DELFT UNIVERSITY OF TECHNOLOGY
I.T.S. Mathematics Departments

Quiz 4: Functions II

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The following functions can be used in this quiz:

acos, asin, atan, cos, cot, exp, ln, log, sin, sqrt, tan,

with acos=arccos, asin=arcsin, atan=arctan, cot=cotan, exp(x)=e^x and sqrt(x)=$\sqrt{x}$. Also the number e is known and one may write $\pi =$ p. Multiplication is denoted by $*$ and powers use ^. For example

$$2e^{\frac{1}{3}\sin(x)} = 2 * e\verb|^|((1/3) * \sin(x)).$$

Click on Begin Quiz to start. Answers are available after End Quiz.

Begin Quiz Answer each of the following questions.

1. $\lim\limits_{n\to\infty} \dfrac{\ln(n!)}{n^2} =$ ______

$n! = n\,.\,(n-1)\ldots 3\,.\,2\,.1$. If necessary use $\infty =$ infinity.

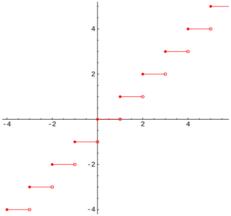
2. Compute the inverse of $f(x) = \ln\left(x + \sqrt{x^2+1}\right)$.

$f^{inverse}(x) =$ ______

Correct answer:

3. The entier-function $f(x) = [x]$ is defined for $x \in \mathbb{R}$ by

$$[x] = \max\{k \in \mathbb{Z}; k \le x\}.$$



Here are some claims:

A. $x \mapsto [x]$ is continuous in a for every $a \in \mathbb{R}\backslash\mathbb{Z}$;
B. $x \mapsto [x]$ is continuous from the right in a for every $a \in \mathbb{R}\backslash\mathbb{Z}$;
C. $x \mapsto [x]$ is continuous from the left in a for every $a \in \mathbb{R}\backslash\mathbb{Z}$;
D. $x \mapsto [x]$ is continuous from the right in a for every $a \in \mathbb{R}$;
E. $x \mapsto [x]$ is continuous in a for every $a \in \mathbb{R}$.

The correct claim(s) are: [ ]

4. $$\lim_{n\to\infty} \frac{\ln(1+2^n)}{n} = \text{[ ]}$$

Correct answer:

5. Does the function $f(x) = \frac{x^2}{x+\cos x}$ have a slant asymptote $\ell(x)$ for $x \to \infty$? Compute $\ell(x)$ or say no:

NB $\ell(x) = ax + b$ is a slant asymptote if $\lim_{x\to\infty} |f(x) = \ell(x)| = 0$.

6. The function $f : \mathbb{R} \to \mathbb{R}^+$ defined by $f(x) = \ln(1 + e^x)$ is .... Copy all appropriate names from:

`onto`, `one-to-one`, `increasing`, `decreasing`, `odd`, `even`,

into:

7. The identity

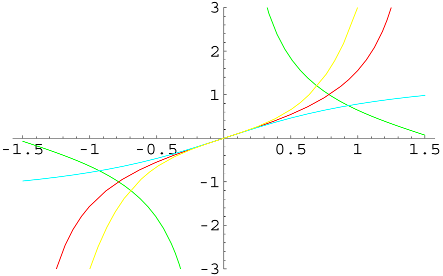
$$\sin(x) + \sin\left(x + \tfrac{2}{3}\pi\right) + \sin\left(x - \tfrac{2}{3}\pi\right) = 0.$$

☐ holds true, ☐ is false.

Correct answer:

8. Here are graphs of

$$t(x) = \tan(x), \qquad c(x) = \cot(x),$$
$$a(x) = \arctan(x), \quad f(x) = x + x^3 + x^5$$



One has, from top to bottom on the left:
green, blue, red, yellow =

(type the letters t, c, a and f in the right order)

Correct answer:

9. $\lim_{n\to\infty} n \sin\left(\frac{2n^2+1}{n}\pi\right) =$ [ ]

10. Here is a list of statements:

I. the sinus function is odd;

II. the tangent function is increasing;

III. $f(x) = (\sin(x))^2$ is periodic with period π;

IV. $\tan\left(x - \frac{1}{2}\pi\right) + \cot(x) = 0$ for all $x \in (0, \pi)$.

I, II, III and IV are respectively

[ ]

(type `false, true, true, true` but then appropriately chosen)

End Quiz | Score: | corrections

Correct answer:

After finishing the quiz one may browse through the solutions on the following pages. Also shift-click on **Ans** *jumps to the answer.*

Solutions to Quiz

Solution to Question 1.

