COMPLETE SOLUTIONS IN THE
THEORY OF ELASTIC MATERIALS
WITH VOIDS—II

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SUMMARY

A system of governing equations for the stress-volume fraction formulation of the
linear theory of homogeneous and isotropic elastic materials with voids is obtained.
A complete solution of this coupled system is developed and its connections with the
complete solutions in the displacement-volume fraction formulation of the theory
are exhibited. The governing equations are decoupled with the aid of this solution.
Analogous results valid in the coupled thermoelasticity theory and the poroelasticity
theory are presented in a unified way.

1. Introduction

This paper is a continuation of the author's earlier paper on the complete
solutions in the linear theory of homogeneous and isotropic elastic materials
containing a distribution of vacuous pores (voids) (1). While the
displacement-volume fraction formulation of the theory has been considered
in (1), the stress-volume fraction formulation is considered in the present
paper. A system of governing equations for the stress-volume fraction
formulation of the theory is deduced in sections 3 and 4 by starting with the
fundamental equations of the theory, summarized in section 2. One of the
equations of this system is a compatibility equation of the Beltrami–Michell
type, and the other is a consequence of the balance of equilibrated force
(2). A complete solution of this system is developed in section 5. This
solution is analogous to and includes as special cases Schaefer's solution in
classical elastostatics (3) and Teodorescu's solution in classical elastodynamics (4). The connection which the solution has with the complete
solutions in the displacement-volume fraction formulation of the theory is
exhibited in section 6. The solution is employed to decouple the governing
equations in section 7.

There exists a close resemblance between the fundamental equations of
the theory of elastic materials with voids and the poroelasticity and
thermoelasticity theories (5). Indeed, one can write down, in a rather ad
hoc way, a set of equations that includes as special cases the fundamental
equations of all the three theories. Generalizations of the results of (1),
obtained by this unified approach, have been presented in (6). Generaliza-
tions of the results of the present paper, obtained in the same fashion, are contained in the Appendix.

2. Fundamental equations

The fundamental equations of the linear theory of homogeneous and isotropic elastic materials with voids are given as follows (2). The kinematic equation is

\[ E = \mathbf{\dot{v}} u = \tfrac{1}{2} (\nabla u + \nabla^T u); \]  

the balance equations are

\[ \text{div} \mathbf{T} + \mathbf{b} = \rho \frac{\partial^2 u}{\partial t^2}, \]

\[ \text{div} \mathbf{h} + g + l = \rho k \frac{\partial^2 \phi}{\partial t^2}; \]

and the constitutive equations are

\[ \mathbf{T} = \left[ \lambda (\text{tr} \mathbf{E}) + \beta \phi \right] \mathbf{I} + 2 \mu \mathbf{E}, \]

or

\[ \mathbf{E} = \frac{1}{E} \left[ (1 + \nu) \mathbf{T} - \{\nu (\text{tr} \mathbf{T}) + \beta (1 - 2 \nu) \phi \} \mathbf{I} \right], \]

\[ \mathbf{h} = \alpha \nabla \phi, \]

\[ g = -\xi \phi - \omega \frac{\partial \phi}{\partial t} - \beta (\text{tr} \mathbf{E}). \]

In these equations, \( \mathbf{E} \) is the strain tensor, \( \mathbf{T} \) is the stress tensor, \( \mathbf{I} \) is the unit tensor (of order two), \( \mathbf{h} \) is the equilibrated stress vector, \( g \) is the intrinsic equilibrated body force, and \( E \) and \( \nu \) are the Young modulus and Poisson ratio, respectively, of the material. The direct tensor notation (7) is adopted in order to make the analysis valid in all orthogonal coordinate systems. The other symbols and notation are as explained in (1). Also, all the assumptions made in (1) are valid here also.

We need the following well-known relations:

\[ \lambda = \frac{E \nu}{(1 + \nu)(1 - 2 \nu)} , \quad \mu = \frac{E}{2(1 + \nu)} , \quad c_1^2 = \frac{2(1 - \nu)}{1 - 2 \nu} c_2^2 . \]  

Using (2.7)_3, the symbol \( a \), defined in (1), can be expressed as follows:

\[ a^2 = \frac{c_2^2}{2c_1^2 (1 - 2 \nu)} = \frac{1}{4(1 - \nu)} . \]

Then (1, (2.13) to (2.15)) become

\[ \Box_1 - \Box_2 = \frac{c_2^2}{1 - 2 \nu} \Delta, \]
3. Equations of motion

Substituting for $E$ from (2.1) in (2.4) and inserting the resulting expression into (2.2), we obtain the following equation, on using (2.7):

$$
\Box_3 D_3 - c_1^2 D_2 = \frac{2\beta^2 (1 - \nu)}{\rho} \Box_2,
$$

(2.10)

$$
D_2 - D_1 \Box_2 = \frac{1}{2(1 - \nu)} D_3 \Delta.
$$

(2.11)

Also, eliminating $g$ and $h$ from equations (2.3), (2.5) and (2.6), and using (2.1), we obtain the equation

$$
D_1 \phi - \beta \text{div} \ u + l = 0.
$$

(3.2)

Equations (3.1) and (3.2), expressed exclusively in terms of $u$ and $\phi$, are the equations of motion in the displacement-volume fraction formulation of the theory. These are the equations for which complete solutions have been developed in (1).

