar{u})] G(u, U^{(n)}(u)) \right\|_{L_{p}(\Omega; \gamma(0, \pi; H, X))}.
\]

By the second part of Proposition 4.2 with $\delta = \theta_G$, $\epsilon = \frac{1}{3} \bar{\epsilon}$, $\beta = \frac{1}{2} - \frac{2}{3} \bar{\epsilon}$, and Theorem 2.2 and (5.8) we have:

\[
\left\| u \mapsto [S(\bar{s} - \bar{u}) - E(\bar{s} - \bar{u})] G(u, U^{(n)}(u)) \right\|_{L_{p}(\Omega; \gamma(0, \pi; H, X))}
\leq n^{-\frac{1}{2} - \theta_{0} + \bar{\epsilon}} \left\| U^{(n)} \right\|_{L_{p}^{\frac{1}{2} - \frac{2}{3} \bar{\epsilon}}([0, T_{0}] \times \Omega; X)}
\leq n^{-\eta} \left\| U^{(n)} \right\|_{L_{p}^{\frac{1}{2} - \frac{2}{3} \bar{\epsilon}}([0, T_{0}] \times \Omega; X)}
\leq n^{-\eta}(1 + \|x_0\|_{L_{p}(\Omega; X)}),
\]

where we also used (G) in the sense of (2.11), the fact that $\alpha > \frac{1}{2} - \frac{2}{3} \bar{\epsilon}$ (whence $\gamma_{\alpha}^{\infty}(0, T_{0}) \times \Omega; X) \rightarrow \gamma_{\frac{1}{2} - \frac{2}{3} \bar{\epsilon}}^{\infty}(0, T_{0}) \times \Omega; X)$, and we used Corollary 3.2. Note that the implied constants are independent of $n$, $T_{0}$ and $x_0$. As $s \in [0, T_{0}]$ was arbitrary, it follows that

\[(5.27) \left\| s \mapsto \int_{0}^{T} [S(\bar{s} - \bar{u}) - E(\bar{s} - \bar{u})] G(u, U^{(n)}(u)) \, dW_{H}(u) \right\|_{L_{\infty}^{\infty}(0, T_{0}; L_{p}(\Omega; X))}
\leq n^{-\eta}(1 + \|x_0\|_{L_{p}(\Omega; X)}),
\]

Next we estimate the part concerning the weighted $\gamma$-radonifying norm. We begin by recalling that, since $X$ is a UMD Banach space, $\gamma(0, t; H, X)$ is a UMD Banach space for any $t > 0$ (by noting that this space embeds into $L^{2}(\bar{\Omega}; X)$ isometrically whenever $(\bar{\Omega}, \bar{\mathbb{P}})$ is a probability space supporting a Gaussian sequence; see, e.g., [27]). Thus, by Theorem 2.1 (applied with state space $\gamma(0, \bar{\mathbb{P}}; X)$ and isomorphism (5.1), for all $t \in [0, T_{0}]$ we obtain

\[
\left\| s \mapsto (t - s)^{-\alpha} \int_{0}^{T} [S(\bar{s} - \bar{u}) - E(\bar{s} - \bar{u})] G(u, U^{(n)}(u)) \, dW_{H}(u) \right\|_{L_{p}(\Omega; \gamma(0, t; X))}
\leq \left\| \int_{0}^{T} [s \mapsto 1_{(0 \leq u \leq \bar{s})}(t - s)^{-\alpha}
\times [S(\bar{s} - \bar{u}) - E(\bar{s} - \bar{u})] G(u, U^{(n)}(u))] \, dW_{H}(u) \right\|_{L_{p}(\Omega; \gamma(0, t; X))}
\approx \left\| u \mapsto (s \mapsto 1_{(0 \leq u \leq \bar{s})}(t - s)^{-\alpha}
\times [S(\bar{s} - \bar{u}) - E(\bar{s} - \bar{u})] G(u, U^{(n)}(u))] \right\|_{L_{p}(\Omega; \gamma(0, t; H, \gamma(0, t; X)))}
\approx \left\| (s, u) \mapsto 1_{(0 \leq u \leq \bar{s})}(t - s)^{-\alpha}
\times [S(\bar{s} - \bar{u}) - E(\bar{s} - \bar{u})] G(u, U^{(n)}(u)) \right\|_{L_{p}(\Omega; \gamma(0, t; H, \gamma(0, t; X))}.
\]
By the second part of Proposition 4.2 with $\delta = \theta_G$, $\epsilon = \frac{1}{2}\epsilon$, $\beta = \frac{1}{2} - \frac{1}{2}\epsilon$, Theorem 2.2, isomorphism $(5.1)$, once again Theorem 2.2 combined with Theorem 2.3, Lemma A.5, Corollary 3.2 and $(5.8)$ we have:

\[
\| (s, u) \mapsto 1_{\{0 \leq u \leq \sigma\}} (t-s)^{-\alpha} \\
\times [S(\sigma - u) - E(\sigma - u)] G(u, U^{(n)}(u)) \|_{L^p(\Omega; [0, T] \times [0, T]; H, X)} \lesssim n^{-\frac{1}{2} - \theta_G + \epsilon} \| (s, u) \mapsto 1_{\{0 \leq u \leq \sigma\}} (t-s)^{-\alpha} \\
\times (\sigma - u)^{-\frac{1}{2} + \frac{1}{2}\epsilon} G(u, U^{(n)}(u)) \|_{L^p(\Omega; [0, T] \times [0, T]; H, X)}
\]

\[
\lesssim n^{-\eta} \| (s, u) \mapsto 1_{\{0 \leq u \leq \sigma\}} (t-s)^{-\alpha} \\
\times (\sigma - u)^{-\frac{1}{2} + \frac{1}{2}\epsilon} G(u, U^{(n)}(u)) \|_{L^p(\Omega; [0, T] \times [0, T]; H, X)}
\]

\[
\lesssim n^{-\eta} \sup_{u \in [0, T]} \left\{ (t + \frac{T}{n} - u)^{\alpha} \| (s) \mapsto (t-s)^{-\alpha} (\sigma - u)^{-\frac{1}{2} + \frac{1}{2}\epsilon} \|_{L^2(\Omega; t)} \right\}
\]

\[
\times \| (u) \mapsto (t + \frac{T}{n} - u)^{-\alpha} G(u, U^{(n)}(u)) \|_{L^p(\Omega; [0, T]; X)}
\]

\[
\lesssim n^{-\eta} (1 + \| U^{(n)} \|^r_{\gamma_{\infty}([0, T] \times \Omega; X)})
\]

\[
\lesssim n^{-\eta} (1 + \| x_0 \|_{L^p(\Omega; X)}).
\]

The implied constants above are independent of $x_0$, $n$, $t$ and $T_0$.

Combining the above with $(5.27)$ above one obtains:

\[
(5.28) \quad \| \int_0^\sigma [S(\sigma - u) - E(\sigma - u)] G(u, U^{(n)}(u)) \|_{\gamma_{\infty}([0, T_0] \times \Omega; X)} \lesssim n^{-\eta} (1 + \| x_0 \|_{L^p(\Omega; X)}),
\]

with implied constant independent of $n$, $T_0$ and $x_0$.

**Part 2g.** The estimate for the eighth and ninth term in $(5.7)$ is similar to Part 2f, except that one needs to keep track of dependence on $T_0$.

We shall prove that for any $\Phi \in L^p(\Omega; \gamma(0, T; H, X_{\theta_G}))$ we have:

\[
(5.29) \quad \| \int_0^\sigma E(\sigma - u) \Phi(u) \|_{\gamma_{\infty}([0, T_0] \times \Omega; X)} \lesssim T_0^{-\frac{1}{2} - \theta_G - \epsilon} \sup_{0 \leq s \leq T_0} \| (s) \mapsto (t-s)^{-\alpha} \Phi(s) \|_{L^p(\Omega; [0, T]; X)}
\]

with implied constant independent of $n$ and $T_0$, provided the right-hand side above is finite.

The estimate for the eighth term in $(5.7)$ follows immediately from $(5.29)$ and $(2.11)$ (i.e., $(G')$):

\[
(5.30) \quad \| \int_0^\sigma E(\sigma - u) [G(u, U^{(n)}(u)) - G(u, V^{(n)}(u))] \|_{\gamma_{\infty}([0, T_0] \times \Omega; X)} \lesssim T_0^{-\frac{1}{2} - \theta_G - \epsilon} \| U^{(n)} - V^{(n)} \|_{\gamma_{\infty}([0, T_0] \times \Omega; X)}.
\]

The estimate for the ninth term in $(5.7)$ follows immediately from $(5.29)$ in combination with $(5.8)$ and Lemma A.4 (i.e., $(G')$), with $B_j = V_j^{(n)}$ noting that
\[ V^{(n)}(u) = V^{(n)}(u) = V^{(n)}_{\frac{n}{T}}; \]

(5.31)

\[
\left\| s \mapsto \int_0^s E(\xi - u) [G(u, V^{(n)}(u)) - G(u, V^{(n)}(u))] \, dW_H(u) \right\|_{L^{\infty, p}(0, T_0) \times \Omega; X} \leq \mathcal{T}_0^{\frac{1}{2} + \theta_G - \epsilon} \sup_{0 \leq t \leq T_0} \| t \mapsto (t - s)^{-\alpha} \Phi(s) \|_{L^p(\Omega; \gamma(0, \pi; X))}.
\]

It remains to prove (5.29). By Theorem 2.1 we have, for \( s \in [0, T_0] \):

\[
\left\| \int_0^s E(\xi - u) \Phi(u) \, dW_H(u) \right\|_{L^p(\Omega; X)} \leq \| u \mapsto E(\xi - u) \Phi(u) \|_{L^p(\Omega; \gamma(0, \pi; X))}.
\]

By the second part of Corollary 4.5 with \( \delta = \theta_G, \epsilon = \frac{1}{3} \epsilon, \beta = \frac{1}{2} - \frac{2}{3} \epsilon \), and Theorem 2.2 we have:

\[
\left\| u \mapsto E(\xi - u) \Phi(u) \right\|_{L^p(\Omega; \gamma(0, \pi; X))} \leq \mathcal{T}_0^{\frac{1}{2} + \theta_G - \epsilon} \| u \mapsto (\xi - u)^{-\alpha} \Phi(u) \|_{L^p(\Omega; \gamma(0, \pi; X \theta_G))},
\]

where we used that \( \alpha > \frac{1}{2} - \frac{2}{3} \epsilon \). Note that the implied constants are independent of \( n \) and \( T_0 \). As \( s \in [0, T_0] \) was arbitrary, it follows that:

