DELFT UNIVERSITY OF TECHNOLOGY
I.T.S. Mathematics Departments

Quiz 2: Complex Numbers II

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The following functions can be used in this quiz:

acos, asin, atan, cos, cot, exp, ln, log, sin, sqrt, tan,

with acos=arccos, asin=arcsin, atan=arctan, cot=cotan, exp(x)=e^x and sqrt(x)=$\sqrt{x}$. Also the number e is known and one may write $\pi =$ p. Multiplication is denoted by $*$ and powers use ^. For example

$$2e^{\frac{1}{3}\sin(x)} = \texttt{2 * e^((1/3) * sin(x))}.$$

Click on Begin Quiz to start. Answers are available after End Quiz.

Begin Quiz Answer each of the following questions.

1. If z is a solution of the equation $\left(z^3+3\right)^2 = -16$ then we have:

$|z| =$ ______

2. Compute the solution of $w^3 = 8i$ that lies in the first quadrant of the complex plane. Write $w =$ a + i * b.

$w =$ ______

Correct answer:

The equation $z^7 + 2z^3 + 2z^2 + 3 = 0$ has …

3. …at least one positive real solution:

☐ true, ☐ false.

4. …at least one negative real solution:

☐ true, ☐ false.

5. …at most 7 complex solutions:

☐ true, ☐ false.

A numerical procedure approximates three roots:
$z \approx -0.771 + 1.052i$, $z \approx 0.245 + 0.876i$ and $z \approx 1.100 + 0.804i$.

6. …exactly 7 different complex solutions:

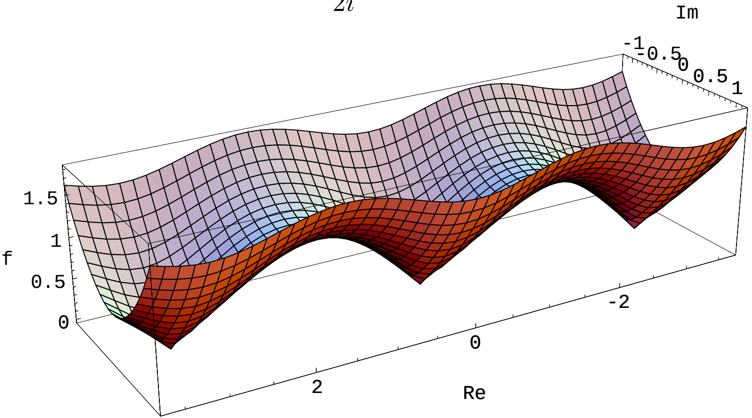
☐ true, ☐ false, ☐ indecisive.

7. How many different solutions in $\mathbb{C}$ has $z^6 - 3z^4 + 9z^2 = 0$?

☐

Correct answer:

Reminder: $\sin(z) = \dfrac{e^{iz} - e^{-iz}}{2i}$ for $z \in \mathbb{C}$.



8. The graph of $f : \mathbb{C} \to \mathbb{R}$ above belongs to $f(z) =$

☐ $|\sin(z)|$ ☐ $|Re\,(sin(z))|$ ☐ $|Im\,(sin(z))|$.

9. Write the solution in $\mathbb{C}$ of $e^z = 1 + i$ that has its argument in $(-2\pi, 0)$ as $z = \texttt{x} + \texttt{i} * \texttt{y}$.

$z =$ [ ]

10. If $\dfrac{z^4 - z^2 + 1}{z^2 + \sqrt{3}z + 1} = z^2 + az + 1$ for all $z \in \mathbb{C}$, then

$a =$ [ ]

End Quiz | Score: | corrections

Correct answer:

After finishing the quiz one may browse through the solutions on the following pages. Also shift-click on **Ans** *jumps to the answer.*

Solutions to Quiz

Solution to Question 1.

Since $\left(z^3+3\right)^2 = -16$ is equivalent with $z^3+3 = \pm 4i$ one finds $z^3 = -3 + \pm 4i$. Hence

$$\left|z^3\right| = \sqrt{3^2 + (\pm 4)^2} = 5$$

which implies $|z| = 5^{1/3}$. Back to Question 1

Solution to Question 2.
Since $|w^3| = 8$ one finds $|w| = 8^{\frac{1}{3}} = 2$.
Since $\arg(w^3) = \frac{1}{2}\pi$ one finds

$$\arg(w) = \frac{1}{6}\pi + \frac{1}{3}2k\pi i \text{ with } k \in \mathbb{Z}.$$

The only solution in the first quadrant is for $k = 0$, hence

$$w = |w|e^{\arg(w)i} = 2e^{\frac{1}{6}\pi i} = \sqrt{3} + i.$$

Back to Question 2

Solution to Question

3. For $z \in \mathbb{R}^+$ one finds $f(z) := z^7 + 2z^3 + 2z^2 + 3 > 3$ so no positive real solution exists.

4. Since $f(x) < 0$ for $x < -10$ and $f(0) = 3 > 0$ there exists a zero of f in $(-10, 0)$, that is, a negative real solution exists.