On the other hand, suppose we eliminate $u$ and $E$ from (2.1), (2.2) and (2.4). Then we obtain the following equation:

$$
2 \hat{\nabla} (\text{div} \ T) - \frac{1}{c_2^2} \frac{\partial^2}{\partial t^2} \left[ T - \frac{\nu}{1 + \nu} (\text{tr} \ T) I \right] + \frac{\beta (1 - 2\nu)}{c_2^2 (1 + \nu)} \frac{\partial^2 \phi}{\partial t^2} I + 2 \hat{\nabla} b = 0.
$$

(3.3)

Also, suppose we eliminate $g$, $h$ and $E$ from (2.3), (2.4), (2.5) and (2.6). Then we obtain the equation

$$
D_1 \phi + \frac{\beta (1 - 2\nu)}{E} (3\phi - \text{tr} \ T) + l = 0.
$$

(3.4)

This equation can also be deduced from (3.2) on substituting for $\text{div} \ u$ computed from (2.1) and (2.4).

Equations (3.3) and (3.4), expressed exclusively in terms of $T$ and $\phi$, are the equations of motion in the stress-volume fraction formulation of the theory. A theorem on the uniqueness of solution of these equations has been obtained in (8).

4. Equation of compatibility

The following well-known compatibility condition is a direct consequence of the kinematic equation (2.1):

$$
\text{curl} \ \text{curl} \ E = 0.
$$
This condition can be put in the following equivalent form (7, p. 41):

$$\Delta E + \nabla \nabla (tr E) - 2 \hat{\nabla} (\text{div} E) = 0. \quad (4.1)$$

Substituting for $E$ from (2.4) in this equation, and eliminating $\hat{\nabla} (\text{div} T)$ from the resulting equation and the equation of motion (3.3), we arrive at the following equation:

$$\Box_2 \left[ T - \frac{1}{1 + \nu} (v(tr T) + (1 - 2\nu)\beta \phi)I \right] +
\frac{c_s^2}{1 + \nu} \left[ \nabla \nabla (tr T) - (1 - 2\nu)\beta \nabla \nabla \phi \right] + 2c_s^2 \hat{\nabla} b = 0. \quad (4.2)$$

Taking the trace of this equation and using (2.7), we obtain

$$\Box_1 (tr T) = \beta (3\Box_2 \phi + c_s^2 \Delta \phi) - \frac{2(1 + \nu)}{1 - 2\nu} c_s^2 \text{div} b. \quad (4.3)$$

With the aid of this expression and (2.9), equation (4.2) reduces to the following form:

$$\Box_2 T + \frac{c_s^2}{1 + \nu} \left[ \nabla \nabla (tr T) + \frac{\nu}{1 - 2\nu} \Delta (tr T)I \right] - \beta \left[ \Box_2 \phi + \frac{\nu}{1 + \nu} c_s^2 \Delta \phi \right] I - \frac{1 - 2\nu}{1 + \nu} \beta c_s^2 \nabla \nabla \phi +
\frac{2\nu}{1 - 2\nu} c_s^2 (\text{div} b)I + 2c_s^2 \hat{\nabla} b = 0. \quad (4.4)$$

This is the desired compatibility equation, expressed exclusively in terms of $T$ and $\phi$.

The compatibility equation (4.4) can also be deduced by starting with the equation of motion (3.1). Taking the divergence of (3.1) and using (2.9), we obtain

$$\Box_1 (\text{div} u) + \frac{\beta}{\rho} \Delta \phi + \frac{1}{\rho} \text{div} b = 0. \quad (4.5)$$

Equations (3.1) and (4.5) yield

$$\frac{E}{1 + \nu} \hat{\nabla} \left[ \Box_2 u + \frac{c_s^2}{1 - 2\nu} \nabla (\text{div} u) + \frac{\beta}{\rho} \nabla \phi + \frac{1}{\rho} b \right] +
\frac{Ev}{(1 + \nu)(1 - 2\nu)} \left[ \Box_1 (\text{div} u) + \frac{\beta}{\rho} \Delta \phi + \frac{1}{\rho} \text{div} b \right] I = 0.$$