(5.32)

\[
\left\| s \mapsto \int_0^s E(\xi - u) \Phi(u) \, dW_H(u) \right\|_{L^\infty(0, T_0; L^p(\Omega))} \leq \mathcal{T}_0^{\frac{1}{2} + \theta_G - \epsilon} \sup_{0 \leq t \leq T_0} \| s \mapsto (t - s)^{-\alpha} \Phi(s) \|_{L^p(\Omega; \gamma(0, \Omega; H, X \theta_G))}.
\]

As for the part concerning the weighted \( \gamma \)-radonifying norm, as before we have, for \( t \in [0, T_0] \):

\[
\left\| (s, u) \mapsto 1_{\{0 \leq u \leq \pi\}} (t - s)^{-\alpha} E(\xi - u) \Phi(u) \right\|_{L^p(\Omega; \gamma([0, \pi]; [0, \pi]; H, X))} \leq \mathcal{T}_0^{\frac{1}{2} + \theta_G - \epsilon} \| (s, u) \mapsto 1_{\{0 \leq u \leq \pi\}} (t - s)^{-\alpha} E(\xi - u) \Phi(u) \|_{L^p(\Omega; \gamma([0, \pi]; [0, \pi]; H, X \theta_G))}.
\]

By the second part of Corollary 4.5 with \( \delta = \theta_G, \epsilon = \frac{1}{3} \epsilon, \beta = \frac{1}{2} - \frac{1}{3} \epsilon \), Theorem 2.2, isomorphism (5.1), once again Theorem 2.2 combined with Theorem 2.3 and Lemma A.5 we have:

\[
\begin{align*}
\left\| (s, u) \mapsto 1_{\{0 \leq u \leq \pi\}} & (t - s)^{-\alpha} E(\xi - u) \Phi(u) \right\|_{L^p(\Omega; \gamma([0, \pi]; [0, \pi]; H, X))} \\
\leq & \mathcal{T}_0^{\frac{1}{2} + \theta_G - \epsilon} \| (s, u) \mapsto 1_{\{0 \leq u \leq \pi\}} (t - s)^{-\alpha} (\xi - u)^{-\frac{1}{2} + \frac{1}{2} \epsilon} \Phi(u) \|_{L^p(\Omega; \gamma([0, \pi]; [0, \pi]; H, X \theta_G))} \\
\leq & \mathcal{T}_0^{\frac{1}{2} + \theta_G - \epsilon} \| u \mapsto (t + \frac{T}{\pi} - u)^{-\alpha} \Phi(u) \|_{L^p(\Omega; \gamma(0, \pi; X))} \\
\leq & \mathcal{T}_0^{\frac{1}{2} + \theta_G - \epsilon} \| u \mapsto (t - u)^{-\alpha} \Phi(u) \|_{L^p(\Omega; \gamma(0, \pi; X))}.
\end{align*}
\]

Taking the supremum over \( t \in [0, T_0] \) and combining the above with (5.32) one obtains (5.29).
Part 2h. As for the final term in (5.7), first observe that because $\theta_G \leq 0$ we have, by (2.6):

$$
\left\| s \mapsto \int_s^T S\left( \frac{t}{n} \right) G(u, U^{(n)}(u)) \, dW_H(u) \right\|_{X_{\infty}^\theta([0,T_0] \times \Omega; X)} \\
\lesssim n^{-\theta_G} \| s \mapsto \int_s^T G(u, U^{(n)}(u)) \, dW_H(u) \|_{X_{\infty}^\theta([0,T_0] \times \Omega; X_{\theta_G})},
$$

with implied constants independent of $n$ and $x_0$.

By (2.4) (take $\hat{\alpha} = \frac{1}{2} - \varepsilon/2$ and $\hat{\varepsilon} = \varepsilon/2$), (2.11), and Corollary 3.2 we have, for $s \in [0, T_0]$:

$$
\left\| \int_s^T G(u, U^{(n)}(u)) \, dW_H(u) \right\|_{L^p(\Omega; X_{\theta_G})} \\
\leq (\pi - s)^{-\frac{1}{2} + \varepsilon} \left\| s \mapsto \int_0^s G(u, U^{(n)}(u)) \, dW_H(u) \right\|_{C^\frac{1}{2} - \varepsilon/2(0,T_0; L^p(\Omega; X_{\theta_G}))} \\
\lesssim n^{-\frac{1}{2} + \varepsilon} \sup_{0 \leq t \leq T_0} \| u \mapsto (t-u)^{-\frac{1}{2} + \frac{\varepsilon}{2}} G(u, U^{(n)}(u)) \|_{L^p(\Omega; H, X_{\theta_G})} \\
\lesssim n^{-\frac{1}{2} + \varepsilon} (1 + \| U^{(n)} \|_{V^{\frac{1}{2} - \varepsilon, p}_{\infty, \theta_G}(0,T_0 \times \Omega, X)}) \\
\lesssim n^{-\frac{1}{2} + \varepsilon} (1 + \| x_0 \|_{L^p(\Omega; X)})
$$

with implied constants independent of $n$, $T_0$, and $x_0$. We have shown that

$$
\left\| s \mapsto \int_s^T G(u, U^{(n)}(u)) \, dW_H(u) \right\|_{L^\infty(0,T_0; L^p(\Omega; X_{\theta_G}))} \leq n^{-\frac{1}{2} + \varepsilon} (1 + \| x_0 \|_{L^p(\Omega; X)}).
$$

Next fix $t \in [0, T_0]$. By Lemma A.2 (with $R = (0, 1)$ and $S = (0, t)$ with the Lebesgue measure, $f(r, u)(s) = (t-s)^{-\alpha}(t-u)^{\alpha}1_{\{s \leq u \leq t\}}$, $\Phi_2 \equiv I$ and $\Phi_1(u) = (t-u)^{-\alpha}G(u, U^{(n)}(u))$) we obtain

$$
\left\| s \mapsto (t-s)^{-\alpha} \int_s^T G(u, U^{(n)}(u)) \, dW_H(u) \right\|_{L^p(\Omega; H, X_{\theta_G})} \\
\lesssim \sup_{u \in [0,t]} \| u \mapsto (t-u)^{-\alpha}1_{\{u \leq s \leq t\}} \|_{L^2(0,t)} \\
\times \| u \mapsto (t-u)^{-\alpha}G(u, U^{(n)}(u)) \|_{L^p(\Omega; H, X_{\theta_G})} \\
\lesssim n^{-\frac{1}{2}} \| u \mapsto (t-u)^{-\alpha}G(u, U^{(n)}(u)) \|_{L^p(\Omega; H, X_{\theta_G})},
$$

with implied constants independent of $x_0$, $n$ and $T_0$.

From here we proceed as in (5.34) and take the supremum over $t \in [0, T_0]$ to arrive at the estimate

$$
\sup_{t \in [0,T_0]} \left\| s \mapsto (t-s)^{-\alpha} \int_s^T G(u, U^{(n)}(u)) \, dW_H(u) \right\|_{L^p(\Omega; H, X_{\theta_G})} \\
\lesssim n^{-\frac{1}{2}} (1 + \| x_0 \|_{L^p(\Omega; X)}).
$$
Combining (5.35) and (5.36) with (5.33) and recalling (5.8) we obtain:

\[
\begin{align*}
\left\| s \to \int_s^\infty S(\frac{T}{n})G(u, U^{(n)}(u))\right\|_{\mathcal{F}_{\infty}^n([0,T_n] \times \Omega; X)} & \lesssim n^{-\frac{1}{2p} + \frac{1}{p} + \varepsilon} (1 + \|x_0\|_{L^p(\Omega; X)}) \\
& \lesssim n^{-\eta} (1 + \|x_0\|_{L^p(\Omega; X)}).
\end{align*}
\]  

(5.37)

**Part 3.** By combining equations (5.7), (5.10), (5.11), (5.15), (5.17), (5.18), (5.26), (5.28), (5.30), (5.31) and (5.37), we obtain that there exist constants \( C > 0 \) and \( \varepsilon > 0 \), independent of \( n \), \( x_0 \) and \( y_0 \), such that for all \( n \geq N \) and \( T_0 \in (0, T) \):

\[
\| U^{(n)} - V^{(n)} \|_{\mathcal{F}_{\infty}^n([0,T_0] \times \Omega; X)} \leq C \| x_0 - y_0 \|_{L^p(\Omega; X)} + C\eta^{-\eta} (1 + \|x_0\|_{L^p(\Omega; X)}).
\]

(5.38)

Define \( c_0 = \frac{1}{2}(2C)^{-\frac{1}{\varepsilon}} \) and let \( N_0 \in \mathbb{N} \) be such that \( N_0 > \max\{N, T/c_0\} \), this implies that for \( n \geq N_0 \) we have \( c_0 \leq \frac{1}{2} c_0 \leq 2c_0 \), and thus \( \frac{1}{c_0} \leq (2c_0)^{-1} = (2C)^{-1} \).

For \( n \geq N_0 \) we obtain by taking \( T_0 = c_0^{-1} \) in (5.38):

\[
\| U^{(n)} - V^{(n)} \|_{\mathcal{F}_{\infty}^n([0,T_0] \times \Omega; X)} \leq 2C \left( \| x_0 - y_0 \|_{L^p(\Omega; X)} + n^{-\eta} (1 + \|x_0\|_{L^p(\Omega; X)}) \right),
\]

and thus there exists a constant \( \bar{C} \) such that for all \( n \geq N \) we have:

\[
\| U^{(n)} - V^{(n)} \|_{\mathcal{F}_{\infty}^n([0,c_0] \times \Omega; X)} \leq \bar{C} \left( \| x_0 - y_0 \|_{L^p(\Omega; X)} + n^{-\eta} (1 + \|x_0\|_{L^p(\Omega; X)}) \right),
\]

which is precisely estimate (5.6). \( \square \)

6. **Proof of Theorems 1.1 and 1.2**

We shall present the proof of Theorem 1.1; the proof of Theorem 1.2 and of the analogues of Theorem 1.2 for classical splitting scheme and the splitting scheme with discretized noise of Example 4.6 is entirely analogous.

Set \( u := (U(t_j^{(n)}))_{j=0}^n \), where \( U \) is the solution to (1.1) with initial datum \( x_0 \in L^p(\Omega, \mathcal{F}_0, X_\eta) \) for some \( p > 2 \) and \( \eta \geq 0 \) such that \( \frac{1}{p} < \eta < \zeta_{\max} \), with \( \zeta_{\max} \) as defined in (5.2).

