

Among all 2-dimensional convex domains the disk is not optimal for the
lifetime of a conditioned Brownian motion.

— *the extended version** —

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February 23, 2001

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*short version accepted for publication ([10])

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1 Introduction

Probabilistic background: Let $\Omega \subset \mathbb{R}^2$ be a convex bounded domain and let τ_Ω denote the lifetime of a Brownian motion in Ω , starting at $y \in \bar{\Omega}$, to be stopped at $x \in \bar{\Omega}$ and conditioned to be killed at the boundary $\partial\Omega$. For $n > 1$ the expectation for this lifetime is known to be equal

$$E_x^y(\tau_\Omega) = \frac{\int_{z \in \Omega} G_\Omega(x, z) G_\Omega(z, y) dz}{G_\Omega(x, y)}, \tag{1}$$

where $G_\Omega(x, y)$ is the Dirichlet Green function for the laplacian (or, in probabilistic setting, $\frac{1}{2}$ laplacian). Indeed, for such a conditioned Brownian motion the corresponding diffusion has transition density

$$p_\Omega^y(t, x, z) = \frac{p_\Omega(t, x, z) G_\Omega(z, y)}{G_\Omega(x, y)},$$

where $p_\Omega(t, x, z)$ is the standard kernel of the parabolic Dirichlet problem. We refer to [6], [14], [3] or [2].

The expectation for the lifetime of conditioned Brownian motion plays an important role in the probabilistic approach to Schrödinger operators. Major estimates on $E_x^y(\tau_\Omega)$ are due to Cranston and McConnell. Indeed, in [4] it is shown that there exists an absolute constant c such that for every two-dimensional domain

$$E_x^y(\tau_\Omega) \leq c |\Omega|, \tag{2}$$

where $|\Omega|$ is Lebesgue measure of Ω . In higher dimensions, $n \geq 3$, Cranston [5] proved $E_x^y(\tau_\Omega) \leq c(\Omega)$ and here $c(\Omega)$ does depend on the Lipschitz nature of the boundary. Local estimates of the type $E_x^y(\tau_\Omega) \leq f_\Omega(x, y)$ are established in [12].

A link to elliptic systems: Our interest in this quantity comes from the fact that λ_c , defined by $\lambda_c^{-1} = \sup \{E_x^y(\tau_\Omega); x, y \in \Omega\}$, appears as a bound for the parameter in noncooperative elliptic systems in order that such a system is positivity preserving (see [12]). For the trivially coupled system

$$\begin{cases} -\Delta u = f - \lambda v & \text{in } \Omega, \\ -\Delta v = f & \text{in } \Omega, \\ u = v = 0 & \text{on } \partial\Omega, \end{cases}$$

$f > 0$ implies $u > 0$ if and only if $\lambda < \lambda_c$. The positivity preserving property for this system implies that property for more generally coupled systems. We refer to [11]. Note that we use the analyst's $-\Delta$ instead of $-\frac{1}{2}\Delta$. All numbers quoted from the references have been rescaled to $-\Delta$. The reader from probability should add the factor 2.

Relation with domain shape: Let us return to known estimates for this expectation. If one looks for the maximum expected lifetime, heuristically this should be attained in a pair of points $(x, y) \in \Omega^2$ which are, in an appropriate metric, as far apart as possible. If Ω is a rectangle, the pair of points which maximize $E_x^y(\tau_\Omega)$ is expected to sit in diagonal corners, if it is an ellipse, they should sit on the long axes. In fact, Griffin, McConnell and Verchota have shown in [8, Corollary 2.4] that for planar domains these points are located at the boundary provided one of them is located at the boundary:

$$\sup \{E_x^y(\tau_\Omega); x \in \partial\Omega, y \in \bar{\Omega}\} = \sup \{E_x^y(\tau_\Omega); x, y \in \partial\Omega\} =: s(\Omega). \tag{3}$$