Rearranging the terms in this equation and using (2.1), (2.4) and (2.7), we arrive at (4.4).
In the absence of voids ($\phi = 0$), equation (4.4) reduces to

$$\Box_2 T + \frac{c_2^2}{1 + \nu} \left[ \nabla \nabla (\text{tr } T) + \frac{\nu}{1 - 2\nu} \Delta (\text{tr } T) I \right] +$$

$$+ \frac{2\nu}{1 - 2\nu} c_2^2 (\text{div } b) I + 2 c_2^2 \nabla \phi = 0. \quad (4.6)$$

This is the well-known Beltrami–Michell equation of classical elastodynamics. Apart from the notation, this equation is identical with (9, equation (5.5.9); 4, equation (2.6)).

In the time-independent case we have $\Box_2 = c_2^2 \Delta$ and equation (4.4) reduces to the following form:

$$\Delta T + \frac{1}{1 + \nu} \left[ \nabla \nabla (\text{tr } T) + \frac{\nu}{1 - 2\nu} \Delta (\text{tr } T) I \right]$$

$$- \frac{\beta}{1 + \nu} \left[ (1 + 2\nu) \Delta \phi I - (1 - 2\nu) \nabla \nabla \phi \right] +$$

$$+ \frac{2\nu}{1 - 2\nu} (\text{div } b) I + 2 \nabla \phi = 0. \quad (4.7)$$

This equation is analogous to (10, equation (2.12)) obtained in the context of the dilatational theory of elasticity (which is just another version of the theory considered here).

Taking the trace of equation (4.7) we obtain

$$\Delta (\text{tr } T) = \frac{(1 + 4\nu)(1 - 2\nu)}{(1 + \nu)(1 - \nu)} \beta \Delta \phi - \frac{1 + \nu}{1 - \nu} \text{div } b. \quad (4.8)$$

This is the three-dimensional counterpart of (11, equation (3.13)).

In the absence of voids, equations (4.7) and (4.8) together yield

$$\Delta T + \frac{1}{1 + \nu} \nabla \nabla (\text{tr } T) + \frac{\nu}{1 - \nu} (\text{div } b) I + 2 \nabla \phi = 0. \quad (4.9)$$

This is the well-known Beltrami–Michell equation of classical elastostatics (7, p. 92).

5. Complete representation for the pair $(T, \phi)$

The equation of motion (3.4) and the compatibility equation (4.4) together represent seven equations for seven unknown functions, namely $\phi$ and the six components of the symmetric tensor $T$. Therefore, these equations may also be regarded as a system of governing equations for the stress–volume fraction formulation of the theory. The solution of this coupled system is facilitated if we first solve equations (3.4) and (4.3) simultaneously for $\phi$ and $\text{tr } T$, and then (after determining $\phi$ and $\text{tr } T$) solve...
equation (4.4) for \( T \). We now proceed to obtain a complete representation for the solution pair \((T, \phi)\) of equations (3.4) and (4.4).

For convenience, we first rewrite (3.4) and (4.4) in the following form:

\[
L_3(\phi, T) + l = 0, \quad (5.1)
\]
\[
L_4(T, \phi) + \frac{2\nu}{1 - 2\nu} c_2^2 (\text{div} b) I + 2 c_2^2 \nabla b = 0. \quad (5.2)
\]

Here

\[
L_3(\phi, T) = \left[ D_1 + \frac{3\beta(1 - 2\nu)}{E} \right] \phi - \frac{\beta(1 - 2\nu)}{E} (\text{tr} T), \quad (5.3)
\]
\[
L_4(T, \phi) = \Box_2 T + \frac{c_2^2}{1 + \nu} \left[ \nabla \nabla (\text{tr} T) + \frac{\nu}{1 - 2\nu} \Delta (\text{tr} T) I \right]
- \beta \left[ \Box_2 \phi + \frac{\nu}{1 + \nu} c_2^2 \Delta \phi \right] I - \frac{1 - 2\nu}{1 + \nu} \beta c_2^2 \nabla \nabla \phi. \quad (5.4)
\]

In classical elastodynamics, a complete representation for \( T \) has been obtained by Teodorescu (4) by solving the Beltrami–Michell equation (4.6). This representation is given as follows:

\[
T = 2[c_2^2 \nabla \nabla s - \nu \Box_2 s I - \nabla c]. \quad (5.5)
\]

Here \( c \) and \( s \) obey the equations

\[
\Box_2 c = c_2^2 b, \quad (5.6)
\]
\[
\Box_1 s = \frac{1}{1 - 2\nu} \text{div} c. \quad (5.7)
\]

This representation is the dynamic counterpart of Schaefer's representation in elastostatics (3) given as follows:

\[
T = (\text{div} c - 2c_2^2 \Delta s) I + 2c_2^2 \nabla \nabla s - 2 \nabla c. \quad (5.8)
\]

Here \( c \) and \( s \) obey the equations

\[
\Delta c = b, \quad (5.9)
\]
\[
\Delta s = (2a^2/c_2^2) \text{div} c. \quad (5.10)
\]

(Our symbols and notation differ slightly from those employed in (3, 4).)