In order to prove Theorem 1.1, we shall in fact consider \( (V_j^{(n)})_{j=0}^n \) being defined by the abstract scheme of Section 5, where \( E(t_j^{(n)}) \) is defined in terms of a family of measures \( (\mu_n)_{n \geq N} \) satisfying (M1), (M2), and (M3). We shall also assume that \( V_0 = y_0 \in L^p(\Omega, \mathcal{F}_0; X) \). This more general case does not require extra arguments and Theorem 1.1 follows immediately by Example 4.7 and by taking \( x_0 = y_0 \).

The proofs of Theorems 1.1 and 1.2 are based on the following version of Kolmogorov’s continuity criterion (see, e.g., [35, Theorem I.2.1]):

**Proposition 6.1** (Kolmogorov’s continuity criterion). Let \( Y \) be a Banach space. For all \( \alpha > 0 \), \( q \in \left( \frac{1}{\eta}, \infty \right) \) and \( 0 \leq \gamma < \alpha - \frac{1}{q} \) there exists a constant \( K \) such that for all \( T > 0 \), \( k \in \mathbb{N} \), and \( u, v \in c_k^{(2\eta)}([0,T]; L^q(\Omega; Y)) \) we have:

\[
\left( \mathbb{E} \| u - v \|_{c_k^{(2\eta)}([0,T]; Y)}^q \right)^{\frac{1}{q}} \leq K \| u - v \|_{c_k^{(2\eta)}([0,T]; L^q(\Omega; Y))},
\]

where \( K \) depends only on \( q \) and \( \gamma \).
Proof of Theorem 1.1. Let $T > 0$ and $n \in \mathbb{N}$ be fixed. Upon replacing $\eta$ by a smaller value, we may assume that $\gamma + \delta + \frac{1}{p} < \eta < \zeta_{\text{max}}$ with $\zeta_{\text{max}}$ as in (5.2).

Let $k \in \mathbb{N}$ be such that $2^{k-1} < n \leq 2^k$. Then $T \leq \frac{2^k T}{n} < 2T$. For $j \in \{0, \ldots, 2n\}$ set

$$d_j^{(n)} := U(t_j^{(n)}) - V_j^{(n)},$$

using that there exists a unique mild solution $U$ to (SEE) on $[0, 2T]$; the definition of $V_j^{(n)}$ for $j \in \{n+1, \ldots, 2n\}$ is straightforward.

By Theorem 5.2 applied to the interval $[0, 2T]$ with $2n$ time steps and with $\eta$ as fixed above we have, because $|t_j^{(n)} - t_i^{(n)}| \geq \frac{T}{n}$,

$$\sup_{0 \leq i < j \leq 2n} \frac{\|d_j^{(n)} - d_i^{(n)}\|_{L^p(\Omega; X)}}{|t_j^{(n)} - t_i^{(n)}|_{\eta - \delta}} \leq \left( \frac{n}{T} \right)^{-\delta} \sup_{0 \leq i < j \leq n} \left( \|d_j^{(n)}\|_{L^p(\Omega; X)} + \|d_i^{(n)}\|_{L^p(\Omega; X)} \right) \lesssim n^{-\delta} n^{-\eta} (1 + \|x_0\|_{L^p(\Omega; X_0)})$$

$$= n^{-\delta} (1 + \|x_0\|_{L^p(\Omega; X_0)}),$$

with implied constant independent of $n$ and $x_0$. In particular:

$$\sup_{0 \leq j \leq 2n} \|d_j^{(n)}\|_{L^p(\Omega; X)} \lesssim \|d_0^{(n)}\|_{L^p(\Omega; X)} + n^{-\delta} (1 + \|x_0\|_{L^p(\Omega; X_0)}).$$

It follows that

$$\left\| \left( d_j^{(n)} \right)_{j=0}^{2^k} \right\|_{L^p_{\eta - \delta}(\mathbb{N}, L^p(\Omega; X))} \lesssim \|x_0 - y_0\|_{L^p(\Omega; X)} + n^{-\delta} (1 + \|x_0\|_{L^p(\Omega; X_{\text{max}})}).$$

Thus, by Kolmogorov’s criterion, using that $\eta - \delta > \gamma + \frac{1}{p}$, and the fact that $T \leq \frac{2^k T}{n} < 2T$;

$$\left( \mathbb{E} \|u - u^{(n)}\|_{L^p_{\gamma}(\mathbb{N}, L^p(\Omega; X))} \right)^{\frac{1}{p}} \leq \left( \mathbb{E} \|d_j^{(n)}\|_{L^p_{\gamma}(\mathbb{N}, L^p(\Omega; X))} \right)^{\frac{1}{p}} \lesssim \left( \mathbb{E} \left( \left( d_j^{(n)} \right)_{j=0}^{2^k} \right)^{\frac{p}{p}} \right)^{\frac{1}{p}} \lesssim \left( \left( d_j^{(n)} \right)_{j=0}^{2^k} \right)^{\frac{1}{p}} \|x_0 - y_0\|_{L^p(\Omega; X)} + n^{-\delta} (1 + \|x_0\|_{L^p(\Omega; X_0)}).$$

Theorem 1.1 now follows from the fact that in there we assumed $y_0 = x_0$. \[\square\]

The corollary below is obtained from Theorem 1.1 by a Borel-Cantelli argument. An analogous corollary may be derived for the general abstract discretization schemes of Section 5, as well as for the modified and classical splitting schemes (under the condition $\gamma + \delta + \frac{1}{p} < \min\{\eta_{\text{max}}, \eta, 1\}$), assuming that $\theta_F, \theta_G \geq 0$ in the case of the latter.

Corollary 6.2. Let $\gamma, \delta \geq 0$, $\eta > 0$, and $p \in [2, \infty)$ be such that $\gamma + \delta + \frac{2}{p} < \min \{\zeta_{\text{max}}, \eta\}$. Suppose that $x_0 = y_0 \in L^p(\Omega, \mathcal{F}_0; X_0)$. Then there exists a random variable $\chi \in L^0(\Omega)$, independent of $x_0$ and $n$ such that:

$$\|u - u^{(n)}\|_{L^p_{\gamma}(\mathbb{N}, L^p(\Omega; X))} \leq \chi n^{-\delta} (1 + \|x_0\|_{L^p(\Omega; X_0)}).$$
Proof. We may assume that $\gamma + \delta + \frac{2}{p} < \eta < \min \{ \zeta_{\text{max}}, 1 \}$. Pick $\delta > 0$ such that $\delta + \frac{1}{p} < \delta < \zeta_{\text{max}} - \gamma - \frac{1}{p}$. By Chebyshev’s inequality and Theorem 1.1 (with $\delta$ instead of $\delta$) we have $\mathbb{P}(\Omega_n) \leq n^{-\frac{(\delta-\delta)^p}{p}}$, where

$$\Omega_n := \left\{ \omega \in \Omega : \|u(\omega) - v^{(n)}(\omega)\|_{c([0,T];X)} > n^{-\delta}(1 + \|x_0\|_{L^p(\Omega;X_n)}) \right\}$$

By assumption we have $\delta - \delta > \frac{1}{p}$, and therefore $\sum_n \mathbb{P}(\Omega_n) < \infty$. The corollary now follows from the Borel-Cantelli lemma.

In particular, if $\gamma, \delta \geq 0$ satisfy $\gamma + \delta < \min \{ \zeta_{\text{max}}, \eta \}$ for the discretization method of Section 5 ($\gamma + \delta < \min \{ \eta_{\text{max}}, \eta, 1 \}$ for the splitting methods), then for initial values $x_0 = y_0 \in L^\infty(\Omega, \mathcal{F}_0; X)$ we obtain:

$$\|u - v^{(n)}\|_{c([0,T];X)} \leq \chi n^{-\delta}(1 + \|x_0\|_{L^\infty(\Omega;X_n)}).$$

Remark 6.3. It is clear from the proof of Theorem 1.1 and Corollary 6.2 that the assertions remain valid if $V^{(n)}$ starts from an initial value $y_0^{(n)} \in L^p(\Omega, \mathcal{F}_0; X)$, provided that for all $n \in \mathbb{N}$ we have $\|x_0 - y_0^{(n)}\|_{L^p(\Omega;X)} \leq n^{-\delta}$.

Remark 6.4. At the cost of some extra work, for the approximations obtained by either the modified or the classical splitting scheme it is possible to obtain pathwise convergence over $[0, T]$ instead of over the grid points, i.e., convergence in $L^p(\Omega; C([0, T]; X))$ and in $C([0, T]; X)$ almost surely. The details are presented in [7].

7. The local case

The pathwise convergence result of Corollary 6.2 remains valid if $F$ and $G$ are merely locally Lipschitz and satisfy linear growth conditions. We shall demonstrate how this extension is obtained for the Euler scheme (or, more generally, the abstract time discretizations of Section 5). Analogous results may be obtained for modified and classical splitting scheme, but in order to do so one needs the extra regularity results mentioned in Remark 6.4.

Consider (SEE) in a umd Banach space $X$ with property $(\alpha)$, where $A$ satisfies $(A)$ and $F$ and $G$ satisfy:

(F$_{\text{loc}}$) For some $\theta_F > -1 + (\frac{1}{\tau} - \frac{1}{2})$, where $\tau$ is the type of $X$, the function $F : [0, T] \times X \to X_{\theta_F}$ is measurable in the sense that for all $x \in X$ the mapping $F(\cdot, x) : [0, T] \to X_{\theta_F}$ is strongly measurable. Moreover, $F$ is locally Lipschitz continuous and uniformly of linear growth in its second variable. That is to say, there exist constants $C_{0,m}, m \in \mathbb{N}$, and $C_1$ such that for all $t \in [0, T]$, all $m \in \mathbb{N}$ and all $x_1, x_2, x \in X$ such that $\|x_1\|, \|x_2\| \leq m$:

$$\|F(t, x_1) - F(t, x_2)\|_{X_{\theta_F}} \leq C_{0,m}\|x_1 - x_2\|_{X},$$

$$\|F(t, x)\|_{X_{\theta_F}} \leq C_1(1 + \|x\|_{X}).$$

(G$_{\text{loc}}$) For some $\theta_G > -\frac{1}{4}$, the function $G : [0, T] \times X \to \mathcal{Z}(H, X_{\theta_G})$ is measurable in the sense that for all $h \in H$ and $x \in X$ the mapping $G(\cdot, x)h : [0, T] \to X_{\theta_G}$ is strongly measurable. Moreover, $G$ is locally $L^2$-Lipschitz continuous and uniformly of linear growth in its second variable. That is to say, there exist constants $K_{G,m}, m \in N$, and $C_1$ such that for all $\alpha \in [0, \frac{1}{2})$, all
\( t \in [0, T], \) all \( m \in \mathbb{N} \) and all simple functions \( \phi_1, \phi_2, \phi : [0, T] \to X \) such that \( \| \phi_1 \|_{L^\infty(0,T;X)}, \| \phi_2 \|_{L^\infty(0,T;X)}, \| \phi \|_{L^\infty(0,T;X)} \leq m \) one has:

\[
\| \phi \|_{C([0,T];X)} + \left( \int_0^T \| \phi(\omega) \|_{L^p(I)} \right)^{\frac{1}{p}} < \infty.
\]

It has been proven in [30] that if one assumes \((F_{\text{loc}})\) and \((G_{\text{loc}})\) instead of \((F)\)
and \((G)\), and moreover assumes that \(x_0 \in L^0(\Omega, X)\), then for every \( p > 2 \) and
\( \alpha \in [0, \frac{1}{2}) \) satisfying \( \frac{1}{p} < \alpha + \theta_G \) equation (1.1) has a unique mild solution on in
\( V_{p,0}^\alpha(0, T, \Omega; X) \) for all \( T > 0 \). The solution is constructed by approximation;
uniqueness is proven separately.

