DELFT UNIVERSITY OF TECHNOLOGY
I.T.S. Mathematics Departments

Quiz 5: Differentiation I

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The following functions can be used in this quiz:

`acos`, `asin`, `atan`, `cos`, `cot`, `exp`, `ln`, `log`, `sin`, `sqrt`, `tan`,

with `acos`=arccos, `asin`=arcsin, `atan`=arctan, `cot`=cotan, `exp`$(x)=e^x$ and `sqrt`$(x)=\sqrt{x}$. Also the number `e` is known and one may write $\pi =$ `p`. Multiplication is denoted by $*$ and powers use ^. For example

$$2e^{\frac{1}{3}\sin(x)} = \texttt{2 * e\^{}((1/3) * sin(x))}.$$

Click on Begin Quiz to start. Answers are available after End Quiz.

Begin Quiz Answer each of the following questions.

1. The function $f(x) = \arctan(x) - \ln\left(1 + x^2\right)$ attains its maximum for $x =$ [ ]

2. Compute the largest number c such that the function f defined on $\mathbb{R}$ by $f(x) = e^x - \arctan\left(1 + e^x\right)$ satisfies

$$f(x) \geq c \text{ for all } x \in \mathbb{R}.$$

Then $c =$ [ ]

Correct answer:

3. Here are some claims for functions defined on $\mathbb{R}$:

A. A differentiable function is continuous;
B. A continuous function is differentiable;
C. An increasing function is continuous;
D. If $x \mapsto f(x)$ is a continuous function, then so is $x \mapsto |f(x)|$;
E. If $x \mapsto f(x)$ is a differentiable function, then so is $x \mapsto |f(x)|$;
F. If $x \mapsto f(x)$ is a continuous function, then $x \mapsto (f(x))^2$ is differentiable.

The correct claim(s) are: [ ]

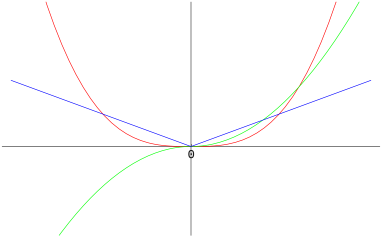
4. Compute the tangent line in $(1, 2)$ to the curve defined by

$$x^3 + 2x^2y^2 - y^3 = 1.$$

Then $y =$ [ ]

Correct answer:

5. Given is some $n \in \mathbb{N}$. The function $f(x) = \sqrt{|x|^n}$ and its first and second derivatives f' and f'' are plotted:



Then $n =$ ☐

6. Let $\ell(x)$ denote the linearization of $f(x) = \sqrt{x}$ around $a = 36$.

$\ell(x) =$ ☐

Correct answer:

7. If $f : \mathbb{R} \to \mathbb{R}$ is a function which is differentiable in $a \in \mathbb{R}$, then it holds that:

$$\text{I.} \quad \lim_{x \to a} f(x) = f(a),$$
$$\text{II.} \quad \lim_{x \to a} f'(x) = f'(a).$$

☐ I and II hold true; ☐ I holds true, II is false;

☐ I is false, II holds true; ☐ I and II are false.

8. Let $f : \mathbb{R} \to \mathbb{R}$ be a differentiable function with a continuous derivative. Suppose $f(0) = 0$, $f(3) = 3$ and $f(5) = 9$. Give the range $[a, b]$ of values of which we are certain that f' attains.

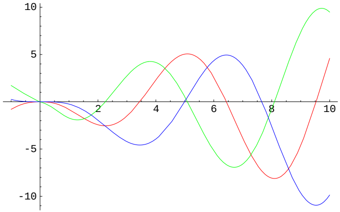
Fill in: `[a,b]` = ☐

9. If $f(x) = \sin\left(e^{\cos(x)}\right) =$ `sin(e∧(cos(x)))`,

then $f'(x) =$ ☐

Correct answer:

10. Here are graphs of some function f and its first and second derivatives f' and f''.



The graphs of f, f' and f'' are colored in

(answer by `red, green, blue` in the correct order)

End Quiz | Score: | corrections

Correct answer:

After finishing the quiz one may browse through the solutions on the following pages. Also shift-click on **Ans** *jumps to the answer.*

Solutions to Quiz

Solution to Question 1.

Since the function is differentiable it suffices to consider points with zero derivative and 'boundary points'. Here there are no boundary points and we have to consider the behaviour for $x \to \pm\infty$.

$$\lim_{x\to\infty}\left(\arctan x - \ln\left(1+x^2\right)\right) = -\infty = \lim_{x\to-\infty}\left(\arctan x - \ln\left(1+x^2\right)\right).$$

We may conclude that there exists a maximum and since $f'(x) = \frac{1}{1+x^2} - \frac{2x}{1+x^2} = \frac{1-2x}{1+x^2}$ the only candidate is $x = \frac{1}{2}$.

One could also proceed by the sign of $f'(x)$.

Back to Question 1

Solution to Question 2.

The derivative of $f(x) = e^x - \arctan(1 + e^x)$ is

$$f'(x) = e^x - \frac{e^x}{1 + (1 + e^x)^2} = e^x \left(1 - \frac{1}{1 + (1 + e^x)^2} \right).$$

Since $f'(x) > 0$ one finds that f is increasing. In other words, if such a c exists then

$$c = \lim_{x \to -\infty} f(x) = -\arctan 1 = -\tfrac{1}{4}\pi.$$

Back to Question 2

Solution to Question 3.

