DELFT UNIVERSITY OF TECHNOLOGY
I.T.S. Mathematics Departments

Quiz 6: Differentiation II

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The following functions can be used in this quiz:

acos, asin, atan, cos, cot, exp, ln, log, sin, sqrt, tan,

with acos=arccos, asin=arcsin, atan=arctan, cot=cotan, exp(x)=e^x and sqrt(x)=$\sqrt{x}$. Also the number e is known and one may write π = p. Multiplication is denoted by * and powers use ^. For example

$$2e^{\frac{1}{3}\sin(x)} = \texttt{2 * e^((1/3) * sin(x))}.$$

Click on Begin Quiz to start. Answers are available after End Quiz.

Begin Quiz Answer each of the following questions.

1. The function $f : [-1, 1] \to \mathbb{R}$ defined by

$$f(x) = \arcsin(x) - \sqrt{1 - x^2}$$

attains its maximum for

$x =$ [ ]

2. The f from question 1 attains its minimum for

$x =$ [ ]

Correct answer:

3. Is f from question 1, with $f(x) = \arcsin(x) - \sqrt{1-x^2}$, ...
... differentiable from the right in $x = -1$?
... differentiable from the left in $x = 1$?

- ☐ twice yes;
- ☐ twice no;
- ☐ yes, resp. no;
- ☐ no, resp. yes.

4. Suppose that $f : \mathbb{R} \to \mathbb{R}$ is differentiable and that $a, b \in \mathbb{R}$ exist such that $\ell(x) = ax + b$ is a slant asymptote for $x \to \infty$. Then:

A. $\lim\limits_{x\to\infty} (f(x) - ax - b) = 0$;

B. $\lim\limits_{x\to\infty} f'(x) = a$;

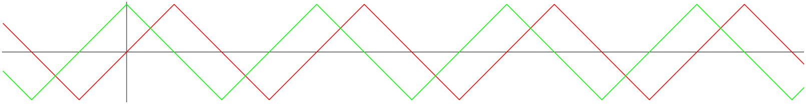
C. $\lim\limits_{x\to\infty} \left(\dfrac{f(x)}{x}\right) = a$.

The correct claim(s) are: ☐

Correct answer:

5. Here are the plots of

$$f(x) = \arcsin(\sin(x)) \text{ and } g(x) = \arcsin(\cos(x)).$$



The function g is differentiable on $\mathbb{R}$ except for all multiples of

6. Let $\ell(x)$ denote the linearisation of $f(x) = \sin x$ around $a = \frac{1}{3}\pi$.

$\ell(x) =$

7. Compute the tangent line in $(x, y) = \left(\frac{1}{6}\pi, \frac{1}{3}\pi\right)$ to the curve implicitely defined by

$$\cos(x) + \cos(y) = \frac{1}{2} + \frac{1}{2}\sqrt{3}$$

Then $y =$

Correct answer:

8. Compute a and b such that the line $\ell(x) = ax + b$ satisfies:

I. ℓ is a tangent line to $f(x) = 2 - x^2$ in some $x_0 \in \mathbb{R}$;

II. $(2, -1)$ lies on the graph of ℓ;

III. $\ell(1) > 2$.

Then $\ell(x) =$ [ ]

9. We consider the function $f : \mathbb{R}^+ \to \mathbb{R}$ defined by $f(x) = x^x$.

Then $\lim_{x \downarrow 0} f(x) =$ [ ]

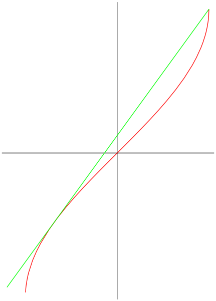
10. For f as in 9, $f'(x) =$ [ ]

11. ... and $\lim_{x \downarrow 0} f'(x) =$ [ ]

Choose from the real numbers or write `+infinity`, `-infinity` or `undefined` (in $\mathbb{R} \cup \{-\infty, \infty\}$).

Correct answer:

12. The graph of the line $\ell : x \mapsto ax+b$ intersects the graph of $f : x \mapsto \arcsin(x)$ exactly twice, namely for $x_1 = 1$ and for some $x_0 \in (-1, 0)$. We consider $\left(-\frac{1}{2}, f(-\frac{1}{2})\right)$ and claim that



☐ $a < \frac{4}{9}\pi$ ☐ $a = \frac{4}{9}\pi$ ☐ $a > \frac{4}{9}\pi$

End Quiz | Score: | corrections

After finishing the quiz one may browse through the solutions on the following pages. Also shift-click on **Ans** *jumps to the answer.*

Solutions to Quiz

Solution to Question 1.