The approximations are obtained as follows. For \( m \in \mathbb{N} \) we define \( F_m(t,x) := F(t, (1 + \frac{m}{|x|}) x) \)
and \( G_m(t,x) := G(t, (1 + \frac{m}{|x|}) x) \). Clearly \( F_m \) and \( G_m \) satisfy
\((F)\) and \((G)\). By [30, Theorem 6.2], for all \( p \in (2, \infty) \) and \( \alpha \in [0, \frac{1}{2}) \) satisfying
\( \frac{1}{p} < \alpha + \theta_G \) there exists a \( U_m \in V_{p,0}^\alpha(0, T, \Omega; X) \) that is a mild solution to:

\[
\begin{align*}
\frac{dU_m}{dt} &= AU_m(t) + F_m(t, U_m(t)) dt + G_m(t, U_m(t)) dW(t); \quad t \in [0, T], \\
U_m(0) &= 1_{\{\|x_0\| \leq m\}} x_0
\end{align*}
\]

Note that by uniqueness this solution corresponds to the solution given by Theorem (2.7).

Fix \( T > 0 \) and set

\( \tau_m^T(\omega) := \inf\{ t \geq 0 : \| U_m(t, \omega) \| \leq m \} \),

with the convention that \( \inf(\emptyset) = T \). By [30, Section 8] we have, due to the linear
growth conditions on \( F \) and \( G \), that

\[
\lim_{m \to \infty} \tau_m^T = T \quad \text{almost surely.}
\]

In fact, due to the fact that this holds for arbitrary \( T > 0 \), there exists a set \( \Omega_0 \subseteq \Omega \)
of measure one such that for all \( \omega \in \Omega_0 \) there exists an \( m_\omega \) such that \( \tau_m^T(\omega) = T \)
for all \( m \geq m_\omega \). Moreover, by a uniqueness argument one may show that for \( m_1 \leq m_2 \)
one has \( U_{m_1}(t) = U_{m_2}(t) \) on \([0, \tau_{m_1}^T] \).

The mild solution \( U \) to (SEE) with \( F \) and \( G \) satisfying \((F_{\text{loc}})\) and \((G_{\text{loc}})\) is
defined by setting:

\[
U(t, \omega) := \lim_{m \to \infty} U_m(t, \omega), \quad t \in [0, T], \ \omega \in \Omega_0,
\]

and \( U(t, \omega) := 0 \) for \( t \in [0, T] \) and \( \omega \in \Omega \setminus \Omega_0 \).

Fix \( x_0 \in L^0(\Omega; X) \) for some \( \eta > 0 \). Let \( \gamma, \delta \geq 0 \) be such that \( \gamma + \delta < \min\{\zeta_{\max}, \eta\} \), with \( \zeta_{\max} \) as defined by (5.2).
Let \( R > 0 \) be such that \( [R/T, \infty) \subseteq \)
If we desire results for the splitting scheme then we are faced with the problem of proving that the solutions to the equations (SEE) and (7.1) exist and are unique, which is a well-known fact. Here we denote by \(u_m\) and \(v_m\) the solutions of the discrete schemes (7.1) and 7.2, respectively, with initial data \(u_m(\cdot, x_0) = f(x_0)\) and \(v_m(\cdot, x_0) = g(x_0)\), where \(f, g\) are functions in \(L_2([0, T]; X)\) and \(L_2([0, T]; X)\), respectively, and \(X\) is a Banach space. Let \(x_0\) be a fixed point in \(X\).

Theorem 7.3. (Existence and uniqueness) Let \(f, g \in L_2([0, T]; X)\) and assume that \(a_2 \in C([0, T]; X)\), \(a_1 \in C([0, T]; X)\), \(b_1 \in C([0, T]; X)\), \(b_2 \in C([0, T]; X)\), and \(g \in L_2([0, T]; X)\). Then the discrete scheme (7.1) and (7.2) has a unique solution \(u_m\) and \(v_m\), respectively, for all \(m \geq 1\). Moreover, \(u_m\) and \(v_m\) converge to the solutions \(u_0\) and \(v_0\) of the continuous problems as \(m \to \infty\).

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first variable. More precisely, there exist constants $L_f$ and $L_g$ such that for all 
$t \in [0, T], \xi \in [0, 1]$, and $x, y \in \mathbb{R}$ we have:

$$|f(t, \xi, x) - f(t, \xi, y)| \leq L_f|x - y|, \quad |g(t, \xi, x) - g(t, \xi, y)| \leq L_g|x - y|.$$  

We also impose the linear growth conditions

$$|f(t, \xi, x)| \leq C(1 + |x|), \quad |g(t, \xi, x)| \leq C(1 + |x|),$$

with constant $C$ independent of $\xi \in [0, 1], t \in [0, T], x \in \mathbb{R}$.

Following the approach of [30, Section 10] we may rewrite this equation to fit in the functional-analytic framework of Section 2.4.1. For $\theta > 0$ and $1 < q < \infty$ we define $H^0,q := H^0,q(0, 1)$ for $0 < \theta < 1 + \frac{1}{q}$ and

$$H^0,q := \{u \in H^0,q(0, 1): b_1, \xi \frac{\partial u}{\partial \xi}(\xi, t) + b_0,\xi u(\xi, t) = 0; \xi \in \{0, 1\}\}$$

for $1 + \frac{1}{q} < \theta < \infty$.

The operator $A : H^0,q \to L^q(0, 1)$ defined by

$$Au := a_2 \frac{\partial^2 u}{\partial \xi^2} + a_1 \frac{\partial u}{\partial \xi}$$

generates an analytic $C_0$-semigroup on $L^q(0, 1)$, see [24, Section 3.1], which is based on [1].

From now on we take $(2, \infty)$ and fix $\beta \in (\frac{1}{2q}, \frac{1}{4})$. The part of $A$ in the space

$$X := H^{2\beta,q}_B = H^{2\beta,q}(0, 1)$$

generates an analytic $C_0$-semigroup in $X$ and $-A$ has bounded imaginary powers in $X$ by [25, Example 4.2.3]. By abuse of notation we shall denote the operator by $A$ again. As a consequence of [25, Theorem 4.2.6] and reiteration of the complex interpolation method, for $\theta \in (0, 1), 2\beta + 2\theta \neq 1 + \frac{1}{q}$, we have, for $0 < \theta < 1$,

$$X_\theta = [X, D(A)]_\theta = [H^{2\beta,q}_B, H^{2\beta+2\theta,q}_B]_\theta = H^{2\beta+2\theta,q}_B.$$

The reason for picking the space $X$ as our state space is two-fold. Firstly, we need a certain amount of space-regularity ($\beta > \frac{1}{2q}$) for proving that the Nemytskii operators $F$ and $G$ induced by $f$ and $g$ satisfy (F) and (G). Secondly, as we shall see in Theorem 8.1, there is a trade-off between the space regularity in which we consider convergence and the convergence rate: as $\beta$ increases to $\frac{1}{4}$, the convergence rate decreases. Beyond this critical value we are no longer able to prove convergence.

Observe that $X$ is a UMD space, and since we assume $q > 2$ the type of $X$ equals $\tau = 2$. Set $H := L^2 := L^2(0, 1)$. For $t \in [0, T], u \in X$, and $h \in H$ we define the Nemytskii operators

$$F(t, u)(\xi) := f(t, \xi, u(\xi));$$

$$(G(t, u)h)(\xi) := g(t, \xi, u(\xi))h(\xi).$$

Set $\theta_F := -\beta$ and pick $\varepsilon > 0$ sufficiently small such that $\theta_G := -\frac{1}{4} - \beta - \varepsilon > -\frac{1}{2}$. Under the above assumptions on $f$ and $g$, it was shown in the proof of [30, Theorem 10.2] (here we use that $\beta > \frac{1}{2q}$) that $F$ defines a mapping from $[0, T] \times X$ to $X_{\theta_F} = L^q(0, 1)$ that satisfies (F) and $G$ defines a mapping from $[0, T] \times X$ to $\gamma(H, X_{\theta_G}) = \gamma(L^2, H_B^{\frac{1}{2}-2\varepsilon,q})$ that satisfies (G): the measurability conditions are satisfied due to the measurability of $f$ and $g$ (in the notation of [30] we take $E = L^q(0, 1)$ and $\eta = \beta$, so that $E_\eta = X$).
Furthermore, the part of the $A$ in $X$ satisfies $(A)$. Modeling the space-time white noise as an $H$-cylindrical Brownian motion $W_H$, we may rewrite (8.1) as follows:

\begin{equation}
\begin{aligned}
dU(t) &= AU(t) \, dt + F(t, U(t)) \, dt + G(t, U(t)) \, dW_H(t); \quad t \in [0, T], \\
U(0) &= u_0.
\end{aligned}
\end{equation}

In order to obtain convergence of the Euler scheme for $U$, we must ensure that $(F') \text{ and } (G')$ are satisfied. This requires extra assumptions on $f$ and $g$. Noting that $\eta_{\max} = \zeta_{\max} = \frac{1}{2} - \beta - \varepsilon$, we assume that there exists a constant $C$ such that for all $s \in [0, 1]$ and all $x \in \mathbb{R}$ we have:

$$
\| t \mapsto f(t, s, x) \|_{C^{\frac{1}{2} - \beta}_{([0, T])}} \leq C(1 + |x|);
$$

and

$$
\| t \mapsto g(t, s, x) \|_{C^{\frac{1}{2} - \beta}_{([0, T])}} \leq C(1 + |x|).
$$

By similar arguments that were used in [30, Section 10] to prove that $F$ and $G$ satisfy $(F)$ and $(G)$, one can use the above to prove that $F$ and $G$ satisfy $(F')$ and $(G')$.

