DELFT UNIVERSITY OF TECHNOLOGY
I.T.S. Mathematics Departments

Quiz 10: Ordinary Differential Equations II

G.S.

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The following functions can be used in this quiz:

acos, asin, atan, cos, cot, exp, ln, log, sin, sqrt, tan,

with acos=arccos, asin=arcsin, atan=arctan, cot=cotan, exp(x)=e^x and sqrt(x)=$\sqrt{x}$. Also the number e is known and one may write π = p. Multiplication is denoted by * and powers use ^. For example

$$2e^{\frac{1}{3}\sin(x)} = \texttt{2 * e^((1/3) * sin(x))}.$$

Click on Begin Quiz to start. Answers are available after End Quiz.

Begin Quiz Answer each of the following questions.

1. The solution of $y'(x) - 2xy(x) = e^{(x^2)}$ that satisfies $y(0) = 1$ is

$y(x) =$ ______

2. The functions $y_1(x) = x$ and $y_2(x) = e^x$ are both solutions of

$$(1 - x)y''(x) + xy'(x) - y(x) = 0.$$

The solution that satisfies $y(0) = 1$ and $y'(0) = 0$ is

$y(x) =$ ______

Correct answer:

3. Here are some claims for the differential equation

$$u'(x) = \frac{1 + \cos(x)}{1 + (u(x))^4}.$$

This differential equation ...

A. is separable;
B. is of first order;
C. is linear;
D. is autonomous;
E. has an increasing solution;
F. only has increasing solutions.

The correct claim(s) are: [ ]

4. Compute the solution of $v''(x) + 4\,v'(x) + 4\,v(x) = 0$ that satisfies $v(0) = 1$ and $v'(0) = 2$.

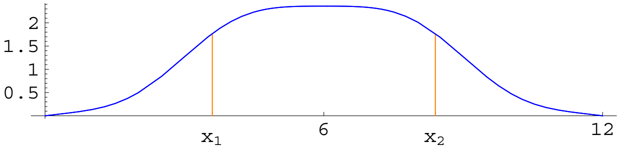
$v(x) =$ [ ]

Correct answer:

5. Compute the solution of $u''(x) + 4\,u'(x) + 4\,u(x) = e^{-2x}$ that satisfies $u(0) = 1$ and $u'(0) = 0$.

$u(x) =$ ☐

6. The following graph depicts the solution $y(x)$ of a differential equation. This function is symmetric in the line $x = 6$.



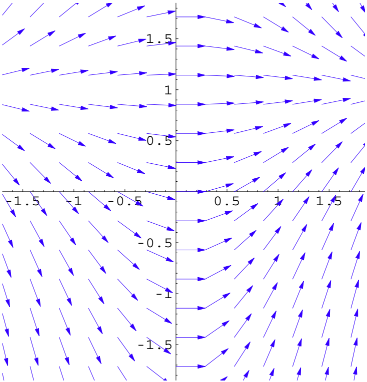
Compare y, y' and y'' in x_1 and x_2, and conclude to which kind of differential equation it could belong.

☐ $y'(x) = f(y(x))$ ☐ $y''(x) = f(y(x))$

☐ $y'(x) = x + f(y(x))$ ☐ $y''(x) = x + f(y(x))$

Correct answer:

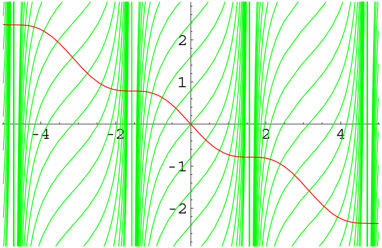
7. Here is the plot of a direction field. Which is the corresponding differential equation.



☐ $y'(x) = x(y(x) - 1)$ ☐ $y'(x) = -x(y(x) - 1)$

☐ $y'(x) = e^{-x^2}(y(x) - 1)$ ☐ $y'(x) = -e^{-x^2}(y(x) - 1)$

8. The green curves correspond to the functions $f_c(x) = \tan(x) + c$ for $c \in \mathbb{R}$. The red curve contains $(0, 0)$ and intersects the graphs of the f_c perpendicular.



