DELFT UNIVERSITY OF TECHNOLOGY
I.T.S. Mathematics Departments

Quiz 8: Integration II

G.S.

Last Revision Date: November 21, 2002

The following functions can be used in this quiz:

acos, asin, atan, cos, cot, exp, ln, log, sin, sqrt, tan,

with acos=arccos, asin=arcsin, atan=arctan, cot=cotan, exp(x)=e^x and sqrt(x)=$\sqrt{x}$. Also the number e is known and one may write π = p. Multiplication is denoted by * and powers use ^. For example

$$2e^{\frac{1}{3}\sin(x)} = \texttt{2 * e\^{}((1/3) * sin(x))}.$$

Click on Begin Quiz to start. Answers are available after End Quiz.

Begin Quiz Answer each of the following questions.

1. The derivative of $f(x) = \int_{-1}^{x^2} te^{-t}\,dt$ equals

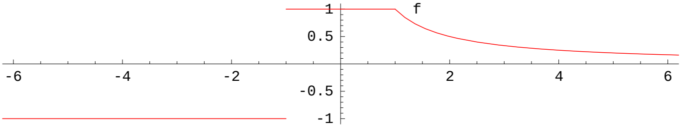
$f'(x) =$ ______

Correct answer:

2. For a function f which is continuous on $[a, b]$ except for a jump in $x_0 \in (a, b)$ the integral over $[a, b]$ is defined by

$$\int_a^b f(s)\,ds = \int_a^{x_0} f(s)\,ds + \int_{x_0}^b f(s)\,ds.$$

Let $f(x) = \begin{cases} -1 & \text{for } x \le -1, \\ 1 & \text{for } -1 \le x \le 1, \\ \frac{1}{x} & \text{for } x > 1. \end{cases}$ and $g(x) = \int_{-2}^{x} f(t)\,dt.$



Here are some claims.

A. g is continuous on $\mathbb{R}$;

B. g is differentiable on $\mathbb{R}$;

C. g has exactly one minimum;

D. $g(1) > 2$;

E. g has exactly one maximum;

F. $g(1) < 2$.

The correct claim(s) are:

Correct answer:

3. Which is the substitution that has been used in

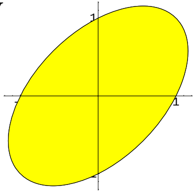
$$\int_{\frac{1}{2}}^{1} \frac{\sqrt{1-x^2}}{x^2}\,dx = \int_{0}^{\frac{1}{3}\pi} (\tan u)^2\,du \text{ ?}$$

$x =$ ______

4. $\displaystyle\int_{\frac{1}{2}}^{1} \frac{\sqrt{1-x^2}}{x^2}\,dx =$ ______

5. Evaluate the area A of the region enclosed by the curve

$$y^2 - xy + x^2 = 1.$$



Hint:

$$\tfrac{1}{2}x - \sqrt{1-\tfrac{3}{4}x^2} < y < \tfrac{1}{2}x + \sqrt{1-\tfrac{3}{4}x^2}.$$

$A =$ ______

Correct answer:

6. Here are two claims for functions f which are continuous on $\mathbb{R}$.

I: If f is even, then $\int_{-x}^{x} f(s)\,ds = 2\int_{0}^{x} f(s)\,ds$ for all $x \in \mathbb{R}$.

II: If $\int_{-x}^{x} f(s)\,ds = 0$ for all $x \in \mathbb{R}$, then f is odd.

☐ Both **I** and **II** hold true.
☐ Only **I** holds true.
☐ Only **II** holds true.
☐ Both **I** and **II** are false.

7. Find the function $F(x)$ such that $F'(x) = (\cos x)^4$ and $F(0) = 0$. Hint: $(\cos x)^2 = 1 - (\sin x)^2$ and recall the formula for $\sin 2x$ and $\cos 2x$.

$F(x) =$ ☐

Correct answer:

8. For the integrals

$$I_1 = \int_1^{\infty} \frac{e^{-\frac{1}{x}}}{x}\, dx \quad \text{and} \quad I_2 = \int_0^1 \frac{e^{-\frac{1}{x}}}{x}\, dx$$

we have that

☐ both are convergent;
☐ I_1 is convergent and I_2 is divergent;
☐ I_1 is divergent and I_2 is convergent;
☐ both are divergent.