Fix $T > 0$ and $n \in \mathbb{N}$, let $U$ be the mild solution of (8.2) on $[0, T]$ and set $u := (U(t_j^{(n)}))_{j=0}^{T/n}$ with $t_j^{(n)} = jT/n$.

**Theorem 8.1.** Let $p > 4$, $q > 2$, $\alpha > 0$, $\beta \in (\frac{1}{2q}, \frac{1}{3})$, and $\gamma$, $\delta \geq 0$ satisfy

$$
\beta + \gamma + \delta + \frac{1}{p} < \min\{1, \alpha\}.
$$

Fix $T > 0$. Let $U^{(n)}$ be defined by the modified splitting scheme with initial value $u_0 \in L^p(\Omega, \mathcal{F}_0; H^{2\alpha,q}(0,1))$, and let $V^{(n)}$ be defined by the implicit Euler scheme. Let $u^{(n)} := (U^{(n)}(t_j^{(n)}))_{j=0}^{T/n}$ and $v^{(n)} := (V_j^{(n)})_{j=0}^{T/n}$. Then:

\begin{align*}
(\mathbb{E}\|u - u^{(n)}\|_{C^{\frac{1}{2} - \beta}_{([0,T];H^{2\alpha,q}(0,1))}})^{\frac{1}{p}} &\lesssim n^{-\delta}(1 + \|u_0\|_{L^p(\Omega; H^{2\alpha,q}(0,1))}); \\
(\mathbb{E}\|u - v^{(n)}\|_{C^{\frac{1}{2} - \beta}_{([0,T];H^{2\alpha,q}(0,1))}})^{\frac{1}{p}} &\lesssim n^{-\delta}(1 + \|u_0\|_{L^p(\Omega; H^{2\alpha,q}(0,1))});
\end{align*}

with implied constant independent of $n$.

**Proof.** This follows from Theorems 1.2 and 1.1 with $X = H^{2\beta,q}$ and $\eta = \alpha - \beta - \frac{1}{2q}$. \(\square\)

By the Borel-Cantelli argument of the previous section, almost sure convergence in $c_{\gamma}(\Omega; H^{2\beta,q}(0,1))$ with rate $n^{-\delta}$ holds for the same initial values under the stronger assumption

(8.3) \quad $\beta + \gamma + \delta + \frac{2}{p} < \min\{\frac{1}{4}, \alpha\}$.

The Sobolev embedding theorem provides a continuous embedding $H^{2\beta,q}(0,1) \hookrightarrow C^\lambda[0,1]$ whenever $2\beta > \lambda + \frac{1}{q}$. Hence, under assumption (8.3) we obtain that for all $\lambda \geq 0$ such that $\lambda + 2\gamma + 2\delta + \frac{2}{p} + \frac{1}{q} < \min\{\frac{1}{4}, 2\alpha\}$, almost surely we have:

$$
\|u - u^{(n)}\|_{c_{\gamma}(\Omega; C^\lambda[0,1])} \lesssim n^{-\delta}(1 + \|u_0\|_{L^p(\Omega; H^{2\alpha,q}(0,1))}).
$$

Let us now take $\gamma = 0$ and suppose that

$$
\lambda + 2\delta < \frac{1}{2}.
$$
Suppose \( u_0 \in L^p(\Omega, \mathcal{F}_0; H^{2/\alpha}(0,1)) \), i.e. we take \( \alpha = \frac{1}{4} \). By picking \( p \) and \( q \) large enough, we have \( \lambda + 2\gamma + 2\delta + \frac{1}{p} + \frac{1}{q} < \frac{1}{2} = \min\{\frac{1}{2}, 2\alpha\} \). By the above we then obtain almost sure uniform convergence (with respect to the grid points \( t_j^{(n)} \)) in the space \( C^\lambda[0,1] \) with rate \( \delta \):

\[
\sup_{0 \leq t \leq n} \| u(t_j^{(n)}) - u_j^{(n)} \|_{C^\lambda[0,1]} \lesssim n^{-\delta}(1 + \| u_0 \|_{L^p(\Omega; H^{2/\alpha}(0,1))}) \quad \text{almost surely.}
\]

**Remark 8.2.** It is proven in [9] that the optimal convergence rate of a time discretization for the heat equation in one dimension with additive space-time white noise based on \( n \) equidistant time steps is \( n^{-\frac{1}{4}} \). This is under the assumption that the noise approximation of the \( n \)-th approximation is based only on linear combinations of \( (W_H(t_j^{(n)}))_{j=0}^n \). In the theorem above we obtain convergence rate \( n^{-\frac{1}{4} + \varepsilon} \) for \( \varepsilon > 0 \) arbitrarily small by taking \( \gamma = 0 \), \( \beta \) sufficiently small and \( p, q \) sufficiently large.

In [9] the authors also provide optimal convergence rates for the heat equation in one dimension with multiplicative space-time white noise, but these results concern simultaneous discretizations of time and space and are therefore not applicable to our situation.

**APPENDIX A. TECHNICAL LEMMAS**

Here we state and prove with two lemmas which give estimates for the \( \gamma \)-radonifying norm of stochastic and deterministic integral processes.

**Lemma A.1.** Let \( q \in [1, \infty] \), \( \frac{1}{q} + \frac{1}{q'} = 1 \), and let \((R, \mathcal{R}, \mu)\) be a finite measure space and \((S, \mathcal{S}, \nu)\) a \( \sigma \)-finite measure space. Let \( Y_1 \) and \( Y_2 \) be Banach spaces, and suppose \( \Psi_1 \in L^q(R, \gamma(S; Y_1)) \) and \( \Psi_2 \in L^{q'}(R, \mathcal{S}(Y_1, Y_2)) \) such that \( (r,s) \mapsto \Psi_2(r)\Psi_1(r,s) \) defines an element of \( L^1(R \times S; Y_2) \). Then:

\[
\left\| s \mapsto \int_R \Psi_2(r)\Psi_1(r,s) \, d\mu(r) \right\|_{\gamma(S; Y_2)} \leq \| \Psi_2 \|_{L^{q'}(R, \mathcal{S}(Y_1, Y_2))} \| \Psi_1 \|_{L^q(R, \gamma(S; Y_1))}.
\]

**Proof.** We first consider the case \( q \in [1, \infty) \). The \( L^1 \)-assumption guarantees that the integral on the left-hand side exists as a Bochner integral in \( Y_2 \) for \( \nu \)-almost all \( s \in S \). By (2.2) and the fact that \( q < \infty \) we may identify \( \Psi_1 \) with an element in \( \gamma(S; L^q(R; Y_1)) \), and by the Hölder inequality \( \Psi_2 \) induces a bounded operator from \( L^q(R; Y_1) \) to \( Y_2 \). Under these identifications, the expression inside the norm at left-hand side equals the operator \( \Psi_2 \circ \Psi_1 \in \gamma(S; Y_2) \) and the desired estimate is noting but the right ideal property for the \( \gamma \)-radonifying norm.

The case \( q = \infty \) now follows by an approximation argument. Suppose first \( \Psi_1 \in L^1 \cap L^{\infty}(R, \gamma(S; Y_1)) \) and \( \Psi_2 \in L^1 \cap L^{\infty}(R, \mathcal{S}(Y_1, Y_2)) \). The above we have:

\[
\left\| s \mapsto \int_R \Psi_2(r)\Psi_1(r,s) \, d\mu(r) \right\|_{\gamma(S; Y_2)} \leq \lim_{q_1, q_2 \uparrow 1^+} \left\| \Psi_2 \right\|_{L^{q'}(S; \mathcal{S}(Y_1, Y_2))} \| \Psi_1 \|_{L^q(R, \gamma(S; Y_1))} = \| \Psi_2 \|_{L^\infty(S; \mathcal{S}(Y_1, Y_2))} \| \Psi_1 \|_{L^1(R, \gamma(S; Y_1))}.
\]

The result for general \( \Psi_1 \in L^{\infty}(R, \gamma(S; Y_1)) \) and \( \Psi_2 \in L^1(R, \mathcal{S}(Y_1, Y_2)) \) follows by approximation. \( \square \)
The above lemma can be applied to prove the following generalization of [30, Proposition 4.5].

**Lemma A.2.** Let $X_1$ and $X_2$ be UMD Banach spaces. Let $(R, \mathcal{A}, \mu)$ be a finite measure space and $(\mathcal{S}, \mathcal{F}, \nu)$ a σ-finite measure space. Let $\Phi_1 : [0, T] \times \Omega \to L'(H, X_1)$, let $\Phi_2 \in L^1(R; L^2(X_1, X_2))$, and let $f \in L^\infty(R \times [0, T]; L^2(\mathcal{S}))$. If $\Phi_1$ is $L^p$-stochastically integrable for some $p \in (1, \infty)$, then

$$\|s \mapsto \int_0^T \int_R f(r, u)(s)\Phi_2(r)\Phi_1(u) \, d\mu(r) \, dW_H(u)\|_{L^p(\Omega; \gamma(S; X_2))} \lesssim \operatorname{ess sup}_{(r, u) \in R \times [0, T]} \|f(r, u)\|_{L^2(S)}\|\Phi_2\|_{L^1(R, L^2(X_1, X_2))}\|\Phi_1\|_{L^p(\Omega; \gamma(0, T; H, X_1))},$$

with implied depending only on $p$, $X_1$, $X_2$, provided the right-hand side is finite.

**Proof.** By [23, Corollary 2.17], for almost all $s \in \mathcal{S}$ the family $\{T_{s,u} : u \in [0, T]\}$ is $\gamma$-bounded in $L^\infty(S, X_2)$, where

$$T_{s,u} x = \int_R f(r, u)(s)\Phi_2(r) x \, d\mu(r).$$

Hence, by the $\gamma$-multiplier theorem (Theorem 2.2), for almost all $s \in \mathcal{S}$ the function $u \mapsto \int_R f(r, u)(s)\Phi_2(r)\Phi_1(u) \, d\mu(r)$ belongs to $L^p_{\nu}(\Omega; \gamma(0, T; H, X_2))$.