5. Since f has degree 7 it has, counting with multiplicities, 7 complex solutions. Without counting multiple zeroes, that is, counting only different solutions, one concludes that there are at most 7 solutions.

6. If $z = a + bi$ is a solution of a real polynomial then also $z = a - bi$ is one. Since the numerical procedure gives 3 complex solutions with positive real part there are also 3 complex solutions with negative real part. With the real solution this makes 7. Back to Question 6

Solution to Question 7.
Since the degree of the polynomial is 6 there are at most 6 solutions. One immediately finds by

$$z^6 - 3z^4 + 9z^2 = z^2\left(z^4 - 3z^2 + 9\right)$$

that $z = 0$ is a double root. Hence there are at most 5 different solutions. To see that indeed 5 different solutions exists one solves $z^4 - 3z^2 + 9 = 0$. Since

$$z^4 - 3z^2 + 9 = \left(z^2 - \tfrac{3}{2}\right)^2 + \tfrac{27}{4}$$

one finds

$$z^2 = \tfrac{3}{2} + \tfrac{3}{2}\sqrt{3}i \text{ or } z^2 = \tfrac{3}{2} - \tfrac{3}{2}\sqrt{3}i.$$

Hence 4 more solutions (4 different arguments) which makes 5 different solutions altogether.

One may even compute these solutions: by $\left|z^2\right| = 3$ and $\arg\left(z^2\right) = \pm\frac{1}{3}\pi$ the four non-zero solutions follow: (with $k \in \mathbb{Z}$)

$$z = \sqrt{3}e^{(k\pm\frac{1}{6})\pi i} = \pm\tfrac{3}{2} \pm \tfrac{1}{2}\sqrt{3}i.$$

Back to Question 7

Solution to Question 8.

Let us write $z = x + iy$.

Considering the real axis ($y = 0$) one finds that

$$\sin(x + iy) = \sin(x).$$

Hence $|\mathrm{I}m\,(sin(x))| = 0$ which kills the third possibility.

Considering the imaginary axis ($x = 0$) one obtains

$$\sin(x + iy) = \frac{e^{-y} - e^{y}}{2i} = i \sinh y$$

and $|\mathrm{R}e\,(sin(iy))| = 0.$ Only the first possibility remains.

By the way, if one wants to make the plot one may rewrite $|\sin(z)|$ to a real function by

$$\sin(x + iy) = \ldots = \sin(x)\cosh(y) + i\cos(x)\sinh(y)$$

and

$$|\sin(x + iy)| = \cdots = \sqrt{\sin(x)^2 + \sinh(y)^2}.$$

Back to Question 8

Solution to Question 9.

One writes $z = x + iy$ and solves for the modulus and the argument separately:

$$\left|e^{x+iy}\right| = e^x \text{ and } |1+i| = \sqrt{2}.$$

Hence $x = \log(\sqrt{2}) = \frac{1}{2}\log(2)$.

$$\arg(e^{x+iy}) = y(\text{mod} 2\pi) \text{ and } \arg(1+i) = \frac{1}{4}\pi.$$

Hence $y = -\frac{7}{4}\pi$ and one concludes with $z = \frac{1}{2}\log(2) - \frac{7}{4}\pi i$.

Back to Question 9

Solution to Question 10.
A direct computation gives:

$$
\begin{array}{rcl}
z^2+\sqrt{3}z+1 & \Big/ & \begin{array}{l}
z^4 - z^2 + 1 \\
\underline{z^4+\sqrt{3}z^3+z^2} \\
-\sqrt{3}z^3 - 2z^2 + 1 \\
\underline{-\sqrt{3}z^3 - 3z^2 - \sqrt{3}z} \\
\qquad\quad z^2+\sqrt{3}z+1 \\
\qquad\quad \underline{z^2+\sqrt{3}z+1} \\
\qquad\qquad\qquad\quad 0
\end{array} \Big\backslash \quad z^2-\sqrt{3}z+1
\end{array}
$$

Hence $a = -\sqrt{3}$.

Back to Question 10