One finds for $n \geq 1$ that

$$\ln(n!) < \ln(n^n) = n\ln(n).$$

Hence

$$0 \leq \frac{\ln(n!)}{n^2} \leq \frac{n\ln(n)}{n^2} \leq \frac{\ln(n)}{n}.$$

Since $\lim_{n\to\infty} \frac{\ln(n)}{n} = 0$ the 'Sandwich'-Theorem gives $\lim_{n\to\infty} \frac{\ln(n!)}{n^2} = 0.$

Back to Question 1

Solution to Question 2.
As usual one sets $y = \ln\left(x + \sqrt{x^2+1}\right)$ and solves for x. The first step uses that the inverse of the logarithm is the exponential function:

$$e^y = x + \sqrt{x^2+1}$$

and one gets rid of the square root by

$$(e^y - x)^2 = x^2 + 1.$$

Hence

$$e^{2y} - 2xe^y + x^2 = x^2 + 1$$

which implies $e^{2y} - 1 = 2xe^y$ and hence

$$x = \frac{e^{2y}-1}{2e^y} = \frac{e^y - e^{-y}}{2}.$$

Hence $f^{inverse}(x) = \dfrac{e^x - e^{-x}}{2}$.

This last function is also known as sinh.

Back to Question 2

Solution to Question 3.

Let n be an integer, then for $x \in [n, n+1)$ one finds $[x] = n$. Since the function is constant on every open interval $(n, n+1)$ it is continuous in x inside such an open interval. That is, $x \mapsto [x]$ is continuous in every $a \in \mathbb{R} \backslash \mathbb{Z}$.

For $n \in \mathbb{Z}$ the function is not continuous:

$$\lim_{x \downarrow n} [x] = n = [n] \text{ and } \lim_{x \uparrow n} [x] = n - 1 \neq= [n].$$

But it shows that $x \mapsto [x]$ is continuous from the right in n (and not continuous from the left). Back to Question 3

Solution to Question 4.

From $\ln(ab) = \ln(a) + \ln(b)$ it follows that

$$\frac{\ln(1+2^n)}{n} = \frac{\ln(2^n) + \ln(2^{-n}+1)}{n} = \frac{n\ln(2) + \ln(2^{-n}+1)}{n}.$$

Using $\ln(1+2^n) > \ln(2^n) = n\ln 2$ and $\ln(2^{-n}+1) < \ln 2$ for $n \geq 1$, one finds

$$\frac{n\ln(2)}{n} < \frac{\ln(1+2^n)}{n} < \frac{(n+1)\ln 2}{n}.$$

One concludes by the 'Sandwich-Theorem':

Since $\lim_{n\to\infty} \frac{n}{n} = 1$ and $\lim_{n\to\infty} \frac{(n+1)\ln 2}{n} = \ln 2$ the limit for $n \to \infty$ of $\frac{\ln(1+2^n)}{n}$ exists and is equal to $\ln 2$.

Back to Question 4

Solution to Question 5.

Suppose $\ell(x) = ax + b$ is the asymptote. Then it should hold that

$$\lim_{x\to\infty}\left(\frac{x^2}{x+\cos x} - \ell(x)\right) = 0.$$

One finds

$$\frac{x^2}{x+\cos x} - (ax+b) = \frac{(1-a)\,x - b - a\cos x - a\frac{\cos x}{x}}{1+\frac{\cos x}{x}}.$$

Since $\lim\limits_{x\to\infty} \frac{\cos x}{x} = 0$ it should hold that

$$\lim_{x\to\infty} (1-a)\,x - b - a\cos x = 0.$$

Necessarily it follows that $1-a=0$ and $b=0$ and $a=0$, which implies a contradiction. Hence no asymptote for $x\to\infty$.

Back to Question 5

Solution to Question 6.