Moreover, by Theorem 2.2 in combination with Theorem 2.3 (note that UMD Banach spaces have non-trivial cotype) we have, for almost all $r \in R$;

$$u \mapsto (s \mapsto f(r, u)(s)\Phi_1(u)) \in L^p_{\nu}(\Omega; \gamma(0, T; \gamma(S, X_1))).$$

By the stochastic Fubini theorem, the isomorphism (2.2) and Lemma A.1 (with $q = \infty$, $Y_1 = L^p(\Omega; X_1)$ and $Y_2 = L^p(\Omega; X_2)$), to $\Psi(r, s) = \int_0^T f(r, u)(s)\Phi_1(u) \, dW_H(u)$ and $\Psi_2 = \Phi_2$ we have:

(A.1)

$$\|s \mapsto \int_0^T \int_R f(r, u)(s)\Phi_2(r)\Phi_1(u) \, d\mu(r) \, dW_H(u)\|_{L^p(\Omega; \gamma(S; X_2))} \lesssim \|s \mapsto \int \Phi_2(r) \int_0^T f(r, u)(s)\Phi_1(u) \, dW_H(u) \, d\mu(r)\|_{\gamma(S; L^p(\Omega; X_2))} \lesssim \|\Phi_2\|_{L^1(R, L^2(X_1, X_2))} \|s \mapsto \int_0^T f(r, u)(s)\Phi_1(u) \, dW_H(u)\|_{L^\infty(R, \gamma(S; L^p(\Omega; X_1)))}.$$

By isomorphism (2.2), Theorem 2.1, and Theorem 2.2 in combination with Theorem 2.3 we have, for almost all $r \in R$ with implicit constants independent of $r$:

$$\|s \mapsto \int_0^T f(r, u)(s)\Phi_1(u) \, dW_H(u)\|_{\gamma(S; L^p(\Omega; X_1))} \lesssim \|u \mapsto (s \mapsto f(r, u)(s)\Phi_1(u))\|_{L^p(\Omega; \gamma(0, T; \gamma(S, X_1)))} \leq \operatorname{ess sup}_{u \in [0, T]} \|f(r, u)\|_{L^2(0, T)}\|\Phi_1\|_{L^p(\Omega; \gamma(0, T; X_1))}. $$

The result now follows by inserting the above estimate into (A.1). \hfill \square

We proceed with two lemmas on Besov embeddings. The proof of the first lemma is closely related to the proof of [30, Lemma 3.1].
Lemma A.3. Suppose $Y$ is a Banach space with type $\tau \in [1, 2)$, and let $\alpha \in [0, \frac{1}{2})$ and $q \in (2, \infty)$ satisfy $\frac{1}{q} < \frac{1}{\tau} - \alpha$. Let $\Phi \in B_{q,\tau}^{\frac{1}{q} - \frac{1}{2}}(0, T; Y) \cap L^\infty(0, T; Y)$ and, for $t \in [0, T]$, define $\Phi_{\alpha, t} : (0, t) \to Y$ by

$$\Phi_{\alpha, t}(s) = (t - s)^{-\alpha} \Phi(s).$$

Then there exists an $\varepsilon_0 > 0$ such that for all $T_0 \in [0, T]$: 

$$\sup_{0 \leq t \leq T_0} \|\Phi_{\alpha, t}\|_{B_{q,\tau}^{\frac{1}{q} - \frac{1}{2}}(0, t; Y)} \lesssim T_0^\varepsilon \|\Phi\|_{L^\infty(0, T_0; Y) \cap B_{q,\tau}^{\frac{1}{q} - \frac{1}{2}}(0, T_0; Y)}.$$

Proof. We shall in fact prove the following stronger result, namely that there exists an $\varepsilon_0 > 0$ such that for all $T_0 \in [0, T]$: 

$$\sup_{0 \leq t \leq T_0} \|\Phi_{\alpha, t}\|_{B_{q,\tau}^{\frac{1}{q} - \frac{1}{2}}(0, t; Y)} \lesssim T_0^\varepsilon \|\Phi\|_{L^\infty(0, T_0; Y) \cap B_{q,\tau}^{\frac{1}{q} - \frac{1}{2}}(0, T_0; Y)}.$$

On the left-hand side above, we think of $\Phi_{\alpha, t}$ as being extended identically zero outside the interval $(0, t)$.

Let $q' \in (1, \infty)$ be such that $\frac{1}{q} + \frac{1}{q'} = \frac{1}{\tau}$. As we assumed $\frac{1}{q} < \frac{1}{\tau} - \alpha$ it follows that $\alpha q' < 1$. Thus we can pick $\varepsilon > 0$ such that $\varepsilon < \min \{\frac{1}{\tau} - \alpha, 1 - \alpha q'\}$.

Fix $t \in [0, T_0]$. Let $\rho \in (0, 1]$ and let $0 < h < \rho$ (we only consider the case $h > 0$; the case $h < 0$ can be dealt with by observing that $\|T_h^\rho f - f\|_{L^p(\mathbb{R}, Y)} = \|T_h^\rho f - f\|_{L^p(\mathbb{R}, Y)}$). Fix the case that $h \leq t$. In that case we have:

$$\|T_h^\rho (\Phi_{\alpha, t}) - \Phi_{\alpha, t}\|_{L^\tau(\mathbb{R}, Y)}$$

$$\leq \|s \mapsto [(t - s - h)^{-\alpha} 1_{[-h,t-h]}(s) - (t - s)^{-\alpha} 1_{[0,t-h]}(s)]\Phi(s + h)\|_{L^\tau(\mathbb{R}, Y)}$$

$$+ \|s \mapsto (t - s)^{-\alpha} 1_{[0,t]}(s)\Phi(s + h)1_{[0,t-h]}(s) - \Phi(s)\|_{L^\tau(\mathbb{R}, Y)}$$

$$\leq \|s \mapsto [(t - s - h)^{-\alpha} 1_{[-h,t-h]}(s) - (t - s)^{-\alpha} 1_{[0,t-h]}(s)]\Phi(\|L^\tau(\mathbb{R}, Y))$$

$$+ \|s \mapsto (t - s)^{-\alpha} 1_{[0,t]}(s)\|_{L^\tau(\mathbb{R}, Y)}\|T_h^\rho (\Phi)1_{[0,t-h]} - \Phi\|_{L^\tau(0,t; Y)}.$$

As $\alpha q' < 1$ we have:

$$\|s \mapsto (t - s)^{-\alpha} 1_{[0,t]}(s)\|_{L^\tau(\mathbb{R}, Y)} \lesssim T_0^{\frac{\tau}{\tau'}}.$$

For $p \geq 1$ and $0 \leq b \leq a$ one has $(a - b)^p \leq a^p - b^p$ and thus:

$$\|s \mapsto [(t - s - h)^{-\alpha} 1_{[-h,t-h]}(s) - (t - s)^{-\alpha} 1_{[0,t-h]}(s)]\|_{L^\tau(\mathbb{R})}$$

$$= \left( \int_{-h}^{t-h} \{(t - s - h)^{-\alpha} - (t - s)^{-\alpha} 1_{[0,t-h]}(s)\}^\tau ds \right)^{\frac{1}{\tau}}$$

$$\leq \left( \int_{-h}^{t-h} [(t - s - h)^{-\alpha\tau} - (t - s)^{-\alpha\tau} 1_{[0,t-h]}(s)] ds \right)^{\frac{1}{\tau}}$$

$$= (1 - \alpha \tau)^{-\frac{1}{\tau}} h^{\frac{1}{\tau} - \alpha} \lesssim h^{\frac{1}{\tau} - \alpha - \varepsilon} T_0^\varepsilon,$$

where the last inequality uses $h \leq t \leq T_0$.

Putting together these estimates,

$$\|T_h^\rho (\Phi_{\alpha, t}) - \Phi_{\alpha, t}\|_{L^\tau(\mathbb{R}, Y)}$$

$$\leq h^{\frac{1}{\tau} - \alpha - \varepsilon} T_0^\varepsilon \|\Phi\|_{L^\infty(0,T_0; Y)} + T_0^{\frac{\tau}{\tau'}} \|T_h^\rho (\Phi)1_{[0,t-h]} - \Phi\|_{L^\tau(0,t; Y)}.$$
Next suppose $h > t$. In that case:

$$\|T_h^\alpha (\Phi_{\alpha,t}) - \Phi_{\alpha,t}\|_{L^\infty(0,T;Y)} = 2\|\Phi_{\alpha,t}\|_{L^\infty(0,T;Y)} \leq 2T_0^\frac{\alpha}{2} h^\frac{1}{2} \leq T_0^\frac{\alpha}{2} t^\frac{1}{2} \leq T_0^\frac{\alpha}{2} T^\frac{\alpha}{2} h^\frac{1}{2} \leq h^\frac{1}{2} \alpha \|\Phi\|_{L^\infty(0,T;Y)},$$

this time using $t^{\frac{1}{2}-\alpha} \leq t^{\frac{1}{2}-\alpha - \varepsilon} T_0^\varepsilon$ and $t \leq h$.