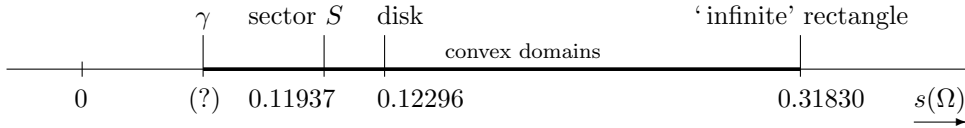
It is believed that a stronger statement holds true:

$$\sup \{E_x^y(\tau_\Omega); x, y \in \bar{\Omega}\} = \sup \{E_x^y(\tau_\Omega); x, y \in \partial\Omega\}. \quad (4)$$

It was also shown in [8, Theorem 3.1] that $s(\Omega)/|\Omega| < 1/\pi = .318309\dots$, and that this upper bound is optimal. In other words the best constant c for (2) is $1/\pi$.

In [13, Theorem 2] Jianming Xu studied estimates from below for $s(\Omega)$. He showed that there exists a universal constant γ for two dimensional convex domains such that $s(\Omega)/|\Omega| \geq \gamma$, and that without a convexity or similar assumption on Ω we cannot expect this result. In fact, Xu showed that amobae-like domains which have many thin necks can have $s(\Omega)/|\Omega|$ as small as we wish. Our paper is sort of a contribution towards determining γ . If a convex domain Ω is “everywhere thin”, in the sense that no two points x and y in Ω are too far apart from each other, then one might expect $s(\Omega)/|\Omega|$ to be small. This heuristic consideration led Griffin, McConnell and Verchota [8, p. 244] to the open question, whether the optimal constant γ is given by the disk, in other words, if $s(\Omega) \geq s(\Omega^*)$ for any convex plane Ω . Here Ω^* denotes the disk of same area as Ω . It is well known that the disk has many isoperimetric properties: it minimizes for instance the diameter, the perimeter or the first Dirichlet-Laplace eigenvalue of a domain of prescribed area. In [8, Prop. 4.2] it was shown (in our notation with Δ) that $s(\Omega^*)/|\Omega^*| = (2 \log 2 - 1)/\pi = 0.12296\dots$, which implies that $\gamma \leq 0.12296\dots$

Main result: We shall show that the disk does not minimize $s(\Omega)$ among all convex planar domains of given area by constructing a counterexample, that is a domain S for which $s(S)/|S| = 3/8\pi = 0.11937\dots$. Therefore the optimal γ must satisfy $\gamma \leq 0.11937\dots < 0.12296\dots$, and so the disk does not minimize $s(\Omega)/|\Omega|$.



Our choice of S was motivated by the consideration that domains with large perimeter but small diameter seem to have small $s(\Omega)/|\Omega|$. A larger perimeter leads to killing more of those Brownian paths that are “wandering around” and tends to decrease the expected lifetime. A good candidate in this respect is an equilateral triangle. Numerical results seemed to confirm, that on an equilateral triangle the maximum of $E_x^y(\tau_\Omega)$ is attained if x and y sit in corners of the triangle and this maximum value is very close to the one of the disk. The result however was to imprecise and hence unconvclusive. A major complication for a direct computation is the fact that we do not have an sufficiently simple explicit Green function for a triangle at our disposal and had to approximate it by a series.

On the other hand, replacing the straight lines of the triangles by convex arcs, even seemed to slightly decrease $s(\Omega)/|\Omega|$. This could hint at a domain like the Reuleaux-triangle [9], a ‘triangle’ with circular arcs, as a potential minimizer. However, the Green function for a Reuleaux-triangle does not seem to be available in a convenient form either. Therefore we looked for a domain which was close to a triangle or Reuleaux-triangle and for which we could calculate the Green function in an sufficiently simple formula. Such a domain indeed exists, namely a sector. To state our results, we define S to be a sector of the unit disk

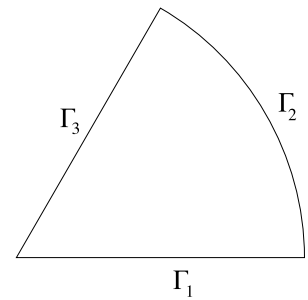
$$S := \{x \in \mathbb{R}^2; |x| < 1 \text{ and } 0 < \arg x < \frac{1}{3}\pi\}.$$

The main work of our paper goes into proving

Theorem 1 $\sup \{E_x^y(\tau_S); x \in \partial S, y \in \bar{S}\} = \frac{1}{16} = \frac{3}{8\pi}|S|$. Notice that $\frac{3}{8\pi} = 0.11937\dots$.