Since $f(x) = \arcsin(x) - \sqrt{1-x^2}$ is continuous on $[-1,1]$ and differentiable on $(-1,1)$ the maximum is attained either if $f'(x) = 0$ or for the boundary points $x = \pm 1$. Since

$$f'(x) = \frac{1}{\sqrt{1-x^2}} + \frac{x}{\sqrt{1-x^2}} = \frac{1+x}{\sqrt{1-x^2}}$$

this means the candidates are $x \in \{-1, 1\}$. Since

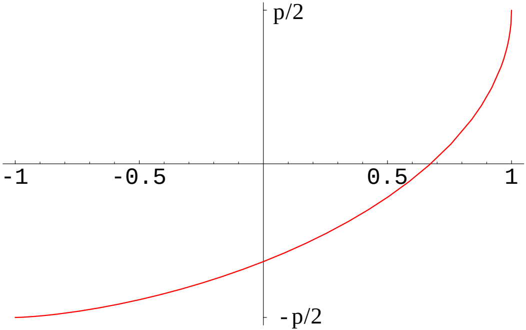
$$f(-1) = -\tfrac{1}{2}\pi,\ f(\tfrac{1}{2}) = \tfrac{1}{6}\pi - \tfrac{1}{2}\sqrt{3} < 0 \text{ and } f(1) = \frac{1}{2}\pi$$

the maximum is attained in 1.

The next page contains a picture of f. Back to Question 1

Solution to Question 2.

Using what we know from 1 the only now remaining candidate for the location of the minimum is $x = -1$. Back to Question 2



Solution to Question 3.

From the picture one might guess that the answer should be yes, respectively no. Indeed this is true. We proceed by the definition.

- First the one on the left.

$$f_r'(-1) = \lim_{x \downarrow -1} \frac{f(x) - f(-1)}{x - (-1)} = \lim_{x \downarrow -1} \frac{\arcsin(x) - \sqrt{1 - x^2} + \frac{1}{2}\pi}{x + 1}.$$

Substituting $\sin y = x$ we get $\sqrt{1 - (\sin y)^2} = \cos y$

$$f_r'(-1) = \lim_{y \downarrow -\frac{1}{2}\pi} \frac{y - \sqrt{1 - (\sin y)^2} + \frac{1}{2}\pi}{\sin y + 1} = \lim_{y \downarrow -\frac{1}{2}\pi} \frac{y - \cos y + \frac{1}{2}\pi}{\sin y + 1}.$$

Since denominator and enumerator both are differentiable and equal 0 for $y = -\frac{1}{2}\pi$ we may use l' Hospital's rule to find, whenever the right hand side exists, that

$$\lim_{y \downarrow -\frac{1}{2}\pi} \frac{y - \cos y + \frac{1}{2}\pi}{\sin y + 1} = \lim_{y \downarrow -\frac{1}{2}\pi} \frac{1 + \sin y}{\cos y}.$$

So before making a conclusing we have to consider this last limit. Since the denominator and enumerator of that one both are again differentiable and equal 0 for $y = -\frac{1}{2}\pi$ we may use l' Hospital's rule to find, whenever the right hand side exists, that

$$\lim_{y \downarrow -\frac{1}{2}\pi} \frac{1 + \sin y}{\cos y} = \lim_{y \downarrow -\frac{1}{2}\pi} \frac{\cos y}{\sin y} = \frac{0}{-1} = 0.$$

Hence $\lim\limits_{y \downarrow -\frac{1}{2}\pi} \frac{1+\sin y}{\cos y}$ exists and equals 0 and hence $\lim\limits_{y \downarrow -\frac{1}{2}\pi} \frac{y - \cos y + \frac{1}{2}\pi}{\sin y + 1}$ exists and equals 0.

- In a similar fashion $f_l'(1)$:

$$f_\ell'(1) = \lim_{x \uparrow 1} \frac{f(x) - f(1)}{x - 1} = \lim_{x \downarrow -1} \frac{\arcsin(x) - \sqrt{1 - x^2} - \frac{1}{2}\pi}{x - 1}$$

and substituting $\sin y = x$ we get $\sqrt{1 - (\sin y)^2} = \cos y$ and with a next substitution $s = \frac{1}{2}\pi - y$

$$f_\ell'(1) = \lim_{y \uparrow \frac{1}{2}\pi} \frac{y - \cos y - \frac{1}{2}\pi}{\sin y - 1} = \lim_{s \downarrow 0} \frac{-s - \cos\left(\frac{1}{2}\pi - s\right)}{\sin\left(\frac{1}{2}\pi - s\right) - 1} =$$

$$= \lim_{s \downarrow 0} \frac{-s - \sin s}{\cos s - 1} = \lim_{s \downarrow 0} \frac{-s - \sin s}{-2 \left(\sin \frac{1}{2} s\right)^2} =$$

$$= \lim_{s \downarrow 0} \left(\frac{\frac{1}{2} s}{\sin \frac{1}{2} s} + \frac{\frac{1}{2} \sin s}{\sin \frac{1}{2} s} \right) \frac{1}{\sin \frac{1}{2} s} = \lim_{s \downarrow 0} \left(\frac{\frac{1}{2} s}{\sin \frac{1}{2} s} + \cos \frac{1}{2} s \right) \frac{1}{\sin \frac{1}{2} s}. \quad (1)$$