It follows that

$$\|\Phi_{\alpha,t}\|_{B_{1,1}^{\frac{1}{2}-\alpha}(0,T;Y)} \leq T_0^\frac{1}{2} \|\Phi\|_{L^\infty(0,T;Y)} + T_0^\frac{1}{2} \sup_{\alpha,t \leq h} \|\Phi_{\alpha,t}\|_{L^\infty(0,T;Y)} \leq T_0^\frac{1}{2} \|\Phi\|_{L^\infty(0,T;Y)} + T_0^\frac{1}{2} \sup_{\alpha,t \leq h} \|\Phi_{\alpha,t}\|_{L^\infty(0,T;Y)}.$$
In order to estimate the Besov norm on the right-hand side of (A.2) we fix $\rho \in (0, 1)$, and let $|h| < \rho$. We have, with $I = [0, T]$, $\|T^I_h \Phi(s) - \Phi(s)\|_{L^p(\Omega; \gamma(H, X_{\xi}))}$

\[
\leq \begin{cases} 
\|G(s + h, B_{2n/T}) - G(s, B_{2n/T})\|_{L^p(\Omega; \gamma(H, X_{\xi}))}, & s + h \neq s, \ s + h \in [0, T], \\
2|\Phi|_{L^\infty(0, T; L^p(\Omega; \gamma(H, X_{\xi}))))}, & \text{otherwise.}
\end{cases}
\]

For $|h| \geq \frac{T}{n}$ one never has $s + h = s$ and thus it follows from the above and (A.3) that

\[
\|T^I_h \Phi - \Phi\|_{L^q(0, T; L^p(\Omega; \gamma(H, X_{\xi}))))} \lesssim n^{-\frac{\zeta_{max}}{\gamma} + \frac{q}{\gamma} \varepsilon} (1 + \sup_{0 \leq j \leq n} \|B_j\|_{L^p(\Omega; X)}) \\
\lesssim |h|^\frac{q}{\gamma} n^{-\frac{\zeta_{max}}{\gamma} + \frac{q}{\gamma} \varepsilon} (1 + \sup_{0 \leq j \leq n} \|B_j\|_{L^p(\Omega; X)}).
\]

On the other hand, for $h < \frac{T}{n}$ and $s + h = s$ we obtain, by (G'):

\[
\|G(s + h, B_{2n/T}) - G(s, B_{2n/T})\|_{L^p(\Omega; \gamma(H, X_{\xi})))} \\
\lesssim |h|^{\frac{q}{\gamma}} n^{-\frac{\zeta_{max}}{\gamma} + \frac{q}{\gamma} \varepsilon} \left(1 + \sup_{0 \leq j \leq n} \|B_j\|_{L^p(\Omega; X)}\right) \\
\lesssim |h|^{\frac{q}{\gamma}} n^{-\frac{\zeta_{max}}{\gamma} + \frac{q}{\gamma} \varepsilon} \left(1 + \sup_{0 \leq j \leq n} \|B_j\|_{L^p(\Omega; X)}\right).
\]

For $|h| < \frac{T}{n}$ observe that $|\{s \in [0, T] : s + h \neq s\}| = n|h|$. Thus for $|h| < \frac{T}{n}$ we have, by the above estimate and (A.3):

\[
\|T^I_h \Phi - \Phi\|_{L^q(0, T; L^p(\Omega; \gamma(H, X_{\xi}))))} \\
\lesssim (T - n|h|)^\frac{q}{\gamma} |h|^{\frac{q}{\gamma}} n^{-\frac{\zeta_{max}}{\gamma} + \frac{q}{\gamma} \varepsilon} \left(1 + \sup_{0 \leq j \leq n} \|B_j\|_{L^p(\Omega; X)}\right) \\
\hspace{2cm} + (n|h|)^\frac{q}{\gamma} n^{-\frac{\zeta_{max}}{\gamma} + \frac{q}{\gamma} \varepsilon} \left(1 + \sup_{0 \leq j \leq n} \|B_j\|_{L^p(\Omega; X)}\right) \\
\lesssim |h|^{\frac{q}{\gamma}} n^{-\frac{\zeta_{max}}{\gamma} + \frac{q}{\gamma} \varepsilon} \left(1 + \sup_{0 \leq j \leq n} \|B_j\|_{L^p(\Omega; X)}\right).
\]

Collecting these estimates we find:

\[
\sup_{|h| \leq \rho} \|T^I_h \Phi - \Phi\|_{L^q(0, T; L^p(\Omega; \gamma(H, X_{\xi}))))} \lesssim \rho^{\frac{q}{\gamma}} n^{-\frac{\zeta_{max}}{\gamma} + \frac{q}{\gamma} \varepsilon} \left(1 + \sup_{0 \leq j \leq n} \|B_j\|_{L^p(\Omega; X)}\right).
\]

Because $\frac{1}{q} > \frac{1}{\gamma} - \frac{1}{2}$ it follows that

\[
\|\Phi\|_{L^q(0, T; L^p(\Omega; \gamma(H, X_{\xi}))))} \\
\lesssim \|\Phi\|_{L^q(0, T; H(\gamma, L^p(\Omega; X_{\xi}))))} + n^{-\frac{\zeta_{max}}{\gamma} + \frac{q}{\gamma} \varepsilon} \left(1 + \sup_{0 \leq j \leq n} \|B_j\|_{L^p(\Omega; X)}\right) \\
\lesssim n^{-\frac{\zeta_{max}}{\gamma} + \frac{q}{\gamma} \varepsilon} \left(1 + \sup_{0 \leq j \leq n} \|B_j\|_{L^p(\Omega; X)}\right).
\]

Inserting the above and (A.3) into (A.2) gives the required result. \(\square\)

The final lemma is an elementary calculus fact.
Lemma A.5. For all $0 \leq \delta, \theta < \frac{1}{2}$ there exists a constant $C$, depending only on $\delta$ and $\theta$, such that for all $0 \leq u \leq t$, all $T > 0$ and all $n \in \mathbb{N}$:

$$ \int_u^t (t-s)^{-2\theta} (\delta-u)^{-2\delta} \, ds \leq C^2 (t+\frac{T}{n} - u)^{1-2\delta-2\theta}. $$

Proof. If $t-u \leq \frac{T}{n}$, then for $s \in [u,t)$ one has $\delta-u = \delta-u = \frac{T}{n}$ so

$$ \int_u^t (t-s)^{-2\theta} (\delta-u)^{-2\delta} \, ds = (1-2\theta)^{-1} (\frac{T}{n})^{-2\delta} (t-u)^{1-2\theta}. $$

Note that $\frac{T}{n} \geq \frac{1}{2} (t+\frac{T}{n} - u)$ and $(t-u)^{1-2\theta} \leq (t+\frac{T}{n} - u)^{1-2\theta}$. Thus:

$$ \int_u^t (t-s)^{-2\theta} (\delta-u)^{-2\delta} \, ds \leq 2^{2\delta} (1-2\theta)^{-1} (t+\frac{T}{n} - u)^{1-2\delta-2\theta}. $$

On the other hand if $t-u > \frac{T}{n}$ then $t-u < t+T/n - u < 2(t-u)$. Moreover, $\delta-u \geq s-u$, and the substitution $v = (s-u)/(t-u)$ gives:

$$ \int_u^t (t-s)^{-2\theta} (\delta-u)^{-2\delta} \, ds \leq \int_u^t (t-s)^{-2\theta} (s-u)^{-2\delta} \, ds 
\leq (t-u)^{1-2\delta-2\theta} \int_0^1 (1-v)^{-2\theta} v^{-2\delta} \, dv 
\leq 2^{(2\delta+2\theta-1)^+} (t+\frac{T}{n} - u)^{1-2\delta-2\theta} \int_0^1 (1-v)^{-2\theta} v^{-2\delta} \, dv. $$

\[ \square \]

APPENDIX B. ESTIMATES FOR STOCHASTIC CONVOLUTIONS

We shall present two estimates for stochastic convolutions. Throughout this section, $Y$ is a umd Banach space and $\tau \in (1,2]$ denotes its type. Moreover, $S$ is an analytic semigroup on $Y$.

Roughly speaking, Lemma B.1 is contained in Step 2 of the proof of [30, Proposition 6.1], but there the space $V^\alpha_p([0,T] \times \Omega; X)$ is considered (see (2.14)). For completeness we give the proof below.

Lemma B.1. Let $\delta \in (-\frac{3}{2} + \frac{1}{p}, \infty)$, $\alpha \in [0, \frac{1}{2})$, and $p \in [2, \infty)$. For all $\Phi \in L^\infty(0,T; L^p(\Omega; Y))$, the convolution $S * \Phi$ belongs to $V^\alpha_p([0,T] \times \Omega; X)$, and for all $T_0 \in [0,T]$ we have:

$$ \|S * \Phi\|_{V^\alpha_p([0,T_0] \times \Omega; Y)} \leq (T_0^{1+\delta} + T_0^{\alpha+\delta}) \|\Phi\|_{L^\infty(0,T_0; L^p(\Omega; Y))}. $$

Proof. By analyticity of the semigroup (equation (2.6)) we have, for $t \in [0,T_0]$

$$ \|S * \Phi\|_{L^p(\Omega; Y)} \leq \int_0^t (t-s)^{\delta \wedge 0} \, ds \|\Phi\|_{L^\infty(0,T_0; L^p(\Omega; Y))} 
\leq T_0^{1+\delta \wedge 0} \|\Phi\|_{L^\infty(0,T_0; L^p(\Omega; Y))}. $$

Taking the supremum over $t \in [0,T_0]$ gives the estimate in $L^\infty(0,T_0; L^p(\Omega; Y))$.

It remains to prove the estimate in the weighted $\gamma$-norm. Fix $t \in [0,T_0]$. As $p \geq 2$, it follows that $L^p(\Omega; Y)$ has type $\tau \in [1,2]$ whenever $Y$ has type $\tau$. Moreover, if we interpret $A$ as an operator on $L^p(\Omega; Y)$ acting pointwise, then $(L^p(\Omega; Y))_{\delta} = \|S * \Phi\|_{V^\alpha_p([0,T_0] \times \Omega; Y)} \leq (T_0^{1+\delta} + T_0^{\alpha+\delta}) \|\Phi\|_{L^\infty(0,T_0; L^p(\Omega; Y))}. $$

\[ \square \]
Thus by [30, Proposition 3.5] with \( E = L^p(\Omega, Y) \), \( \eta = 0 \), and \( \theta = -\delta \) we have, as \( \delta > -\frac{3}{2} + \frac{1}{7} \):

\[
\| s \mapsto (t - s)^{-\alpha}(S \ast \Phi)(s)\|_{L^p(0; \Omega; L^p(\Omega, Y))} \lesssim T_0^{\frac{1}{2} - \alpha} \| \Phi \|_{L^\infty(0; T_0; L^p(\Omega; Y_\delta))}.
\]

Taking the supremum over \( t \in [0, T_0] \) gives the desired estimate. \( \square \)

We proceed with the second Lemma.

**Lemma B.2.** Let \( \delta \in (-\frac{1}{2}, \infty) \) and \( \alpha \in [0, \frac{1}{2}) \). Suppose \( \Phi : [0, T] \times \Omega \to \mathcal{L}(H, Y_\delta) \) is strongly measurable and adapted and satisfies

\[
(B.1) \quad \sup_{0 \leq t \leq T} \| s \mapsto (t - s)^{-\alpha}\Phi(s)\|_{L^p(\Omega; \gamma(0; t; H; Y_\delta))} < \infty,
\]

for some \( p \in (1, \infty) \).