Both the functions $x \mapsto e^x$ and $y \mapsto \ln y$ are increasing and hence $f : x \mapsto \ln(1 + e^x)$ is increasing, Since f is even strictly increasing this function is one-to-one.

Since $1 + e^x > 1$ the logarithm is well-defined for all $x \in \mathbb{R}$ and the combination f of $x \mapsto e^x$, $z \mapsto z + 1$ and $y \mapsto \ln y$ is continuous. Then the function f is onto if

$$\lim_{x \downarrow -\infty} f(x) = 0 \text{ and } \lim_{x \uparrow \infty} f(x) = \infty.$$

Indeed $\lim_{x \downarrow -\infty} \ln(1 + e^x) = \ln 1 = 0$ and $\lim_{x \uparrow \infty} \ln(1 + e^x) = \infty$.

The function is neither odd $(f(x) \neq -f(-x))$ nor even $(f(x) \neq f(-x)$, except for $x = 0)$. Back to Question 6

Solution to Question 7.
The addition formula $\sin(\alpha+\beta) = \sin(\alpha)\cos(\beta) + \cos(\alpha)\sin\beta$ gives

$$\begin{aligned}\sin\left(x \pm \tfrac{2}{3}\pi\right) &= \sin(x)\cos\left(\tfrac{2}{3}\pi\right) \pm \cos(x)\sin\left(\tfrac{2}{3}\pi\right) \\ &= -\tfrac{1}{2}\sin x \pm \tfrac{1}{2}\sqrt{3}\cos(x).\end{aligned}$$

Hence

$$\sin\left(x + \tfrac{2}{3}\pi\right) + \sin\left(x - \tfrac{2}{3}\pi\right) = -\sin(x).$$

Back to Question 7

Solution to Question 8.
Except $\cot(x)$ all these functions are 0 at 0. Hence cot is 'green'.

For $x = 1$ one finds

$$\begin{aligned} \arctan(1) &= \tfrac{1}{4}\pi < 1 = \tan\left(\tfrac{1}{4}\pi\right) < \tan(1), \\ \tan(1) &< \tan\left(\tfrac{1}{3}\pi\right) = \sqrt{3} < 3 = f(1), \end{aligned}$$

and hence f is 'yellow', t is 'red' and a is 'blue'.

Back to Question 8

Solution to Question 9.
Notice that

$$\sin\left(\frac{2n^2+1}{n}\pi\right) = \sin\left(2n\pi + \frac{1}{n}\pi\right) = \sin\left(\frac{1}{n}\pi\right)$$

and by

$$\lim_{n\to\infty} n\sin\left(\frac{2n^2+1}{n}\pi\right) = \lim_{n\to\infty} n\sin\left(\frac{1}{n}\pi\right) = \lim_{n\to\infty} \frac{\sin\left(\frac{1}{n}\pi\right)}{\frac{1}{n}\pi}\pi = \pi$$

using the standard limit $\lim\limits_{x\to 0} \frac{\sin x}{x} = 1$. Back to Question 9

Solution to Question 10.

I. Since $\sin(-x) = -\sin(x)$ the sinus function is odd.

II. Restricted to $(-\frac{1}{2}\pi, \frac{1}{2}\pi)$ it is but since nobody mentioned that restriction the tangent function is not increasing: $\tan(\frac{1}{4}\pi) > \tan(\pi)$.

III. Yes, $x \mapsto \sin(x)^2$ is periodic with period π. If this is not obvious try $\sin(x)^2 = \frac{1}{2} + \frac{1}{2}\cos(2x)$.

IV. We start with the formula's for sin and cos:

$$\begin{aligned} \sin\left(x - \tfrac{1}{2}\pi\right) &= \cos\left(x\right), \\ \cos\left(x - \tfrac{1}{2}\pi\right) &= -\sin\left(x\right). \end{aligned}$$

Then

$$\tan\left(x - \tfrac{1}{2}\pi\right) = \frac{\sin\left(x - \frac{1}{2}\pi\right)}{\cos\left(x - \frac{1}{2}\pi\right)} = -\frac{\cos\left(x\right)}{\sin\left(x\right)} = -\cot\left(x\right).$$

Back to Question 10