For the red curve: $y =$ ☐

9. The solution of $xy'(x) = \dfrac{y(x) - 1}{y(x) + 1}$ that satisfies $y(2) = 0$ cannot be explicitly written. The inverse however can be computed:

$x(y) =$ ☐

Correct answer:

10. The functions $y_1(x) = 1$, $y_2(x) = x$ and $y_3(x) = e^{-x} + 1$ all solve

$$x\, y''(x) + (x-1)\, y'(x) - y(x) + 1 = 0. \tag{1}$$

The general solution of (1) is:

- ☐ $y(x) = c_1\, y_1(x) + c_2\, y_2(x) + y_3(x)$;
- ☐ $y(x) = c_1\, y_1(x) + y_2(x) + c_3\, y_3(x)$;
- ☐ $y(x) = y_1(x) + c_2\, y_2(x) + c_3\, y_3(x)$;
- ☐ $y(x) = c_1\, y_1(x) + c_2\, y_2(x) + c_3\, y_3(x)$;
- ☐ $y(x) = c_1\, y_1(x) + c_2\, y_2(x) + (1 - c_1 - c_2)\, y_3(x)$;

with $c_i \in \mathbb{R}$.

End Quiz | Score: | corrections

After finishing the quiz one may browse through the solutions on the following pages. Also shift-click on **Ans** *jumps to the answer.*

Solutions to Quiz

Solution to Question 1.

This differential equation is linear so we first solve the homogeneous equation $y'(x) - 2xy(x) = 0$. Separating x and y we get

$$\frac{y'(x)}{y(x)} = 2x$$

and find $y(x) = ce^{x^2}$. Next by variation of constants (substituting $y(x) = c(x)e^{x}$):

$$y'(x) - 2xy(x) = c'(x)e^{x^2}$$

we have to solve $c'(x) = 1$, that is, $c(x) = x + C$ for some $C \in \mathbb{R}$ and

$$y(x) = (x + C)e^{x^2}.$$

Plugging in the initial condition:

$$y(x) = (x + 1)e^{x^2}.$$

Back to Question 1

Solution to Question 2.
This differential equation is linear so the general solution is

$$y(x) = c_1 x + c_2 e^x$$

with $c_i \in \mathbb{R}$. So the appropriate constants follow by plugging in the initial conditions:

$$y(0) = c_2 = 1 \text{ and } y'(0) = c_1 + c_2 = 0.$$

One finds $y(x) = e^x - x$. Back to Question 2

Solution to Question 3.

• Separable since it has the form

$$u'(x) = f(u(x)).g(x).$$

• First order because of u' as highest order appearing.

• It is not linear since it cannot be written as $u'(x) = a(x).u(x) + b(x)$.

• It is not autonomous because of the $\cos(x)$.

•/• The right hand side is positive for any x and u and hence u' is positive implying that every solution is increasing.

Back to Question 3

Solution to Question 4.
The first step for linear equations with constant coefficients is to test with $v(x) = e^{\lambda x}$. One finds

$$\lambda^2 + 4\lambda + 4 = 0.$$

Splitting off squares one obtains $(\lambda + 2)^2 = 0$ and the solutions of this quadratic equation are $\lambda = -2$ with multiplicity 2. Then the general form of the solution is

$$v(x) = c_1 e^{-2x} + c_2 x e^{-2x}.$$

with $c_1, c_2 \in \mathbb{R}$. The initial conditions $v(0) = 1$ and $v'(0) = 2$ imply $c_1 = 1$ and $-2c_1 + c_2 = 2$. Hence

$$v(x) = e^{-2x} + 4xe^{-2x}.$$

Back to Question 4

Solution to Question 5.
From the previous question we know that the general solution of

$$v''(x) + 4\,v'(x) + 4\,v(x) = 0$$

is

$$v(x) = c_1 e^{-2x} + c_2 x e^{-2x}.$$

Since the right hand side again contains e^{-2x} one should try as a special solution $u_p(x) = \alpha x^2 e^{-2x}$. Plugging this in the differential equation we find that indeed this is a solution for $\alpha = \frac{1}{2}$. Hence

$$u(x) = \tfrac{1}{2} x^2 e^{-2x} + c_1 e^{-2x} + c_2 x e^{-2x}$$

is the general solution. The initial condition gives $c_1 = 1$ and $c_2 = 2$.

Back to Question 5

Solution to Question 6.
First $y'(x) = f(y(x))$. Since $f(y(x_1)) = f(y(x_2))$ the derivative of y in x_1 and x_2 should be equal, a contradiction since $y'(x_1) > 0 > y(x_2)$. A similar argument contradicts $y'(x) = x + f(y(x))$. The d.e. implies:

$$y'(x_1) = x_1 + f(y(x_1)) < x_2 + f(y(x_2)) = y'(x_2).$$

The symmetry around $x = 6$ means $y(x) = y(12 - x)$ and hence

$$y'(x) = -y'(12 - x) \text{ and } y''(x) = y''(12 - x).$$

Hence not only $y(x_1) = y(x_2)$ but also $y''(x_1) = y''(x_2)$ and this gives a contradiction to the differential equation:

$$y''(x_1) = x_1 + f(y(x_1)) < x_2 + f(y(x_2)) = y''(x_2).$$

Back to Question 6

Solution to Question 7.
For $x > 0$ and $y > 1$ the direction field points downwards and hence there $y'(x)$ and hence the right hand side of the differential equation should be negative. Hence either

$$y'(x) = -x(y(x) - 1)) \text{ or } y'(x) = -e^{-x^2}(y(x) - 1)$$

For $x < 0$ and $y > 1$ the direction field points upwards and hence there $y'(x)$ and hence the right hand side of the differential equation should be positive. Only $y'(x) = -x(y(x) - 1))$ remains.