9. Set $I = \int_{-1}^{1} \frac{1}{x^2}\, dx$.

☐ $I = 0$; ☐ $I = -2$; ☐ $I = 2$; ☐ $I \notin \mathbb{R}$.

10. Here are some claims for functions f and g that are differentiable on $\mathbb{R}$.

A. $\displaystyle\int_0^x sf'(s)\,ds = xf(x) - \int_0^x f(s)\,ds$ for all $x \in \mathbb{R}$.

B. $\displaystyle\int_0^x (x-s)f(s)\,ds = \int_0^x \int_0^t f(s)\,ds\,dt$ for all $x \in \mathbb{R}$.

C. $\displaystyle\int_{-x^2}^{x^2} f(s)\,ds = \int_{-x}^{x} f(s^2)\,2s\,ds$ for all $x \in \mathbb{R}$.

D. If $\displaystyle\int_0^x f(s)\,ds = \int_2^x g(s)\,ds$ for all $x \in \mathbb{R}$,

then $f(x) = g(x)$ for all $x \in \mathbb{R}$.

E. If $\displaystyle\int_0^x f(s)\,ds = \int_2^x g(s)\,ds$ for all $x \in \mathbb{R}$,

then $f(x) - f(0) = g(x) - g(2)$ for all $x \in \mathbb{R}$.

The correct claim(s) are: [ ]

Correct answer:

11. A first step in evaluating the integral of a rational function is simplifying to partial fractions:

$$\int_\alpha^\beta \frac{x^4+1}{x^3-x}\,dx = \int_\alpha^\beta \left(c_1 x + c_2 + \frac{c_3}{x} + \frac{c_4}{x-1} + \frac{c_5}{x+1}\right) dx.$$

Then $c_4 =$ ______

12. Evaluate $\int_{\frac{1}{3}}^{\frac{2}{3}} \frac{x^4+1}{x^3-x}\,dx =$ ______

Correct answer:

End Quiz | Score: | corrections

After finishing the quiz one may browse through the solutions on the following pages. Also shift-click on **Ans** *jumps to the answer.*

Solutions to Quiz

Solution to Question 1.

The fundamental theorem of calculus states that for a continuous function g the derivative of

$$f(t) = \int_a^t g(s)\, ds$$

equals $g(t)$. In our case we have $f(x^2)$ and we find with the chain rule for differentiation that

$$\frac{d}{dx} f(x^2) = f'(x^2)\, 2x = g(x^2)\, 2x = x^2\, e^{-x^2}\, 2x =$$

$$= \texttt{2*(x^3)*e^(-(x^2)}.$$

Back to Question 1

Solution to Question 2.

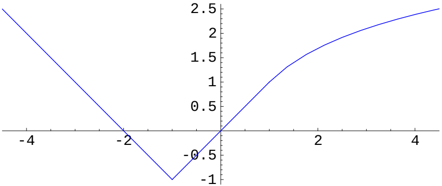
• One may not directly apply the fundamental theorem of calculus since the function involved is not continuous in -1. Outside of -1 there is no problem but let us proceed by direct computation. For $x \leq -1$ we have $g(x) = -x - 2$. For $x \in (-1, 1]$ one finds:

$$g(x) = \int_{-2}^{-1} f(s)\,ds + \int_{-1}^{x} f(s)\,ds = g(-1) + (x + 1) = x.$$

For $x > 1$ one finds:

$$g(x) = \int_{-2}^{1} f(s)\,ds + \int_{1}^{x} f(s)\,ds = g(1) + \ln x = 1 + \ln x.$$

The function g is continuous on $\mathbb{R}$. Here is a picture:



• One sees that g is not smooth in $x = 1$ and hence one doesn't expect g to be differentiable in $x = -1$. Indeed, the left and right derivative are different:

$$g'_\ell(-1) = \lim_{h\uparrow 0} \frac{g(-1+h) - g(-1)}{h} = -1,$$

$$g'_r(-1) = \lim_{h\downarrow 0} \frac{g(-1+h) - g(-1)}{h} = 1.$$

• Extremes may exists only for x such that g has a zero derivative or where g is not differentiable. That gives only one candidate: $x = -1$. Indeed there g has a minimum. It is the only minimum.

• There is no maximum.

•/• Since $g(0) = 1$ the last estimate is appropriate.