In the context of the theory considered in this paper, we seek the following generalized version of (5.8):

\[
T = \left[ \text{div} c - (\Box_2 + c_2^2 \Delta) s \right] I + 2c_2^2 \nabla \nabla s - 2 \nabla c. \quad (5.11)
\]

Here \( c \) and \( s \) are to be determined appropriately.

Substituting for \( T \) from (5.11) in the right-hand side of (5.4), we find, on
using (2.9), that
\[ L_4(T, \phi) = \left[ \frac{1 - 2\nu}{1 + \nu} c_2^2 \nabla \nabla + \mathbf{I} \left\{ \mathbf{\square}_2 + \frac{\nu}{1 + \nu} c_2^2 \Delta \right\} \right] \times \]
\[ \times \left[ \frac{1}{1 - 2\nu} \text{div } c - \mathbf{\square}_1 \mathbf{s} - \beta \phi \right] \]
\[ - 2 \left[ \mathbf{\nabla} + \mathbf{I} \frac{\nu}{1 - 2\nu} \text{div } \mathbf{\square}_2 c. \right] \quad (5.12) \]

Clearly, if we set
\[ \beta \phi = \frac{1}{1 - 2\nu} \text{div } c - \mathbf{\square}_1 \mathbf{s} \quad (5.13) \]

and assume that \( c \) obeys the equation
\[ \mathbf{\square}_2 c = c_2^2 \mathbf{b}, \quad (5.14) \]

then equation (4.4) is readily satisfied.

Substituting for \( T \) and \( \phi \) from (5.11) and (5.13) into the right-hand side of (5.3), and using (2.7), (2.8) and (2.9), we obtain
\[ \beta L_3(\phi, T) = \frac{(2a^2/c_2^2)}{D_3(\text{div } c) - D_2s}. \quad (5.15) \]

We readily see that if \( s \) is assumed to obey the equation
\[ D_2s = \frac{2a^2}{c_2^2} D_3(\text{div } c) + \beta l, \quad (5.16) \]

then equation (3.4) is also satisfied.

Thus, if \( c \) and \( s \) are arbitrary functions obeying equations (5.14) and (5.16), then (5.11) and (5.13) constitute a solution of the equations (3.4) and (4.4).

We now show that this solution is complete in the sense that every solution \( (T, \phi) \) of the system (3.4), (4.4), which is consistent with the kinematic and constitutive relations of the theory, admits a representation as described by (5.11), (5.13), (5.14) and (5.16).

Suppose that \( (T, \phi) \) is an arbitrary solution of equations (3.4) and (4.4) which is consistent with the kinematic and constitutive equations of the theory. Then \( T \) and \( \phi \) are to be related with each other through an equation of the form
\[ T = [\lambda \text{div } \mathbf{u} + \beta \phi] \mathbf{I} + 2\mu \mathbf{\nabla} \mathbf{u} \quad (5.17) \]

for some displacement vector \( \mathbf{u} \); see (2.1) and (2.4)\). In view of the Helmholtz representation of a vector field, there exist functions \( p \) and \( q \) such that
\[ \mathbf{u} = \nabla p + \text{curl } q. \quad (5.18) \]
We set
\[ c = -\mu (\text{curl } q + \nabla \Lambda_0), \quad (5.19) \]
\[ s = \rho (p - \Lambda_0), \quad (5.20) \]
where \( \Lambda_0 \) is such that
\[ \Box_2 \Lambda_0 = (\beta / \rho) \phi + \Box_1 p. \quad (5.21) \]

An explicit form of such a \( \Lambda_0 \) is given by (1, (4.8)).

Substituting for \( p \) and \( \text{curl } q \) from (5.20) and (5.19) in (5.18), we get
\[ \lambda \text{ div } u + \beta \phi = (\Box_2 + c_2^2 \Delta) s - \text{ div } c. \quad (5.22) \]

Also, together with (2.9), the relations (5.18) to (5.21) yield
\[ \lambda \text{ div } u + \beta \phi = (\Box_2 + c_2^2 \Delta) s - \text{ div } c. \quad (5.23) \]

With the aid of (5.22) and (5.23), the expression (5.17) becomes (5.11).

