DELFT UNIVERSITY OF TECHNOLOGY
I.T.S. Mathematics Departments

Quiz 7: Integration I

G.S.

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The following functions can be used in this quiz:

acos, asin, atan, cos, cot, exp, ln, log, sin, sqrt, tan,

with acos=arccos, asin=arcsin, atan=arctan, cot=cotan, exp(x)=e^x and sqrt(x)=$\sqrt{x}$. Also the number e is known and one may write π = p. Multiplication is denoted by * and powers use ^. For example

$$2e^{\frac{1}{3}\sin(x)} = 2 * e^{\wedge}((1/3) * \sin(x)).$$

Click on Begin Quiz to start. Answers are available after End Quiz.

Begin Quiz Answer each of the following questions.

1. The derivative of $f(x) = \int_{-1}^{x} t^2 e^{-t^2}\, dt$ equals

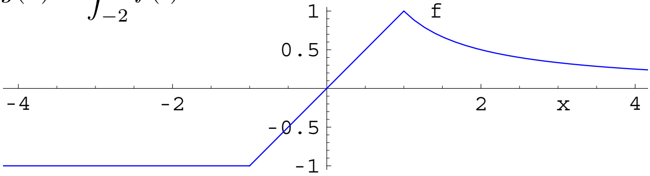
$f'(x) =$ ______

Correct answer:

2. The function f is defined by

$$f(x) = \begin{cases} -1 & \text{for } x \leq -1, \\ x & \text{for } -1 \leq x \leq 1, \\ \frac{1}{x} & \text{for } x > 1. \end{cases}$$

and $g(x) = \int_{-2}^{x} f(t)\,dt.$



Here are some claims.

A. g is continuous on $\mathbb{R}$;
B. g is differentiable on $\mathbb{R}$;
C. g has exactly one minimum;
D. $g(1) = 2$;
E. g has exactly one maximum;
F. $g(3) < 0$.

The correct claim(s) are:

Correct answer:

3. Rewriting, one finds:

$$\int_0^2 \frac{x}{1+\sqrt{x}}\,dx =$$

☐ $\int_0^{\sqrt{2}} \frac{t^2}{1+t}\,dt$; ☐ $\int_0^{2} \frac{t^2}{1+t}\,dt$; ☐ $\int_0^{4} \frac{t^2}{1+t}\,dt$;

☐ $\int_0^{\sqrt{2}} \frac{2t^3}{1+t}\,dt$; ☐ $\int_0^{2} \frac{2t^3}{1+t}\,dt$; ☐ $\int_0^{4} \frac{2t^3}{1+t}\,dt$.

4. Evaluate:

$$\int_0^1 \frac{x^2+x}{x^2+1}\,dx = \boxed{}$$

Correct answer:

5. Evaluate the area A of the region enclosed by the lines $y = -1$, $y = 10$, the y-axis and the graph of the function $f(x) = \ln x$.

$A =$ ☐

6. We consider functions f that are continuous on $\mathbb{R}$. Here are two claims.

I: If f is even on $[-2, 2]$ and $f(0) \neq 0$, then $\int_{-2}^{2} f(x)\,dx \neq 0$.

II: If $\int_{-2}^{2} f(x)\,dx = 0$, then f is odd on $[-2, 2]$.

☐ Both **I** and **II** hold true. ☐ Only **I** holds true.

☐ Only **II** holds true. ☐ Both **I** and **II** are false.

Correct answer:

7. For the integrals

$$I_1 = \int_1^{\infty} \frac{e^x}{x^2}\,dx \quad \text{and} \quad I_2 = \int_0^1 \frac{e^x}{x^2}\,dx$$

we have that

☐ both are convergent;

☐ I_1 is convergent and I_2 is divergent;

☐ I_1 is divergent and I_2 is convergent;

☐ both are divergent.

8. Find the anti-derivative $F(t)$ of $f(t) = 9t^2 \ln t$ (= `9*(t^2)*ln(t)`) that has a zero for $t = 1$.

$F(t) =$ ☐

Correct answer:

9. Here are some claims for functions f and g that are continuous on $\mathbb{R}$.

A. $\int_0^x (f(s)+g(s))\,ds = \int_0^x f(s)\,ds + \int_0^x g(s)\,ds$ for all $x \in \mathbb{R}$.

B. $\int_0^x f(s)\,g(s)\,ds = \int_0^x f(s)\,ds \cdot \int_0^x g(s)\,ds$ for all $x \in \mathbb{R}$.

C. $\int_0^x 3f(s)\,ds = 3\int_0^x f(s)\,ds$ for all $x \in \mathbb{R}$.

D. $\int_0^{x^2} f(s)\,ds = \int_0^x f(s^2)\,ds$ for all $x \in \mathbb{R}$.

E. If $\int_0^x f(s)\,ds = \int_0^x g(s)\,ds$ for all $x \in \mathbb{R}$,

then $f(x) = g(x)$ for all $x \in \mathbb{R}$.