Back to Question 2

Solution to Question 3.
Since $\tan x = \frac{\sin x}{\cos x}$ one might guess that $x = \cos u =$ `cos(u)`. Indeed it fits:

$$\int \frac{\sqrt{1-x^2}}{x^2}\,dx = \int \frac{\sqrt{1-(\cos u)^2}}{(\cos u)^2}\,d\cos(u) =$$

$$= \int \frac{\sin u}{(\cos u)^2} \,.-\sin u\,du = -\int (\tan u)^2\,du.$$

The boundary conditions take care of the minus sign: since $\cos 0 = 1$ and $\cos(\frac{1}{3}\pi) = \frac{1}{2}$ one finds that $u = 0$ coincides with $x = 1$ and $u = \frac{1}{3}\pi$ with $x = \frac{1}{2}$. Hence

$$\int_{\frac{1}{2}}^{1} \frac{\sqrt{1-x^2}}{x^2}\,dx = -\int_{\frac{1}{3}\pi}^{0} (\tan u)^2\,du = \int_{0}^{\frac{1}{3}\pi} (\tan u)^2\,du.$$

Back to Question 3

Solution to Question The substitution from the previous question helps us on our way:

$$\int_{\frac{1}{2}}^{1} \frac{\sqrt{1-x^2}}{x^2}\,dx = \int_0^{\frac{1}{3}\pi} (\tan u)^2\,du =$$

$$= \int_0^{\frac{1}{3}\pi} \frac{(\sin u)^2}{(\cos u)^2}\,du = \int_0^{\frac{1}{3}\pi} \left(\frac{1}{(\cos u)^2} - 1\right) du =$$

$$= [\tan u - u]\Big|_0^{\frac{1}{3}\pi} = \tan(\frac{1}{3}\pi) - \frac{1}{3}\pi = \sqrt{3} - \pi/3 = \texttt{sqrt(3)-p/3}.$$

Back to Question 4

Solution to Question 5.
The equation $y^2 - xy + x^2 = 1$ can be solved for y by splitting off squares:

$$\left(y - \tfrac{1}{2}x\right)^2 + \tfrac{3}{4}x^2 = 1 \text{ and next } y - \tfrac{1}{2}x = \pm\sqrt{1 - \tfrac{3}{4}x^2}.$$

The expression $\sqrt{1 - \frac{3}{4}x^2}$ is well-defined for $x \in \left[-\frac{2}{3}\sqrt{3}, \frac{2}{3}\sqrt{3}\right]$.
The surface area equals

$$A = \int_{-\frac{2}{3}\sqrt{3}}^{\frac{2}{3}\sqrt{3}} \left(\left(\tfrac{1}{2}x + \sqrt{1 - \tfrac{3}{4}x^2}\right) - \left(\tfrac{1}{2}x - \sqrt{1 - \tfrac{3}{4}x^2}\right)\right) dx =$$

$$= \int_{-\frac{2}{3}\sqrt{3}}^{\frac{2}{3}\sqrt{3}} 2\sqrt{1 - \tfrac{3}{4}x^2}\, dx = \tfrac{4}{3}\sqrt{3} \int_{-1}^{1} \sqrt{1 - t^2}\, dt =$$

$$= \tfrac{2}{3}\sqrt{3}\pi = \texttt{2*(3^-.5)*p}. \qquad (1)$$

The last identity one might recall since the last integral describes the surface $\frac{1}{2}\pi$ of the semi disk with radius 1. One may also proceed by direct computation of this integral. Substituting $t = \sin u$ and hence

$dt = \cos u \, du$ one gets:

$$(1) = \tfrac{4}{3}\sqrt{3}\int_{-\frac{1}{2}\pi}^{\frac{1}{2}\pi} \sqrt{1-(\sin u)^2}\cos u \, du = \tfrac{4}{3}\sqrt{3}\int_{-\frac{1}{2}\pi}^{\frac{1}{2}\pi} (\cos u)^2 \, du =$$

$$= \tfrac{4}{3}\sqrt{3}\int_{-\frac{1}{2}\pi}^{\frac{1}{2}\pi} \tfrac{1}{2}\left(1+\cos 2u\right) \, du = \tfrac{4}{3}\sqrt{3}\left[\tfrac{1}{2}u + \tfrac{1}{4}\sin 2u\right]\Big|_{-\frac{1}{2}\pi}^{\frac{1}{2}\pi} = \tfrac{2}{3}\sqrt{3}\pi.$$

Back to Question 5

Solution to Question 6.