Eliminating \( p \) from (5.20) and (5.21) and using (2.12) and (5.19), we obtain (5.13).

Substituting for \( T \) and \( \phi \) from (5.11) and (5.13) into the right-hand sides of (5.3) and (5.4) and using (2.7), (2.8) and (2.9), we obtain (5.15) and
\[ L_4(T, \phi) = -2 \left[ \hat{\phi} + \frac{\nu}{1 - 2\nu} \text{ div } c \right] \Box_2 c. \quad (5.24) \]

Since \( (T, \phi) \) is a solution of equations (5.1) and (5.2), it follows that \( s \) obeys (5.16) and that \( c \) obeys the equation
\[ \Box_2 c = c_2^2 b + c^*, \quad (5.25) \]
where \( c^* \) is an arbitrary vector field such that \( \hat{\phi} c^* = 0 \) and \( \text{ div } c^* = 0 \). But for \( c^* \), the equation (5.25) is identical with (5.14). It may be verified that \( c^* \) makes no contribution to the stress and volume-fraction fields; we may therefore set \( c^* = 0 \) without loss of generality. This proves the completeness of the solution described by (5.11), (5.13), (5.14) and (5.16).

In the absence of voids \( (\beta = l = \phi = 0) \), we have \( (1) (1/\alpha) D_2 = \Delta \Box_1 \) and \( (1/\alpha) D_3 = c_2^2 \Delta \). Then (5.13) gives
\[ \Box_1 s = \frac{1}{1 - 2\nu} \text{ div } c \quad (5.26) \]
and (5.16) is identically satisfied. Thus, in the context of classical elastodynamics, (5.11) gives a complete representation for \( T \) when \( c \) and \( s \) obey equations (5.14) and (5.26). With the aid of (2.9) and (5.26), expression (5.11) can be rewritten as
\[ T = 2[c_2^2 \nabla s - \hat{\phi} c - \nu \Box_2 s]. \quad (5.27) \]
Equations (5.27), (5.14) and (5.26) are precisely (5.5), (5.6) and (5.7)
respectively. Teodorescu's solution in classical elastodynamics is thus recovered.

In the time-independent case, (5.11), (5.13), (5.14) and (5.16) become

\[ \mathbf{T} = [\text{div} \mathbf{c} - 2c_2^2 \Delta s] \mathbf{I} + 2c_2^2 \nabla \nabla s - 2 \hat{\nabla} \mathbf{c}, \quad (5.28) \]
\[ \beta \phi = \frac{1}{1 - 2\nu} \text{div} \mathbf{c} - c_2^2 \Delta s, \quad (5.29) \]
\[ \Delta \mathbf{c} = \mathbf{b}, \quad (5.30) \]
\[ D_4 \Delta s = \frac{2a^2}{c_2^2} \left[ D_4 + \frac{\beta^2}{\rho} (1 - 2\nu) \right] \text{div} \mathbf{c} + \beta l. \quad (5.31) \]

Thus, in time-independent problems, (5.28) and (5.29) constitute a complete representation for \((\mathbf{T}, \phi)\) (in the presence of voids) when \(\mathbf{c}\) and \(s\) obey (5.30) and (5.31).

In the absence of voids (5.28) to (5.30) reduce to (5.8) to (5.10). Equation (5.31) is then identically satisfied, and Schaefer's solution in classical elastostatics is thus recovered.

6. Connections with the representations for \((\mathbf{u}, \phi)\)

By using the kinematic relation (2.1) and the constitutive equation (2.4), it is straightforward to verify that

\[ \mathbf{u} = \left( 1/\mu \right) \left( c_2^2 \nabla s - \mathbf{c} \right) \quad (6.1) \]

is the displacement field associated with \(\mathbf{T}\) and \(\phi\) represented by (5.11) and (5.13). This displacement field is unique up to a rigid-body motion.

Suppose that we set

\[ \mathbf{\Omega} = - \frac{1}{\mu} \mathbf{c}, \quad (6.2) \]
\[ \Lambda = \frac{1}{\mu} \left[ \mathbf{r} \cdot \mathbf{c} - \frac{c_2^2}{a^2} s \right]. \]

Then (6.1) and (5.13) become

\[ \begin{aligned} \mathbf{u} &= \mathbf{\Omega} - a^2 \nabla (\Lambda + \mathbf{r} \cdot \mathbf{\Omega}), \\
(\beta/\rho) \phi &= a^2 [\square_1 (\Lambda + \mathbf{r} \cdot \mathbf{\Omega}) - 2c_2^2 \text{div} \mathbf{\Omega}]. \end{aligned} \quad (6.3) \]

Also, the governing equations (5.14) and (5.16) for \(\mathbf{c}\) and \(s\) imply that \(\mathbf{\Omega}\) and \(\Lambda\) obey the following equations:

\[ \square_2 \mathbf{\Omega} = -(1/\rho) \mathbf{b}, \quad (6.4) \]
\[ D_2 (\Lambda + \mathbf{r} \cdot \mathbf{\Omega}) = 2D_2 (\text{div} \mathbf{\Omega}) - (\beta l/\rho a^2). \]

The relations (6.3) and (6.4) describe the Boussinesq–Papkovitch–Neuber
type representation for \((u, \phi)\) obtained in (1). By reversing the algebra, one can recover the relations (5.11), (5.13), (5.14) and (5.16) starting with (6.3) and (6.4).