F. If $\int_0^x f(s)\,ds = \int_0^x g(s)\,ds$ for some $x \in \mathbb{R}$,

then $f(x) = g(x)$ for some $x \in \mathbb{R}$.

The correct claim(s) are: [ ]

Correct answer:

10. The integral

$$\int_1^{\infty} \frac{\ln x}{1+x^2}\,dx.$$

☐ is convergent; ☐ is divergent.

11. Evaluate the integral

$$I = \int_0^{\infty} \frac{\ln x}{1+x^2}\,dx.$$

Hint: use the substitution $u = 1/x$. You may choose your answer from the real numbers, `infinity`, `-infinity` and `non-defined` in $\mathbb{R} \cup \{-\infty, \infty\}$.

$I =$ ☐

Correct answer:

End Quiz | Score: | corrections

After finishing the quiz one may browse through the solutions on the following pages. Also shift-click on **Ans** *jumps to the answer.*

Solutions to Quiz

Solution to Question 1.

The fundamental theorem of calculus states that for a continuous function g the derivative of

$$f(x) = \int_a^x g(t)\,dt$$

equals $g(x)$. In our case g is the (continuous!) function defined by $g(x) = x^2 e^{-x^2}$.

Back to Question 1

Solution to Question 2.

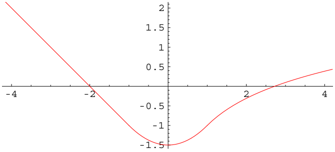
•/• Without computation one concludes by the fact that f is continuous and the fundamental theorem of calculus that g is differentiable and hence continuous.

•/• Since $g(x) = 0$ for exactly one point implies that f has exactly one extreme which is located in this one point ($x = 0$). Since $g''(0) = f'(0) = 1$ it is a minimum. There is no maximum.

•/• The last two claims need either the direct computation above or measuring up the areas:

$$g(1) = \int_{-2}^{-1}(-1)\,ds + \int_{-1}^{1} s\,ds = -1 + 0 = -1,$$

$$g(3) = g(1) + \int_{1}^{3} \frac{1}{s}ds = -1 + \ln 3 > -1 + \ln e = 0.$$



Back to Question 2

Solution to Question 3.

Comparing the integral it seems that $\sqrt{x}$ coincides with t. Hence one has probably used the transformation $\sqrt{x} = t$. With this transformation $x = 0$, respectively $x = 4$ coincide with $t = \sqrt{0} = 0$ and $t = \sqrt{2}$. Moreover, dx should be replaced by $dt^2 = 2t\,dt$. Hence

$$\int_0^2 \frac{x}{1+\sqrt{x}}\,dx = \int_0^{\sqrt{2}} \frac{t^2}{1+t} 2t\,dt,$$

which is the fourth possibility. Back to Question 3

Solution to Question We need to rewrite the function in a form that we can integrate:

$$\frac{x^2+x}{x^2+1} = \frac{x^2+1+x-1}{x^2+1} =$$

$$= 1 + \frac{x-1}{x^2+1} = 1 + \frac{x}{x^2+1} - \frac{1}{x^2+1},$$

and

$$\int_0^1 \frac{x^2+x}{x^2+1}\,dx = \int_0^1 \left(1 + \frac{x}{x^2+1} - \frac{1}{x^2+1}\right) dx =$$

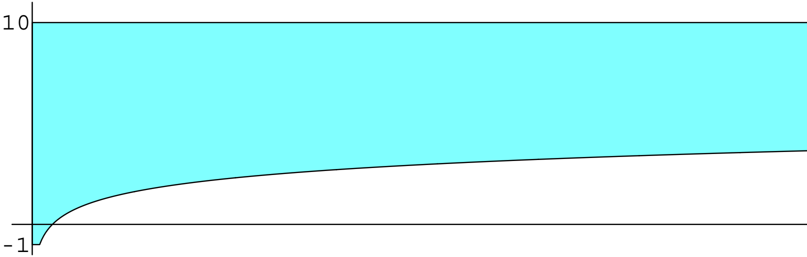
$$= [x]\Big|_0^1 + \tfrac{1}{2}\int_0^1 \frac{2x}{x^2+1}\,dx - [\arctan(x)]\,\Big|_0^1 =$$

$$= 1 + \tfrac{1}{2}\left[\ln(x^2+1)\right]\Big|_0^1 - \tfrac{1}{4}\pi = 1 + \tfrac{1}{2}\ln 2 - \tfrac{1}{4}\pi.$$

Back to Question 4

Solution to Question 5.
By reflecting the picture in the line $x = y$ it might be easier to visualize that the surface area coincides with the integral

$$\int_{-1}^{10} e^y \, dy = [e^y] \Big|_{-1}^{10} = e^{10} - e^{-1}.$$



Back to Question 5

Solution to Question 6.