An immediate consequence is

Corollary 2 Among all convex two-dimensional domains Ω of given area the disk does not minimize $\sup \{E_x^y(\tau_S); x \in \partial\Omega, y \in \bar{\Omega}\}$.



Outline of the proof: In Section 2 we compute the explicit Green function for S . We are not able to explicitly calculate the iterated Green's function, i.e. the enumerator of (1), but due to (3) it is sufficient to analyze the behaviour of (1) for x and y on the boundary ∂S .

Note that both the enumerator and denominator in (1) go to zero on the boundary ∂S . In order to compute it for x and y on the boundary ∂S , we divide both terms in Section 3 in such way, that not only they converge to a nonvanishing function on the boundary. Moreover, we choose the divisor in such a way, that the remaining integrand becomes a rational function of z_1 and z_2 . Due to symmetry we have to distinguish just four cases for the locations of x and y , each of which is treated in a separate section, i.e. Sections 4 to 7.

We proceed by rewriting the integral in terms of polar coordinates (θ, r) . Since in each case the integrand is rational, it allows us to perform the integration with respect to θ by means of a contour integral in the complex plane. In fact, by grouping appropriate terms together, we are able to come to a closed contour. Distinguishing numerous subcases, we can in each subcase identify the poles of the integrand and apply the residue theorem to evaluate the integral with respect to θ . The resulting expressions contain rational functions and logarithms in r which allows us to perform an explicit integration with respect to r .

As one might guess, the expected lifetime has local maxima when x and y are two distinct corner points. The supremum is indeed attained, when x is at the vertex and y on any place of the circular sector of ∂S . If one restricts attention to these particular x and y only, the computations are much simpler. However, we felt the need to give a proof, that the supremum is attained in those points.

Since all these calculations are very elaborate but hardly contribute to the basic understanding, only a shorter version of this manuscript will be published ([10]).

2 The Green function

The Green function on the unit disk is as follows:

$$G_O(x, y) = \frac{1}{2\pi} \log \left(\frac{[X \ Y]}{|x - y|} \right),$$

where $[X \ Y] = |x|y| - y|y|^{-1}| = \sqrt{|x|^2|y|^2 - 2x \cdot y + 1}$. The function defined by

$$u(x) = \int_{|y| < 1} G_O(x, y) f(y) dy$$

solves, with $\Omega = O = \{x \in \mathbb{R}^2; |x| < 1\}$,

$$\begin{cases} -\Delta u = f & \text{in } \Omega, \\ u = 0 & \text{on } \partial\Omega. \end{cases}$$

With this function we may build the Green function for the sector S . Using the notation \mathcal{R} for rotating with $\frac{2}{3}\pi$ around $(0, 0)$ and \mathcal{S} for reflecting in $x_2 = 0$ we obtain

$$\begin{aligned} G_S(x, y) &= G_O(x, y) + G_O(\mathcal{R}x, y) + G_O(\mathcal{R}^2x, y) + \\ &\quad - G_O(\mathcal{S}x, y) - G_O(\mathcal{R}\mathcal{S}x, y) - G_O(\mathcal{R}^2\mathcal{S}x, y) \\ &= \frac{1}{2\pi} \log \left(\frac{[X \ Y][\mathcal{R}X \ Y][\mathcal{R}^2X \ Y] |Sx - y| |\mathcal{R}Sx - y| |\mathcal{R}^2Sx - y|}{|x - y| |\mathcal{R}x - y| |\mathcal{R}^2x - y| [\mathcal{S}X \ Y][\mathcal{R}\mathcal{S}X \ Y][\mathcal{R}^2\mathcal{S}X \ Y]} \right). \end{aligned}$$

Notice that we may rewrite $G_O(x, y)$ as

$$G_O(x, y) = \frac{1}{4\pi} \log \left(1 + \frac{(1 - |x|^2)(1 - |y|^2)}{|x - y|^2} \right) = \frac{1}{2\pi} \log \left| \frac{1 - \bar{\mathbf{x}}\mathbf{y}}{\mathbf{x} - \mathbf{y}} \right|$$

with $\mathbf{x} = x_1 + ix_2 \in \mathbb{C}$ etc.

