DELFT UNIVERSITY OF TECHNOLOGY
I.T.S. Mathematics Departments

Quiz 9: Ordinary Differential Equations I

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The following functions can be used in this quiz:

acos, asin, atan, cos, cot, exp, ln, log, sin, sqrt, tan,

with acos=arccos, asin=arcsin, atan=arctan, cot=cotan, exp(x)=e^x and sqrt(x)=$\sqrt{x}$. Also the number e is known and one may write π = p. Multiplication is denoted by * and powers use ^. For example

$$2e^{\frac{1}{3}\sin(x)} = \texttt{2 * e\^{}((1/3) * sin(x))}.$$

Click on Begin Quiz to start. Answers are available after End Quiz.

Begin Quiz Answer each of the following questions.

1. The solution of $y'(x) = y(x) + \sin(x)$ that satisfies $y(0) = 0$ is

$y(x) =$ ______

2. Compute the solution of $y'(x) = (y(x))^2 + 1$ that is defined for $x \in (0, \pi)$.

$y(x) =$ ______

Correct answer:

3. Here are some claims for the differential equation

$$u'(x) + u(x) = e^x u(x).$$

This differential equation ...

A. is separable;
B. is of first order;
C. is linear;
D. is autonomous;
E. is a logistic equation;
F. is not linear but named after Bernoulli;
G. has a bounded solution.

The correct claim(s) are: ☐

4. Compute the solution of $v''(x) + 4\,v'(x) + 5\,v(x) = 0$ that satisfies $v(0) = 1$ and $v'(0) = 2$.

$$v(x) = \square$$

Correct answer:

5. Let a and b denote two nonzero differentiable functions defined on $\mathbb{R}$. Suppose that $y_1(x)$ and $y_2(x)$ are two different solutions of

$$y'(x) = a(x)y(x) + b(x) \tag{1}$$

A. The functions $y_1(x)$ and $y_2(x)$ intersect each other exactly once.

B. The functions $y_1(x)$ and $y_2(x)$ do not intersect each other.

C. The function $y(x) = 0.8\,y_1(x) - 0.5\,y_2(x)$ is a solution of (1).

D. The function $y(x) = 0.3\,y_1(x) + 0.7\,y_2(x)$ is a solution of (1).

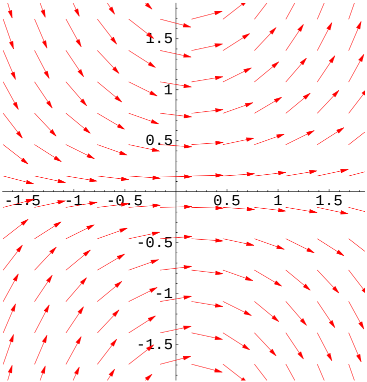
E. For every $c_1, c_2 \in \mathbb{R}$ is the function $y(x) = c_1\,y_1(x) + c_2\,y_2(x)$ a solution of (1).

F. For every solution $y(x)$ of (1) there exist $c_1, c_2 \in \mathbb{R}$ such that $y(x) = c_1\,y_1(x) + c_2\,y_2(x)$ for all $x \in \mathbb{R}$.

The correct claim(s) are: [ ]

Correct answer:

6. Here is the plot of a direction field. Which is the corresponding differential equation?



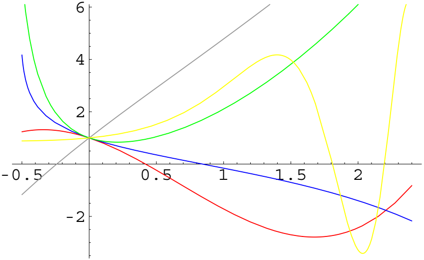
☐ $y'(x) = x\,y(x)$ ☐ $y'(x) = -x\,y(x)$

☐ $y'(x) = x\,(y(x))^2$ ☐ $y'(x) = -x\,(y(x))^2$

7. One of the graphs is a picture for the solution of

$$y'(x) = y(x) - e^{y(x)}$$

that satisfies $y(0) = 1$.



The corresponding color is:

☐ green ☐ blue ☐ red ☐ yellow ☐ grey

(Hint: is the solution increasing, decreasing or oscillating?)

8. Consider the differential equation $u'(x) = 1 - (u(x))^2$.

A. There is exactly one solution that satisfies $u(0) = 0$.

B. There is exactly one solution that satisfies $\lim_{x \to \infty} u(x) = 1$.

C. There is exactly one solution that satisfies $\lim_{x \to -\infty} u(x) = 1$.

D. There is exactly one solution that satisfies $u(-1) = u(1)$.

The correct claim(s) are: [ ]

9. How many solutions $y : \mathbb{R} \mapsto \mathbb{R}$ of

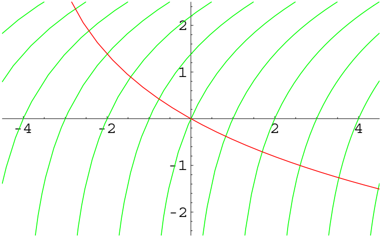
$$xy'(x) + y(x) = 2x$$

are defined on all of $\mathbb{R}$? [ ]

Write an integer number or `infinity`.