We used that $\cos x = 1 - 2 \left(\sin \frac{1}{2} x\right)^2$ and $\sin x = 2 \cos(\frac{1}{2} x) \sin(\frac{1}{2} x)$. Since

$$\lim_{s \downarrow 0} \frac{\frac{1}{2} s}{\sin \frac{1}{2} s} = 1 = \lim_{s \downarrow 0} \cos \frac{1}{2} s \text{ and } \quad \lim_{s \downarrow 0} \frac{1}{\sin \frac{1}{2} s} = +\infty$$

the limit in (1) does not exist in $\mathbb{R}$. Hence f is not differentiable from the left in 1. Back to Question 3

Solution to Question 4.

The function f has a slant asymptote at ∞ if there exists a linear function $\ell(x) = ax + b$ such that

$$\lim_{x\to\infty} |f(x) - \ell(x)| = 0. \tag{2}$$

Hence A holds true by definition.

Consider $f(x) = x + \frac{1}{x}\sin\left(x^2\right)$. This function has $\ell(x) = x$ for an asymptote but $f'(x)$ does not converge to 1 when $x \to \infty$. B does not hold.

If (2) holds then

$$\lim_{x\to\infty} \frac{f(x)}{x} = \lim_{x\to\infty} \frac{1}{x}(f(x) - (ax + b)) + a + \frac{b}{x} = 0.0 + a + 0 = a.$$

C holds true. Back to Question 4

Solution to Question 5.
The function arcsin is the inverse of the sinus function on $\left[-\frac{1}{2}\pi, \frac{1}{2}\pi\right]$.

Hence if $-\pi \le x \le 0$, then $-\frac{1}{2}\pi \le x + \frac{1}{2}\pi \le \frac{1}{2}\pi$ and

$$\arcsin(\cos x) = \arcsin\left(\sin\left(x + \tfrac{1}{2}\pi\right)\right) = x + \tfrac{1}{2}\pi.$$

We find $g'(x) = 1$ for $x \in (-\pi, 0)$ and $g'_\ell(0) = 1 = g'_r(-\pi)$.

If $0 \le x \le \pi$, then $-\frac{1}{2}\pi \le -x + \frac{1}{2}\pi \le \frac{1}{2}\pi$ and

$$\arcsin(\cos x) = \arcsin(\cos(-x)) = \arcsin\left(\sin\left(-x + \tfrac{1}{2}\pi\right)\right) = -x+\tfrac{1}{2}\pi.$$

We find $g'(x) = -1$ for $x \in (0, \pi)$ and $g'_\ell(\pi) = -1 = g'_r(0)$.

Since $x \mapsto \cos x$ is 2π-periodic so is $x \mapsto \arcsin(\cos x)$ and we may conclude that g is differentiable on $\mathbb{R}$ except for all multiples of π.

Back to Question 5

Solution to Question 6.
The linearisation of $f(x) = \sin x$ at $a = \frac{1}{3}\pi$ is

$$\ell(x) = f\left(\tfrac{1}{3}\pi\right) + \left(x - \tfrac{1}{3}\pi\right) f'\left(\tfrac{1}{3}\pi\right).$$

Hence

$$\ell(x) = \sin\left(\tfrac{1}{3}\pi\right) + \left(x - \tfrac{1}{3}\pi\right)\cos\left(\tfrac{1}{3}\pi\right) = \tfrac{1}{2}\sqrt{3} + \tfrac{1}{2}\left(x - \tfrac{1}{3}\pi\right).$$

Back to Question 6

Solution to Question 7.

Supposing that y is a function of x we differentiate

$$\cos(y(x)) + \cos x = \tfrac{1}{2} + \tfrac{1}{2}\sqrt{3}$$

to find

$$-\sin(y(x))\, y'(x) - \sin x = 0.$$

For $(x, y(x)) = \left(\tfrac{1}{6}\pi, \tfrac{1}{3}\pi\right)$ it follows that

$$-\sin\left(\tfrac{1}{3}\pi\right) y'(\tfrac{1}{6}\pi) - \sin\left(\tfrac{1}{6}\pi\right) = 0$$

implying

$$y'(\tfrac{1}{6}\pi) = -\frac{\sin\left(\tfrac{1}{6}\pi\right)}{\sin\left(\tfrac{1}{3}\pi\right)} = -\tfrac{1}{3}\sqrt{3}.$$

Hence the tangent line becomes $\ell(x) = \tfrac{1}{3}\pi - \tfrac{1}{3}\sqrt{3}\left(x - \tfrac{1}{6}\pi\right)$.