In an analogous way, one can deduce from (5.11), (5.13), (5.14) and (5.16) the Green–Lamé type and the Cauchy–Kovalevski–Somigliana type representations for \((u, \phi)\), obtained in (1), and vice-versa, by making use of the following relations:

\[
\begin{align*}
\varphi &= \frac{1}{\rho} s + \Gamma, \\
\text{curl } \psi &= -\left(\nabla \Gamma + \frac{1}{\mu} c\right), \\
D_2 G &= -\frac{\alpha}{\mu} c, \\
\frac{\beta}{\rho} H &= -\left[\frac{\alpha}{\rho a^2} s + 2D_3 (\text{div } \mathbf{G})\right].
\end{align*}
\] (6.5) (6.6)

Here \(\Gamma\) is as defined by (1, 6.2)).

Thus, the Schaefer–Teodorescu type representation for \((T, \phi)\), obtained in section 5, implies and is implied by each of the three representations for \((u, \phi)\) obtained in (1). The completeness of the latter representations substantiates the completeness of the former representation, and vice-versa.

In classical elastodynamics, Teodorescu (4) has asserted the completeness of his representation for \(T\) (in the absence of body force) on the basis of the connection that the representation has with the Cauchy–Kovalevski–Somigliana representation for \(u\). The relations he has employed in this regard can be recovered as particular cases of (6.6).

Suppose that we set

\[
\tilde{\Lambda} = \frac{1}{2}(\Lambda + \mathbf{r} \cdot \mathbf{\Omega}).
\] (6.7)

Then (6.3) and (6.4) become

\[
\begin{align*}
\mathbf{u} &= \mathbf{\Omega} - 2a^2 \nabla \tilde{\Lambda}, \\
\frac{\beta}{\rho} \phi &= 2a^2 [\nabla_1 \tilde{\Lambda} - c_1^2 \text{div } \mathbf{\Omega}], \\
\Box_2 \mathbf{\Omega} &= -\frac{1}{\rho} \mathbf{b}, \\
D_2 \tilde{\Lambda} &= D_3 (\text{div } \mathbf{\Omega}) - \frac{\beta l}{2\rho a^2}.
\end{align*}
\] (6.8) (6.9)
In the absence of voids, \((6.8)_2\) yields
\[\Box_i \bar{\lambda} = c_i^2 \text{div} \Omega\]  
and \((6.9)_2\) is identically satisfied.

In the time-independent case, \((6.9)_1\) and \((6.10)\) become
\[
\begin{align*}
\Delta \Omega &= -(1/\mu) b, \\
\Delta \bar{\lambda} &= \text{div} \Omega.
\end{align*}
\]  

Thus, in classical elastostatics, \((6.8)_1\) gives a complete representation for \(\mathbf{u}\) when \(\Omega\) and \(\bar{\lambda}\) obey equations \((6.11)\). This is precisely the Naghdi–Hsu solution in classical elastostatics \((12)\). Equations \((6.8)_1\), \((6.9)_1\) and \((6.10)\) describe the counterpart of this solution in classical elastodynamics.

The connection between the complete solution described by \((6.8)\) and \((6.9)\) (which may be referred to as the Naghdi–Hsu type solution) and the Schaefer–Teodorescu type solution can be exhibited through the following relations:
\[
\begin{align*}
\Omega &= -\frac{1}{\mu} \mathbf{c}, \\
\bar{\lambda} &= -\frac{1}{2\rho a^2} s.
\end{align*}
\]  

These are just alternative versions of \((6.2)\).

7. Uncoupling of \(\phi\) and \(T\)

In what follows, we decouple the equations \((3.4)\) and \((4.4)\) by making use of the representations for \(T\) and \(\phi\) obtained in section 5.