(i) If \( 0 \leq \beta < \min\{\frac{1}{2} - \alpha, \frac{1}{2} + \delta\} \), then there exists an \( \epsilon > 0 \) such that for all \( T_0 \in [0, T] \):

\[
\sup_{0 \leq t \leq T_0} \| s \mapsto (t - s)^{-\alpha - \beta}\int_0^s S(s - u)\Phi(u) \, dW_H(s)\|_{L^p(\Omega; \gamma(0; t; Y))} \lesssim T_0^\epsilon \sup_{0 \leq t \leq T_0} \| s \mapsto (t - s)^{-\alpha}\Phi(s)\|_{L^p(\Omega; \gamma(0; t; H; Y_\delta))}.
\]

(ii) If, moreover, \( \alpha > -\delta \), then there exists an \( \epsilon > 0 \) such that for all \( T_0 \in [0, T] \):

\[
\| s \mapsto \int_0^s S(s - u)\Phi(u) \, dW_H(u)\|_{Y^{\alpha + \beta, p}(0, T_0)]} \lesssim T_0^\epsilon \sup_{0 \leq t \leq T_0} \| s \mapsto (t - s)^{-\alpha}\Phi(s)\|_{L^p(\Omega; \gamma(0; t; H; Y_\delta))}.
\]

**Proof.** Fix \( t \in [0, T_0] \). Let \( \epsilon > 0 \) be such that \( \epsilon < \frac{1}{2} - \delta - \beta \). Here \( \delta^- = (-\delta) \vee 0 \). We apply Lemma A.2 with \( X_1 = Y_\delta \), \( X_2 = Y \), \( R = S = [0, t] \), and the functions \( \Phi_1(u) = (t - u)^{-\alpha}\Phi(u) \), \( \Phi_2(r) = \frac{d}{dr}[r^\beta + \epsilon S(r)] \), and \( f(r, u)(s) = (t - s)^{-\alpha - \beta}(s - u)^{-\delta - \epsilon}1_{0 \leq r \leq s - u} \). By (2.6) we have \( \| \Phi_2(r)\|_{\mathcal{L}(X_1, X)} \lesssim r^{-1 + \epsilon} \) for \( r \in [0, T] \). From the lemma it follows that:

\[
\| s \mapsto (t - s)^{-\alpha - \beta}\int_0^s S(s - u)\Phi(u) \, dW_H(u)\|_{L^p(\Omega; \gamma(0; t; Y))} \lesssim t^{\frac{1}{2} - \beta - \delta^-} \| s \mapsto (t - s)^{-\alpha}\Phi(s)\|_{L^p(\Omega; \gamma(0; t; H; Y_\delta))}.
\]

Taking the supremum over \( t \in [0, T_0] \) we obtain (i).

For the estimate in \( Y^{\alpha + \beta, p}\)-norm it remains, by part (i), to prove the estimate in \( L^\infty(0, T_0; L^p(\Omega, Y_\delta)) \). Let \( \epsilon < \min\{\alpha + \delta, \frac{1}{2} - \delta^- - \beta\} \). By Lemma 2.4 (apply part (1) if \( \delta \in (-\frac{3}{2}, 0] \) and part (2) if \( \delta \in [0, \infty) \)) the operators \( r^\alpha S(r), r \in [0, t] \), are \( \gamma \)-bounded from \( Y_\delta \) to \( Y \), with \( \gamma \)-bound at most \( Ct^{\alpha + \delta} \) with \( C \) independent of \( t \in [0, T] \). Hence, by the \( \gamma \)-multiplier theorem, for all \( t \in [0, T] \),

\[
\left\| \int_0^t S(t - s)\Phi(s) \, dW_H(s) \right\|_{L^p(\Omega; Y)} \lesssim t^{\alpha + \delta} \| s \mapsto (t - s)^{-\alpha}\Phi(s)\|_{L^p(\Omega; \gamma(0; t; H; Y_\delta))}.
\]

The norm estimate in \( L^\infty(0, T; L^p(\Omega; Y_\delta)) \) is obtained by taking the supremum over \( t \in [0, T] \). \( \square \)
APPENDIX C. Existence and uniqueness

The aim of this section is to outline the proof of Theorem 2.7. The setting is always that of Section 2.

Proof of Theorem 2.7. Assume first that \( \alpha \in [0, \frac{1}{2}) \) is so large that \( \alpha + \theta_G > \eta \). Let \( p \in [2, \infty) \) and \( T_0 \in [0, T] \) be fixed. For \( \Phi \in \mathcal{Y}_\infty^{\alpha, p}([0, T_0] \times \Omega; X_\eta) \) define

\[
L(\Phi)(t) := S(t)\alpha + \int_0^t S(t-s)F(s, \Phi(s))\,ds + \int_0^t S(t-s)G(s, \Phi(s))\,dW_H(s).
\]

Copying Step 1 of the proof of [30, Proposition 6.1] without changes, and substituting Steps 2 and 3 by the Lemmas B.1 and B.2 above, we find that there exists an \( \varepsilon_0 > 0 \) and a \( C > 0 \) such that \( L : \mathcal{Y}_\infty^{\alpha, p}([0, T_0] \times \Omega; X_\eta) \to \mathcal{Y}_\infty^{\alpha, p}([0, T_0] \times \Omega; X_\eta) \) and

\[
\|L(\Phi)\|_{\mathcal{Y}_\infty^{\alpha, p}([0, T_0] \times \Omega; X_\eta)} \leq C\|x_0\|_{X_\eta} + CT_0^{\alpha}\|F(\cdot, \Phi(\cdot))\|_{L^\infty(0, T_0; L^p(\Omega; X_{\eta p}))} + CT_0^{\alpha} \sup_{0 \leq t \leq T_0} \|s \mapsto (t-s)^{-\alpha} G(s, \Phi(s))\|_{L^p(\Omega; \gamma(0, t; X_{\eta G}))}
\]

\[
\leq C\|x_0\|_{X_\eta} + C(M(F) + M(G))T_0^{\alpha}(1 + \|\Phi\|_{\mathcal{Y}_\infty^{\alpha, p}([0, T_0] \times \Omega; X_\eta)}),
\]

where in the last line we used (F) and (2.11). Moreover,

\[
\|L(\Phi_1) - L(\Phi_2)\|_{\mathcal{Y}_\infty^{\alpha, p}([0, T_0] \times \Omega; X_\eta)} \leq C(\text{Lip}(F) + \text{Lip}_q(G))T_0^{\alpha}\|\Phi_1 - \Phi_2\|_{\mathcal{Y}_\infty^{\alpha, p}([0, T_0] \times \Omega; X_\eta)},
\]

where in the last line we used (2.10).

Thus by a fixed-point argument, for sufficiently small \( T_0 \) there exists a unique process \( \Phi \in \mathcal{Y}_\infty^{\alpha, p}([0, T_0] \times \Omega; X_\eta) \) satisfying (2.12) on the interval \([0, T_0] \). By repeating this construction a finite number of times, each time taking the final value of the previous step as the initial value of the next, we obtain a solution on \([0, T] \).

So far, we have proved existence and uniqueness under the additional assumption \( \alpha + \theta_G > \eta \). Existence in \( \mathcal{Y}_\infty^{\alpha, p}([0, T] \times \Omega; X_\eta) \) for arbitrary \( \alpha \in [0, \frac{1}{2}) \) follows by from (2.8). It remains to prove uniqueness for arbitrary \( \alpha \in [0, \frac{1}{2}) \).

Let \( \alpha \in [0, \frac{1}{2}) \) be arbitrary and let \( \Phi \in \mathcal{Y}_\infty^{\alpha, p}([0, T] \times \Omega; X_\eta) \). Viewing \( F \) as a mapping from \([0, T] \times X_\eta \) to \( X_{\eta p} \) (as \( \eta \geq 0 \)), we have \( F(\cdot, \Phi(\cdot)) \in \mathcal{Y}_\infty^{\alpha, p}([0, T] \times \Omega; X_{\eta p}) \). Then, by Lemma B.1 with \( \delta = \theta_F - \eta \) and \( Y = X_\eta \) (and \( \tilde{\alpha} = \alpha + \beta \)), we find \( S \ast F(\cdot, \Phi(\cdot)) \in \mathcal{Y}_\infty^{\alpha + \beta, p}([0, T] \times \Omega; X_\eta) \) for all \( \beta \in [0, \frac{1}{2} - \delta] \).

By Lemma B.2 (with \( Y = X_\eta \) and \( \delta = \theta_G - \eta \) and (2.11) we have, for all \( \beta \in (0, \frac{1}{2} - \alpha) \) such that \( \beta < \frac{1}{2} + \theta_G - \eta \):

\[
\sup_{0 \leq t \leq T} \left\| s \mapsto (t-s)^{-\alpha - \beta} \int_0^s S(s-u)G(u, \Phi(u))\,dW_H(u) \right\|_{L^p(\Omega; \gamma(0, t; X_{\eta G}))} \leq \sup_{0 \leq t \leq T} \left\| s \mapsto (t-s)^{-\alpha} G(s, \Phi(s)) \right\|_{L^p(\Omega; \gamma(0, t; H_{\eta G}))} \leq \|\Phi\|_{\mathcal{Y}_\infty^{\alpha, p}([0, T] \times \Omega; X_\eta)}.
\]

Since also \( S(\cdot)x_0 \in \mathcal{Y}_\infty^{\alpha + \beta, p}([0, T] \times \Omega; X_\eta) \) for all \( \beta \in [0, \frac{1}{2} - \alpha) \), we see that if \( \alpha \in [0, \frac{1}{2}) \) and \( \Phi \in \mathcal{Y}_\infty^{\alpha, p}([0, T] \times \Omega; X_\eta) \) satisfies (2.12), then \( \Phi \in \mathcal{Y}_\infty^{\alpha + \beta, p}([0, T] \times \Omega; X) \) for all \( \beta \in [0, \frac{1}{2} - \alpha) \) such that \( \beta < \frac{1}{2} + \theta_G - \eta \). Repeating this argument a finite number of steps if necessary, we obtain that \( \Phi \in \mathcal{Y}_\infty^{\alpha + \beta, p}([0, T] \times \Omega; X) \) for all
$\beta \in [0, \frac{1}{2} - \alpha)$. As uniqueness of a process in $\mathcal{Y}^{\alpha,p}(0, T) \times \Omega; X)$ satisfying (2.12) has been established for $\alpha > \eta - \theta_{G}$ this completes the proof. □

**Remark C.1.** Inspection of the proofs of the main theorems reveals that uniqueness is only used for large $\alpha \in [0, \frac{1}{2})$. As a consequence, the last part of the above proof is not needed for our purposes. It has been included for completeness reasons.

**References**