Correct answer:

10. The green curves correspond to the functions $f_c(x) = 2\ln(x-c)$ for $c \in \mathbb{R}$. The red curve contains $(0,0)$ and intersects the graphs of the f_c perpendicular.



For the red curve: $y =$ [ ]

Correct answer:

End Quiz | Score: | corrections

After finishing the quiz one may browse through the solutions on the following pages. Also shift-click on **Ans** *jumps to the answer.*

Solutions to Quiz

Solution to Question 1.

This differential equation is linear so we first solve the homogeneous equation $y' = y$ to find $y(x) = ce^x$. Next we may proceed by 'clever-guessing' or by variation of constants (substituting $y(x) = c(x)e^x$):

$$y'(x) = c'(x)e^x + c(x)e^x \text{ and } y(x) + \sin x = c(x)e^x + \sin x$$

yielding $c'(x)e^x = \sin x$ and hence

$$c'(x) = e^{-x}\sin(x).$$

We find $c(x) = -\frac{1}{2}e^{-x}(\sin(x) + \cos(x)) + c_1$ (using integration by parts twice). Hence, for $c_1 \in \mathbb{R}$,

$$y(x) = c(x)e^x = -\frac{1}{2}(\sin(x) + \cos(x)) + c_1e^x.$$

Plugging in the initial condition:

$$y(x) = \frac{1}{2}(e^x - \sin(x) - \cos(x)).$$

Back to Question 1

Solution to Question 2.
This differential equation is separable so we rewrite to

$$\frac{y'(x)}{1+(y(x))^2}=1$$

(and notice that we do not loose anything since $1+y^2\neq 0$). A primitive of the left hand side is $\arctan(y(x))$. Hence

$$\arctan(y(x))=x+c$$

which we may change to $y(x)=\tan(x+c)$ for $x+c\in\left(-\frac{1}{2}\pi,\frac{1}{2}\pi\right)$. By the condition that the domain contains $(0,\pi)$ we find $c=-\frac{1}{2}\pi$:

$$y(x)=\tan\left(x-\tfrac{1}{2}\pi\right)=-\cot(x).$$

Back to Question 2

Solution to Question 3.

• Separable since we may rewrite to $u'(x) = f(u(x)).g(x)$:

$$u'(x) = (e^x - 1).u(x).$$

• First order because of u' as highest order appearing.

• Linear since we may rewrite to $u'(x) = a(x).u(x) + b(x)$: take $a(x) = e^x - 1$ and $b(x) = 0$.

• It is not autonomous since e^x appears.

• It doesn't look like a logistic equation $u'(x) = c\,x\,(1-x)$.

• Bernoulli gave his name to $u'(x) = a(x).u(x) + b(x).(u(x))^p$.

• The function $u(x) = 0$ is a solution and is bounded.

Back to Question 3

Solution to Question 4.

The first step for linear equations with constant coefficients is to test with $v(x) = e^{\lambda x}$. One finds

$$\lambda^2 + 4\lambda + 5 = 0.$$

Splitting off squares one obtains $(\lambda + 2)^2 = -1$ and the solutions of this quadratic equation are $\lambda = -2 \pm i$. In complex form

$$v(x) = \alpha e^{(-2-i)x} + \beta e^{(-2+i)x}.$$

We prefer the real form

$$v(x) = c_1 e^{-2x} \sin x + c_2 e^{-2x} \cos x.$$

with $c_1, c_2 \in \mathbb{R}$. The initial conditions $v(0) = 1$ and $v'(0) = 2$ imply $c_2 = 1$ and $c_1 - 2c_2 = 2$. Hence

$$v(x) = 4e^{-2x} \sin x + e^{-2x} \cos x.$$

Back to Question 4

Solution to Question 5.

•/• If two solutions intersect then they satisfy the same initial value problem. A first order linear differential equation as in (1) with a given initial value has a unique solution. Hence, if two solutions intersect then they are identical which contradicts the statement that the solutions are different.

• Plugging y into the differential equation one finds

$$y'(x) = a(x)y(x) + .3\,b(x)$$

and $.3\,b(x) \neq b(x)$.

• Now the computation fits.

• The computation only fits when $c_1 + c_2 = 1$ and not for all c_i.

• The difference $v(x) = y_1(x) - y_2(x)$ solves $v'(x) = a(x)v(x)$. Hence the general form of the solution of (1) is

$$y(x) = y_1(x) + C\,v(x) = (1 + C)y_1(x) - C\,y_2(x),$$

which can be written in the given form. Back to Question 5

Solution to Question 6.
In the first quadrant the direction field points upwards which means that the right hand side of the differential equation should be positive. This is the case for

$$y'(x) = x\,y(x) \text{ and } y'(x) = x\,(y(x))^2$$

In the fourth quadrant the direction field is pointing downwards which is the case for

$$y'(x) = x\,y(x) \text{ and } y'(x) = -x\,(y(x))^2$$

Hence the only candidate left is $y'(x) = x\,y(x)$.