Back to Question 7

Solution to Question 8.
Given that ℓ is a tangent line to f at x_0 implies

$$\begin{aligned}\ell(x) &= f(x_0) + (x - x_0)\, f'(x_0) \\ &= \left(2 - x_0^2\right) - (x - x_0)\, 2x_0 = 2 + x_0^2 - 2x_0\, x.\end{aligned}$$

Next $\ell(2) = -1$ implies $2 + x_0^2 - 2x_0\, 2 = -1$ and this second order equation for x_0 can be simplified to

$$x_0^2 - 4x_0 + 2 = -1$$

and is solved by $x_0 = 1$ and $x_0 = 3$.
If $x_0 = 1$ then $\ell(x) = 3 - 2x$ and $\ell(1) = 1 < 2$.
If $x_0 = 3$ then $\ell(x) = 11 - 6x$ and $\ell(1) = 5 > 2$. Hence

$$\ell(x) = 11 - 6x.$$

Back to Question 8

Solution to Question 9.

The function $f(x) = x^x$ can be rewritten as $f(x) = e^{x \log x}$. Then

$$\lim_{x \downarrow 0} e^{x \log x} = e^{\lim_{x \downarrow 0} x \log x} = e^0 = 1.$$

10. Differentiating gives

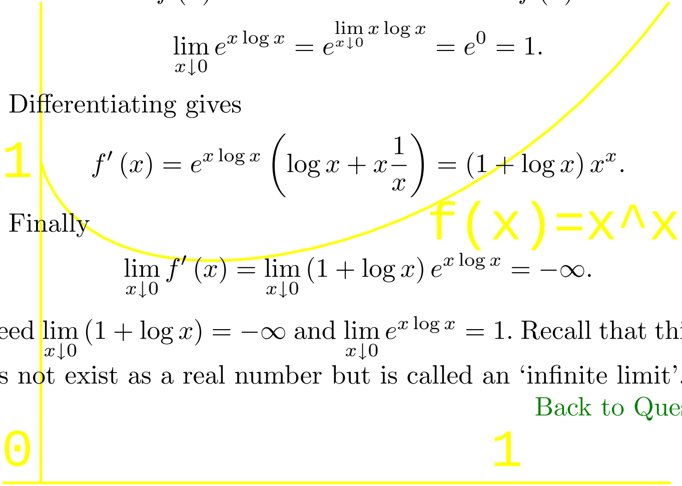
$$f'(x) = e^{x \log x}\left(\log x + x\frac{1}{x}\right) = (1 + \log x)\, x^x.$$

11. Finally

$$\lim_{x \downarrow 0} f'(x) = \lim_{x \downarrow 0} (1 + \log x)\, e^{x \log x} = -\infty.$$

Indeed $\lim_{x \downarrow 0} (1 + \log x) = -\infty$ and $\lim_{x \downarrow 0} e^{x \log x} = 1$. Recall that this limit does not exist as a real number but is called an ‘infinite limit’.

Back to Question 9



Solution to Question 12.

From $\arcsin\left(-\frac{1}{2}\right) = -\frac{1}{6}\pi$ we find

$$\left(-\tfrac{1}{2}, f\left(-\tfrac{1}{2}\right)\right) = \left(-\tfrac{1}{2}, -\tfrac{1}{6}\pi\right) := P.$$

The line ℓ satisfies either one of the following three options:

i. ℓ lies below P; ii. ℓ contains P, or iii. ℓ lies above P.

If ℓ lies below, this line either intersects f twice between -1 and 0 or not at all. If ℓ contains P it has to be tangent to f in P and

$$a = f'\left(-\tfrac{1}{2}\right) = \frac{1}{\sqrt{1-\left(-\frac{1}{2}\right)^2}} = \tfrac{2}{3}\sqrt{3}.$$

Also the difference quotient for the points $\left(-\frac{1}{2}, f\left(-\frac{1}{2}\right)\right)$ and $(1, f(1))$ should fit:

$$a = \frac{f(1) - f\left(-\frac{1}{2}\right)}{1 - \left(-\frac{1}{2}\right)} = \frac{\frac{1}{2}\pi - \left(-\frac{1}{6}\pi\right)}{\frac{3}{2}} = \tfrac{4}{9}\pi.$$

Since $\frac{2}{3}\sqrt{3} \neq \frac{4}{9}\pi$ this gives a contradiction. The remaining possibility is that ℓ lies above $\left(-\frac{1}{2}, -\frac{1}{6}\pi\right)$. Then we have that a is less than the difference quotient, that is, $a < \frac{4}{9}\pi$.

Back to Question 12