Back to Question 7

Solution to Question 8.
The green curves belong to $f_c(x) = \tan(x) + c$. The red curve $y = g(x)$ in (x, y) has direction perpendicular to the $y = f_c(x)$ through that point. Hence

$$g'(x).f_c'(x) = -1.$$

Since $f_c'(x) = (\cos(x))^{-2}$ we find $-(g'(x))^{-1} = (\cos(x))^{-2}$. In other words: we have to solve

$$g'(x) = -(\cos(x))^2 = -\tfrac{1}{2}\cos(2x) - \tfrac{1}{2}.$$

We find

$$g(x) = -\tfrac{1}{4}\sin(2x) - \tfrac{1}{2}x + C$$

and using $g(0) = 0$ we find $C = 0$. The red line satisfies

$$y = -\tfrac{1}{4}\sin(2x) - \tfrac{1}{2}x.$$

Back to Question 8

Solution to Question 9.

The differential equation $xy'(x) = \dfrac{y(x)-1}{y(x)+1}$ is separable and hence our first step is

$$\frac{y(x)+1}{y(x)-1}y'(x) = \frac{1}{x}$$

and next we simplify the fraction

$$\left(1 + \frac{2}{y(x)-1}\right) y'(x) = \frac{1}{x}.$$

Integrating gives an implicit form of the solution,

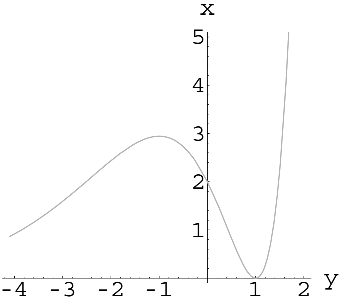
$$y(x) + \ln\left(y(x)-1\right)^2 = \ln|x| + c_1$$

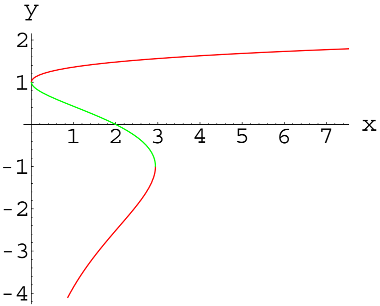
and $x = c_2\left(y(x)-1\right)^2 e^{y(x)}$. The initial condition $y(2) = 0$ implies $c_2 = 2$ and we find for the inverse function, where it exists,

$$x(y) = 2\left(y-1\right)^2 e^y.$$

Some more reflection on the next page.

If one studies the function $y \mapsto 2\,(y-1)^2\, e^y$ one finds that the interval around 0 where this function is monotone, is $[-1,1]$. Hence only there it is the inverse of the solution $y(x)$ of the differential equation. See the graphs down here. The grey one on the left depicts $y \mapsto 2\,(y-1)^2\, e^y$ and the right one its reflection in $y = x$. Only the green part can be used for the function $x \mapsto y(x)$ that solves the differential equation.





Back to Question 9

Solution to Question 10.
Since $y_1(x) = 1$ is a solution to the linear differential equation

$$x\,y''(x) + (x-1)\,y'(x) - y(x) + 1 = 0 \tag{2}$$

one finds that $v_a(x) = y_2(x) - y_1(x) = x - 1$ and $v_b(x) = y_3(x) - y_1(x) = e^{-x}$ solve the homogeneous equation

$$x\,v''(x) + (x-1)\,v'(x) - v(x) = 0. \tag{3}$$

Hence the general solution to (3) can be written as

$$v(x) = c_1\,v_a(x) + c_2\,v_b(x).$$

The general solution to (2) turns out to be

$$y(x) = y_1(x) + c_1\,v_a(x) + c_2\,v_b(x) =$$

$$= (1 - c_1 - c_2)\,y_1(x) + c_1\,y_2(x) + c_2\,y_3(x).$$

Also any cyclical change of y_1, y_2, y_3 will do.

Instead of these theoretical considerations one could just have plugged the given combinations in. Back to Question 10