A. True, check the definitions and use

$$|f(x) - f(y)| = \left|\frac{f(x) - f(y)}{x - y}\right| (x - y).$$

B. A standard counter example is $f(x) = |x|$ which is continuous but not differentiable in 0.

C. Think of f defined by $f(x) = 1$ for $x > 0$ and $f(x) = 0$ for $x \leq 0$.

D. If $x \mapsto g(x)$ and $x \mapsto f(x)$ are continuous on $\mathbb{R}$ then so is $x \mapsto g(f(x))$. Here $g(x) = |x|$.

E. Again think of $|x|$.

F. Here the function $x \mapsto |x|$ doesn't supply a counterexample. The function $g : x \mapsto |x| + 1$ does:

$$(g(x))^2 = |x|^2 + 2|x| + 1 = x^2 + 2|x| + 1$$

has right-derivative equal to 2 at $x = 0$ and left-derivative -2.

Back to Question 3

Solution to Question 4.
If y is implicitly defined as a function of x by $x^3 + 2x^2y^2 - y^3 = 1$, then differentiating both sides of

$$x^3 + 2x^2 y\,(x)^2 - y\,(x)^3 = 1$$

gives

$$3x^2 + 4x\, y(x)^2 + 4x^2\, y(x)\, y'(x) - 3y(x)^2 y'(x) = 0.$$

Hence we may solve $y'(1)$ knowing $(x, y\,(x)) = (1, 2)$:

$$3 + 16 + 8y'(1) - 12y'(1) = 0,$$

implying $y'(1) = \frac{19}{4}$. The tangent line is defined by

$$\ell(x) = y(1) + (x - 1)\, y'(1) = 2 + (x - 1)\, \tfrac{19}{4}.$$

Going back to coordinates x and y we find

$$y = 2 + \tfrac{19}{4}\,(x - 1)\,.$$

Back to Question 4

Solution to Question 5.
For $x > 0$ one has $\sqrt{|x|^n} = x^{\frac{1}{2}n}$ and since one of the functions is a straight line for $x > 0$, either $f(x) = x$, $f'(x) = 2x$ or $f''(x) = 6x$. Hence $\frac{1}{2}n$ is either 1, 2 or 3.

In the first case $f'(x) = 1$ and $f''(x) = 0$ for $x > 0$; in the second case $f(x) = x^2$ and $f''(x) = 2$; in the third case $f(x) = x^3$ and $f'(x) = 3x^2$. Only the third case is appropriate, that is, $n = 6$. The function can be simplified to

$$f(x) = \sqrt{x^6} = |x|^3.$$

For $x < 0$ one has $f(x) = -x^3$ and hence $f'(x) = -3x^2$ and $f''(x) = -6x$. Hence f is red, f' is green and f'' is blue.

Back to Question 5

Solution to Question 6.
The linearisation of $f(x) = \sqrt{x}$ at $a = 36$ is

$$\ell(x) = f(36) + (x - 36) f'(36) = 6 + \tfrac{1}{12}(x - 36).$$

Notice that $f'(x) = \frac{1}{2}x^{-\frac{1}{2}}$ implies $f'(36) = \frac{1}{12}$.

Back to Question 6

Solution to Question 7.
If f is differentiable it is continuous, and continuous in a means $\lim_{x\to a} f(x) = f(a)$.

Differentiable does not imply that the derivative is continuous. It is not easy to find a counter example by oneself. A standard counter example which can be found in most books on calculus is

$$f(x) = \begin{cases} x^2 \sin\left(\frac{1}{x}\right) & \text{for } x \neq 0, \\ 0 & \text{for } x = 0. \end{cases}$$

For $x \neq 0$ one proceeds by standard differentiation rules:

$$f'(x) = 2x \sin\left(\frac{1}{x}\right) - \cos\left(\frac{1}{x}\right).$$

For $x = 0$ one cannot avoid the definition:

$$f'(0) = \lim_{h\to 0} \frac{f(h) - f(0)}{h} = \lim_{h\to 0} \frac{h^2 \sin\left(\frac{1}{h}\right) - 0}{h} = \lim_{h\to 0} h \sin\left(\frac{1}{h}\right) = 0.$$

Notice that $\lim_{x\to 0} f'(x)$ does not exist. Back to Question 7

Solution to Question 8.
By the Mean Value Theorem we may conclude that there is $x_1 \in (1, 3)$ such that $f'(x_1) = \frac{3-0}{3-0} = 1$. Similarly, there is a $x_2 \in (3, 5)$ such that $f'(x_2) = \frac{9-3}{5-3} = 3$.

By the Intermediate Value Theorem for f', a continuous function, all values between $f'(x_1)$ and $f'(x_2)$ are attained. The answer is $[1, 3]$. Of other values for the derivative we cannot be certain.

Back to Question 8

Solution to Question 9.

A straightforward but rather headache producing application of the chain-rule:

$$\frac{d}{dx}\left(\sin\left(e^{\cos x}\right)\right) = \cos\left(e^{\cos x}\right)\frac{d}{dx}\left(e^{\cos x}\right) =$$

$$= \cos\left(e^{\cos x}\right)e^{\cos x}\frac{d}{dx}\cos x = -\cos\left(e^{\cos x}\right)e^{\cos x}\sin x.$$

Back to Question 9

Solution to Question 10.
Where f is decreasing f' is negative and similarly where f' is decreasing f'' is negative.

Now consider green; its derivative in 0 is negative. Since both red and blue are 0 they cannot be green's derivative. Hence green is f''.

If f'' (green) equals 0, then f' has a horizontal tangent. Hence f' is red. This fits with blue: if f' (red) equals 0, blue has a horizontal tangent.

Back to Question 10