Eliminating \(c\) and \(s\) from \((5.13)\) and \((5.16)\) with the aid of \((5.14)\) and \((2.10)\), we obtain the following equation that contains \(\phi\) as the only unknown function:
\[D_2 \phi + (\beta/\rho)(\text{div} \mathbf{b}) + \Box_i l = 0.\]  

Together with \((5.14)\), \((5.16)\) and \((2.11)\), the expression \((5.11)\) yields
\[D_2 (\text{tr} T) - (c_2^2 D_1 - 6a^2 D_3)(\text{div} \mathbf{b}) + (3\Box_2 + c_2^2 \Delta) \beta l = 0.\]

Equation \((7.1)\) is identical with \((7.1)\) of \((1)\). Also, \((7.2)\) is analogous to \((7.2)\) of \((1)\). From \((7.2)\) we note that in the absence of \(\mathbf{b}\) and \(l\), \(\text{tr} T\) obeys the same equation as that obeyed by \(\varphi\), \(\phi\) and \(\text{div} \mathbf{u}\), namely \(D_2 F = 0\), see \((1)\).

With the aid of \((7.1)\) and \((7.2)\), equation \((4.4)\) yields the following equation that contains \(T\) as the only unknown function:
\[D_2 \Box_2 T + \frac{c_2^2}{1 + \nu} \nabla \nabla [D_0 (\text{div} \mathbf{b}) - 2(1 + \nu) \beta \Box_2 l] +
\]
\[+ \left[\frac{c_2^2 \nu}{(1 + \nu)(1 - 2\nu)} \right] (D_7 (\text{div} \mathbf{b}) - D_8 l) + D_9 \left\{\frac{\beta}{\rho} (\text{div} \mathbf{b}) + \Box_i l\right\} I +
\]
\[+ 2c_2^2 D_2 \nabla \mathbf{b} = 0.\]  

\(\Box_i l\)
Here
\[
\begin{align*}
D_6 &= (c_2^2 D_1 - 6a^2 D_3) + (1 - 2\nu) \frac{\beta^2}{\rho}, \\
D_7 &= 2(1 + \nu) D_2 + (c_2^2 D_1 - 6a^2 D_3) \Delta, \\
D_8 &= \beta (3\Box_2 + c_2^2 \Delta) \Delta, \\
D_9 &= \beta \left[ \Box_2 + \frac{\nu}{1 + \nu} c_2^2 \Delta \right].
\end{align*}
\]

The coupled system of equations (3.4) and (4.4) has thus been decoupled into two independent equations (7.1) and (7.3) governing \( \phi \) and \( T \) respectively. Whereas each of the equations in the coupled system (3.4), (4.4) is of order two, in the uncoupled system equation (7.1) is of order two and (7.3) is of order six. We note that in the absence of \( \mathbf{b} \) and \( l \), \( u \) and \( T \) both obey the equation \( D_2 \Box_2 F = 0 \); see (1, (7.3)) and (7.3) above.

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REFERENCES


APPENDIX

Consider the following generalization of equation (2.6):
\[
g = -\xi \phi - \omega \frac{\partial \phi}{\partial t} - \beta \left( \delta - \delta' \frac{\partial}{\partial t} \right) (\text{tr} \ E). \tag{2.6A}
\]
For \( \delta = 1 \) and \( \delta' = 0 \), this equation becomes identical with (2.6).
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For
\[ \xi = \delta = k = 0, \]  
(A.1)
equations (2.1) to (2.5) and (2.6A) represent the basic equations of linear coupled thermoelasticity (13, 14), if new meanings are given to some of the symbols, keeping the meanings of the other symbols unchanged. Under the new definitions, \( \omega \) is the specific heat per unit volume, \( \beta \) is the stress-temperature modulus, \( \delta' \) is the initial temperature, \( \phi \) is temperature change, \( \alpha \) is thermal conductivity, \( l \) is heat supply, \( (-h) \) is the heat flux, and \( (-g) \) is the product of the initial temperature and entropy per unit volume.

For
\[ \xi = \delta = k = l = 0, \quad \delta' = -1, \]  
(A.2)
equations (2.1) to (2.5) and (2.6A) represent the basic equations of the theory of elastic materials with fluid-filled pores (poroelasticity) (14, 15), if new meanings are given to some of the symbols, keeping the meanings of the other symbols unchanged. Thus \( T \) is now total stress, \( \phi \) is fluid pressure, \( \alpha \) is the permeability coefficient, \( g \) is the rate of increment of the fluid content of a porous element, and \( \beta, \omega \) are material constants characterizing the porosity of the material.

Thus, the coupled thermoelasticity theory, the theory of elastic materials with fluid-filled pores, as well as the theory of elastic materials with vacuous pores (voids) may be treated in a unified way by considering equations (2.1) to (2.5) and (2.6A).