Back to Question 6

Solution to Question 7.
One should remark that $y - e^y < 0$ for all $y \in \mathbb{R}$. Hence any solution to the differential equation is decreasing. Only the blue graph has that property. Back to Question 7

Solution to Question 8.
The differential equation $u'(x) = 1 - (u(x))^2$ is separable. So the first step means separating:

$$\frac{u'(x)}{1 - (u(x))^2} = 1 \text{ or } 1 - (u(x))^2 = 0.$$

The second option gives two solutions: $u(x) = 1$ and $u(x) = -1$. The fraction in the first option is split as follows:

$$\left(\frac{\frac{1}{2}}{1 - u(x)} + \frac{\frac{1}{2}}{1 + u(x)} \right) u'(x) = 1.$$

The anti-derivative gives

$$-\frac{1}{2} \ln |1 - u(x)| + \frac{1}{2} \ln |1 + u(x)| = x + C$$

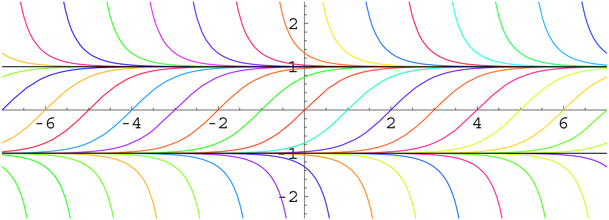
and further reductions give

$$\ln \left| \frac{1 + u(x)}{1 - u(x)} \right| = 2x + C \text{ and } \frac{1 + u(x)}{1 - u(x)} = \pm e^{2x+C}.$$

Finally with $\tilde{c} = \pm e^C$ we obtain

$$u(x) = \frac{\tilde{c}e^{2x} - 1}{\tilde{c}e^{2x} + 1} \text{ with } \tilde{c} \in \mathbb{R}.$$

Here are the graphs of some solutions:



- There is one solution with $u(0) = 0$, namely $u(x) = \frac{e^{2x}-1}{e^{2x}+1}$.
- There are many solutions with $\lim_{x\to\infty} u(x) = 1$.
- Only one solution exists with $\lim_{x\to-\infty} u(x) = 1$, namely $u(x) = 1$.
- And two solutions exist with $u(-1) = u(1)$, namely $u(x) = 1$ and $u(x) = -1$. All others are strictly monotone. Back to Question 8

Solution to Question 9.
The differential equation $x\,y'(x) + y(x) = 2x$ is linear. Hence we first solve $x\,y'(x) + y(x) = 0$ as a separable d.e.:

$$\frac{y'(x)}{y(x)} = -\frac{1}{x} \text{ implying } \ln|y(x)| = -\ln|x| + c_0$$

which simplifies to $y\,(x) = C\,\frac{1}{x}$. Substituting $y(x) = C(x)\frac{1}{x}$ gives

$$x\left(C'(x)\frac{1}{x} - C(x)\frac{1}{x^2}\right) + C(x)\frac{1}{x} = 2x$$

that is $C'(x) = 2x$. We find $C(x) = x^2 + c_1$ and the general form of the solution becomes

$$y\,(x) = \left(x^2 + c_1\right)\frac{1}{x} = x + c_1\frac{1}{x}.$$

Except for $c_1 = 0$ all these functions have an asymptote at $x = 0$. The only solution that is defined on $\mathbb{R}$ is for $c_1 = 0$:

$$y(x) = x.$$

Back to Question 9

Solution to Question 10.
The green curves belong to $f_c(x) = 2\ln(x-c)$. The red curve $y = g(x)$ in (x,y) has direction perpendicular to the $y = f_c(x)$ through that point. Hence

$$g'(x).f_c'(x) = -1.$$

Since $y = 2\ln(x-c)$ implies $x - c = e^{\frac{1}{2}y}$ we obtain from $f_c'(x) = \frac{2}{x-c}$ that

$$g'(x) = \frac{-1}{f_c'(x)} = -\frac{1}{2}(x-c) = -\frac{1}{2}e^{\frac{1}{2}y} = -\frac{1}{2}e^{\frac{1}{2}g(x)}.$$

This differential equation for g is separable, $-2e^{-\frac{1}{2}g(x)}g'(x) = 1$, and is solved by $4e^{-\frac{1}{2}g(x)} = x + C$. Simplifying leads to

$$g(x) = -2\ln\left(\frac{1}{4}(x+C)\right)$$

and using $g(0) = 0$ we find $C = 4$. The red line satisfies

$$y = -2\ln\left(\frac{1}{4}x + 1\right).$$

Back to Question 10