Consider the even function $f(x) = x^2 - \frac{4}{3}$ and notice that this function is not odd (only the zero function is both even and odd).

The integral is as follows,

$$\int_{-2}^{2} f(x)\,dx = \int_{-2}^{2} \left(x^2 - \frac{4}{3}\right) dx = \left[\tfrac{1}{3}x^3 - \tfrac{4}{3}x\right] \Big|_{-2}^{2} = 0,$$

and contradicts both claims. Back to Question 6

Solution to Question 7.

In both cases we use the comparison test.

For I_1, notice that $e^x > x > 0$ for $x > 0$. So

$$\frac{e^x}{x^2} > \frac{1}{x} > 0 \quad \text{for} \quad x > 0.$$

Since $\int_1^\infty \frac{1}{x}\,dx$ is divergent, so is I_1.

For I_2 use the fact that $e^x > 1$ on $[0,1]$, so

$$\frac{e^x}{x^2} > \frac{1}{x^2} > 0 \quad \text{on } [0,1].$$

Since $\int_0^1 \frac{1}{x^2}\,dx$ is divergent, so is I_2.

Other methods: I_1 must be divergent, since $\lim_{x\to\infty} \frac{e^x}{x^2} \neq 0$. For I_2 we can use the comparison test with the same inequality as used for I_1 in the above solution.

Back to Question 7

Solution to Question 8.

By a repeated integration by parts we find:

$$\int 9t^2 \ln t \, dt = \int \ln t \, d(3t^3) = 3t^3 \ln t - \int 3t^3 \, d(\ln t) =$$

$$= 3t^3 \ln t - \int 3t^2 \, dt = 3t^3 \ln t - t^3 + c.$$

For $t = 1$ this expression should be zero, so $c = 1$ and the requested anti-derivative becomes

$$F(t) = 3t^3 \ln(t) - t^3 + 1.$$

. Back to Question 8

Solution to Question 9.

• One of the properties of the integral.

• This is certainly not a general property of integrals. Try $f(x) = g(x) = 1$. One obtains $\frac{1}{2}x^2$ on the left hand side and x^2 on the right.

• Another property of the integral.

• By the substitution rule one finds with $s = t^2$ that

$$\int_0^{x^2} f(s)\,ds = \int_0^x f\left(t^2\right)\,2t\,dt = \int_0^x f\left(s^2\right)\,2s\,ds.$$

• If f is continuous the fundamental theorem of calculus states that $F(x) = \int_0^x f(s)\,ds$ is a differentiable function with $F'(x) = f(x)$. If G is the similar thing for g, then the assumption says $F(x) = G(x)$ for all $x \in \mathbb{R}$. Hence $f(x) = F'(x) = G'(x) = g(x)$ for all $x \in \mathbb{R}$.

• The same statement with 'for some' is not true. Consider for example $f(x) = 0$ and $g(x) = e^x$ and notice that $\int_0^0 f(s)\,ds = 0 = \int_0^0 g(s)\,ds$.

Back to Question 9

Solution to Question 10.
Notice that this integral is improper at ∞. Using $\ln x < \sqrt{x}$ and $1 + x^2 > x^2$ we may use the estimates

$$0 \leq \frac{\ln x}{1+x^2} < \frac{\sqrt{x}}{x^2} = x^{-\frac{3}{2}} \text{ for } x \in [1, \infty).$$

Since $\int_1^\infty x^{-\frac{3}{2}}\,dx$ converges so does $\int_1^\infty \frac{\ln x}{1+x^2}\,dx$.

Back to Question 10

Solution to Question 11.
The integral $\int_0^\infty \frac{\ln x}{1+x^2}\,dx$ is improper for $x = 0$ and for '$x = \infty$'. Hence we have to consider both improper parts separately and first show that separate integrals converge. From the previous question we know that the integral near ∞ converges. Near 0 remains. By the substitution $u = x^{-1}$ we find:

$$\lim_{\varepsilon\downarrow 0}\int_\varepsilon^1 \frac{\ln x}{1+x^2}\,dx = -\lim_{M\to\infty}\int_1^M \frac{-\ln u}{1+u^{-2}}\,\frac{-1}{u^2}\,du =$$

$$= -\lim_{M\to\infty}\int_1^M \frac{\ln u}{1+u^2}\,du.$$

The last integral converges and hence $\lim\limits_{\varepsilon\downarrow 0}\int_\varepsilon^1 \frac{\ln x}{1+x^2}\,dx$ exists. Since also $\int_1^\infty \frac{\ln x}{1+x^2}\,dx$ converges we may conclude that $\int_0^\infty \frac{\ln x}{1+x^2}\,dx$ converges. Moreover, since the contribution from 0 to 1 cancels the contribution from 1 to ∞, we find $\int_0^\infty \frac{\ln x}{1+x^2}\,dx = 0$. Back to Question 11