Below, we present the modified versions of the results obtained in sections 5 to 7 that are valid in a (fictitious) continuum theory based on equations (2.1) to (2.5) and (2.6A). The corresponding results valid in the thermoelasticity theory and poroelasticity theory can be deduced from these results by invoking (A.1) and (A.2) respectively.

It is easily seen that equations (3.1), (3.3) and (4.4) do not depend on equation (2.6), and hence remain valid when (2.6) is replaced by (2.6A). But, then, equation (3.2) assumes the following modified form:

\[ D_1 \phi - \beta \left( \delta - \delta' \frac{\partial}{\partial t} \right) \text{div} \mathbf{u} + l = 0. \]  
(3.2A)

Results of (1) have been extended to the system (3.1), (3.2A) in (6). The Naghdi–Hsu type solution for this system is given by (6.8) above with \( \Omega \) obeying (6.9), and \( \Lambda \) obeying the following equation:

\[ D_2 \hat{\Lambda} = D_3^*(\text{div} \mathbf{u}) - \frac{\beta l}{2\rho a^2}. \]  
(6.9A)_2

Here

\[ D_2^* = D_1 + \frac{\beta^2}{\rho} \left( \delta - \delta' \frac{\partial}{\partial t} \right) \Delta, \]  
(A.3)

\[ D_3^* = c_i^2 D_1 + \frac{\beta^2}{2\rho a^2} \left( \delta - \delta' \frac{\partial}{\partial t} \right). \]  
(A.4)

Computing \( \text{div} \mathbf{u} \) from (2.1) and (2.4)_2 and substituting the resulting expression in (3.2A), we obtain the following modified form of (3.4):

\[ L^*_5(\phi, T) + l = 0. \]  
(5.1A)

Here

\[ L^*_5(\phi, T) = D_1 \phi + \frac{\beta (1-2\nu)}{E} \left( \delta - \delta' \frac{\partial}{\partial t} \right) [3\phi - (\text{tr} \mathbf{T})]. \]  
(5.3A)
Following step-by-step the arguments made in section 5, we find that a complete solution \((T, \phi)\) for the coupled system (4.4) and (5.1A) is described by (5.11), (5.13), (5.14) and the following equation:

\[
D_s^2 T = \frac{2a^2}{c_s^2} D_s^2 (\text{div} \mathbf{e}) + \beta l. \tag{5.16A}\]

The connection which this solution has with the complete solutions for the system (3.1) and (3.2A) presented in (6) and the Naghdi–Hsu type solution described by (6.8) (6.9), and (6.9A) can be obtained by making use of (6.2), (6.5), (6.12) and the following equations:

\[
\begin{align*}
D_s^2 \mathbf{G} &= -\frac{\alpha}{\mu} \mathbf{e}, \\
\frac{\beta}{\rho} H &= -\left[ \frac{\alpha}{\rho a^2 s} + 2D_s^2 (\text{div} \mathbf{G}) \right].
\end{align*} \tag{6.6A}\]

The modified forms of equations (7.1) and (7.2) read as follows:

\[
\begin{align*}
D_s^2 \phi + \frac{\beta}{\rho} \left( \delta - \delta' \frac{\partial}{\partial t} \right) (\text{div} \mathbf{b} + \Box, l) &= 0, \tag{7.1A}

D_s^2 (\text{tr} \mathbf{T}) - (c_s^2 D_s - 6a^2 D_s^2) \text{div} \mathbf{b} + (3\Box_2 + c_s^2 \Delta) \beta l &= 0. \tag{7.2A}
\end{align*}
\]

The modified version of equation (7.3) is given as follows:

\[
\begin{align*}
D_s^2 \Box_2 \mathbf{T} + \frac{c_s^2}{1 + \nu} \nabla \nabla [D_s^2 (\text{div} \mathbf{b}) - 2(1 + \nu) \beta \Box_2 l] &+ \\
&+ \left[ \frac{c_s^2 \nu}{(1 + \nu)(1 - 2\nu)} \left( D_s^2 (\text{div} \mathbf{b}) - D_s \mathbf{a} \right) \right. \\
&\quad + D_s \left[ \frac{\beta}{\rho} \left( \delta - \delta' \frac{\partial}{\partial t} \right) (\text{div} \mathbf{b} + \Box, l) \right] \right] + 2c_s^2 D_s^2 \hat{\mathbf{v}} \mathbf{b} &= 0. \tag{7.3A}
\end{align*}
\]

Here

\[
\begin{align*}
D_s &= (c_s^2 D_s - 6a^2 D_s^2) + \frac{\beta^2 (1 - 2\nu)}{\rho} \left( \delta - \delta' \frac{\partial}{\partial t} \right), \\
D_s^* &= 2(1 + \nu)D_s + (c_s^2 D_s - 6a^2 D_s^2) \Delta.
\end{align*}
\]