• If f is even then by substituting $-s = t$ and using $f(-t) = f(t)$ one finds:

$$\int_{-x}^{0} f(s)\,ds = -\int_{x}^{0} f(-t)\,dt = \int_{0}^{x} f(-t)\,dt = \int_{0}^{x} f(t)\,dt,$$

and hence $\displaystyle\int_{-x}^{x} f(s)\,ds = \int_{-x}^{0} f(s)\,ds + \int_{0}^{x} f(s)\,ds = 2\int_{0}^{x} f(t)\,dt.$

• Since $x \mapsto f(x)$ (and hence also $x \mapsto f(-x)$) is continuous the fundamental theorem of calculus says that

$$\int_{0}^{x} f(s)\,ds \quad \text{and} \quad \int_{-x}^{0} f(s)\,ds = -\int_{0}^{x} f(-s)\,ds$$

are both differentiable, with derivatives $f(x)$ respectively $-f(-x)$. Hence, from $\frac{d}{dx}(0) = 0$,

$$\frac{d}{dx}\left(\int_{-x}^{x} f(s)\,ds\right) = f(x) - f(-x) = 0,$$

meaning that f is odd.

Back to Question 6

Solution to Question 7.
Recall that $\cos 2t = 2(\cos t)^2 - 1 = 1 - 2(\sin t)^2$ and hence

$$\int (\cos x)^4 \, dx = \int (\cos x)^2 \left(1 - (\sin x)^2\right) dx =$$

$$= \int \left((\cos x)^2 - (\sin x \cos x)^2\right) = \int \left((\cos x)^2 - \tfrac{1}{4}(\sin 2x)^2\right) =$$

$$= \int \left(\left(\tfrac{1}{2} + \tfrac{1}{2}\cos 2x\right) - \tfrac{1}{4}\left(\tfrac{1}{2} - \tfrac{1}{2}\cos 4x\right)\right) dx =$$

$$= \tfrac{3}{8}x + \tfrac{1}{4}\sin 2x + \tfrac{1}{32}\sin 4x + C.$$

In order to have 0 in 0 one sets $C = 0$ to find

$$F(x) = \tfrac{3}{8}x + \tfrac{1}{4}\sin 2x + \tfrac{1}{32}\sin 4x =$$

= `(3/8)*x+(1/4)*sin(2*x)+(1/32)*sin(4*x).`

Back to Question 7

Solution to Question 8.

• For the denominator that appears in the integral $\int_1^\infty \frac{e^{-\frac{1}{x}}}{x}dx$ the following estimate holds. If $1 \le x < \infty$ then $-1 \le -\frac{1}{x} < 0$ and since $t \mapsto e^t$ is increasing $e^{-1} \le e^{-\frac{1}{x}} \le 1$. Hence

$$\int_1^M \frac{e^{-\frac{1}{x}}}{x}dx \ge \int_1^M \frac{e^{-1}}{x}\,dx = e^{-1}\log M$$

and since $\lim_{M\to\infty} e^{-1}\log M = \infty$ the integral diverges.

• The second one is harder. Using $e^t > t$ for all t one finds $e^{\frac{1}{x}} > \frac{1}{x}$ for all $x \ne 0$. Hence for all $x > 0$ it follows that

$$e^{-\frac{1}{x}} = \frac{1}{e^{\frac{1}{x}}} < \frac{1}{\frac{1}{x}} = x.$$

Now we may a comparison test for $\int_0^1 \frac{e^{-\frac{1}{x}}}{x}dx$ from above by $\int_0^1 \frac{x}{x}dx = 1$. Since $0 \le \frac{e^{-\frac{1}{x}}}{x} \le \frac{x}{x} = 1$ the integral $\int_0^1 \frac{e^{-\frac{1}{x}}}{x}dx$ also converges.

Back to Question 8

Solution to Question 9.
The integral is improper at zero, one might even say both from the left and from the right. A correct evaluation should consider both singularities separately:

$$\int_{-1}^{1} \frac{1}{x^2}\, dx = \lim_{a \downarrow 0} \int_{a}^{1} \frac{1}{x^2}\, dx + \lim_{b \uparrow 0} \int_{-1}^{b} \frac{1}{x^2}\, dx.$$

Both integrals diverge. For the one on the right:

$$\lim_{a \downarrow 0} \int_{a}^{1} \frac{1}{x^2}\, dx = \lim_{a \downarrow 0} \left[\frac{-1}{x} \right] \Big|_{a}^{1} = \infty.$$

Back to Question 9

Solution to Question 10.

• A partial integration gives:

$$\int_0^x sf'(s)\,ds = \int_0^x s\,df(s) = [sf(s)]\,\Big|_0^x - \int_0^x f(s)\,ds.$$

• By rewriting and using the fundamental theorem of calculus one finds for the left hand side:

$$\tfrac{d}{dx}\int_0^x (x-s)\,f(s)\,ds = \tfrac{d}{dx}\left(x\int_0^x f(s)\,ds - \int_0^x sf(s)\,ds\right) =$$

$$= \int_0^x f(s)\,ds + xf(x) - xf(x) = \int_0^x f(s)\,ds.$$

For the right hand side $\tfrac{d}{dx}\int_0^x \left(\int_0^t f(s)\,ds\right)dt = \int_0^x f(s)\,ds.$

Since left and right hand side have the same derivative they may differ at most by a constant. Since

$$\int_0^0 (0-s)\,f(s)\,ds = 0 = \int_0^0 \int_0^t f(s)\,ds$$

this constant is 0 and hence left and right hand side are equal.

• Note that on the right hand side, substituting $s = -t$ for $s < 0$:

$$\int_{-x}^{x} f\left(s^2\right) 2s\,ds = \int_{0}^{x} f\left(s^2\right) 2s\,ds + \int_{-x}^{0} f\left(s^2\right) 2s\,ds =$$

$$= \int_{0}^{x} f\left(s^2\right) 2s\,ds - \int_{0}^{x} f\left(t^2\right) 2t\,dt = 0.$$

Unless the function is odd the left hand side is not identically zero.

• For the one but last claim one proceeds again by the fundamental theorem of calculus. One may differentiate and finds:

$$f(x) = \tfrac{d}{dx} \int_{0}^{x} f(s)\,ds = \tfrac{d}{dx} \int_{2}^{x} g(s)\,ds = g(x).$$

• As before one finds by differentiating that $f(x) = g(x)$ so the conclusion $f(x) - f(0) = g(x) - g(2)$ can only hold if $f(0) = g(2)$. If one doesn't make this assumption the last claim is wrong.

Back to Question 10

Solution to Question 11.
Before going to partial fractions one has to take care that the degree of the denominator is less than the one of the numerator:

$$\frac{x^4+1}{x^3-x} = \frac{x(x^3-x)+x^2+1}{x^3-x} = x + \frac{x^2+1}{x^3-x}.$$

Next, knowing that $x^3 - x = x(x-1)(x+1)$, one splits

$$\frac{x^2+1}{x^3-x} = \frac{A}{x} + \frac{B}{x-1} + \frac{C}{x+1} = \frac{A(x^2-1)+B(x^2+x)+C(x^2-x)}{x(x-1)(x+1)}.$$

Combining the same degrees one obtains

$$(A+B+C)x^2 + (B-C)x - A = x^2 + 1 \text{ for all } x \in \mathbb{R}$$

implying $A+B+C=1$, $B-C=0$ and $A=1$, which is solved by $A=-1$, $B=C=1$. Then $c_4 = B = 1$. Back to Question 11

Solution to Question 12.
With the partial fractions of the previous question:

$$\int_{\frac{1}{3}}^{\frac{2}{3}} \frac{x^4+1}{x^3-x}\,dx = \int_{\frac{1}{3}}^{\frac{2}{3}} \left(x - \frac{1}{x} + \frac{1}{x-1} + \frac{1}{x+1} \right) dx =$$

$$= \left[\tfrac{1}{2}x^2 - \log(x) + \log(1-x) + \log(x+1) \right] \Big|_{\frac{1}{3}}^{\frac{2}{3}} =$$

$$= \tfrac{1}{2}\left((\tfrac{2}{3})^2 - (\tfrac{1}{3})^2 \right) - \log\left(\frac{\frac{2}{3}}{\frac{1}{3}} \right) + \log\left(\frac{1-\frac{2}{3}}{1-\frac{1}{3}} \right) + \log\left(\frac{\frac{2}{3}+1}{\frac{1}{3}+1} \right) =$$

$$= \frac{1}{6} + \log\frac{5}{16} = \texttt{(1/6)+log(5)-4*log(2)}.$$

Back to Question 12