3 Four boundary combinations and limits of the Green function

We introduce the notation

$$\begin{aligned}\Gamma_1 &= \{(a, 0); 0 \leq a \leq 1\}, \\ \Gamma_2 &= \{(\cos \theta, \sin \theta); 0 \leq \theta \leq \frac{1}{3}\pi\}, \\ \Gamma_3 &= \{(\tau \cos \frac{1}{3}\pi, \tau \sin \frac{1}{3}\pi); 0 \leq \tau \leq 1\}.\end{aligned}$$

Depending on the location of x and y in one of these Γ_i we have nine cases of which we distinguish the following four, since the remaining ones follow by symmetry.

$$1) x, y \in \Gamma_1, \quad 2) x \in \Gamma_1, y \in \Gamma_2, \quad 3) x, y \in \Gamma_2, \quad 4) x \in \Gamma_1, y \in \Gamma_3. \quad (5)$$

In order to evaluate the enumerator of (1), we take the limit inside the integral. Without this procedure the integrand contains logarithms which we are unable to handle. Subsequently we derive expressions for G as one of the arguments approaches the boundary. This will also help us to control the limiting behaviour. Since there are straight and circular parts of the boundary we distinguish two cases. For later reference let us record these.

a) For x near Γ_1 one has x_2 small and one finds by direct calculus that

$$\begin{aligned}G_S(x, y) &= \frac{1}{4\pi} \log \left(1 + \frac{4x_2 y_2}{|x - y|^2} \right) - \frac{1}{4\pi} \log \left(1 + \frac{4x_2 y_2}{[XY]^2} \right) + \\ &+ \frac{1}{4\pi} \log \left(1 - \frac{4x_2 (\frac{1}{2}y_2 + \frac{1}{2}\sqrt{3}y_1)}{|x - \mathcal{R}y|^2} \right) - \frac{1}{4\pi} \log \left(1 - \frac{4x_2 (\frac{1}{2}y_2 + \frac{1}{2}\sqrt{3}y_1)}{[XRY]^2} \right) + \\ &+ \frac{1}{4\pi} \log \left(1 - \frac{4x_2 (\frac{1}{2}y_2 - \frac{1}{2}\sqrt{3}y_1)}{|x - \mathcal{R}^2 y|^2} \right) - \frac{1}{4\pi} \log \left(1 - \frac{4x_2 (\frac{1}{2}y_2 - \frac{1}{2}\sqrt{3}y_1)}{[X\mathcal{R}^2 Y]^2} \right)\end{aligned} \quad (6)$$

with

$$\begin{aligned}\mathcal{R}y &= \left(-\frac{1}{2}y_1 - \frac{1}{2}\sqrt{3}y_2, \frac{1}{2}\sqrt{3}y_1 - \frac{1}{2}y_2 \right), \\ \mathcal{R}^2 y &= \left(-\frac{1}{2}y_1 + \frac{1}{2}\sqrt{3}y_2, -\frac{1}{2}\sqrt{3}y_1 - \frac{1}{2}y_2 \right).\end{aligned}$$

b) For x near Γ_2 one has $1 - |x|$ small and it follows that

$$\begin{aligned}G_S(x, y) &= \frac{1}{4\pi} \log \left(1 - \frac{(1 - |x|^2)(1 - |y|^2)}{|x - y|^2} \right) + \\ &+ \frac{1}{4\pi} \log \left(1 - \frac{(1 - |x|^2)(1 - |y|^2)}{|\mathcal{R}x - y|^2} \right) + \frac{1}{4\pi} \log \left(1 - \frac{(1 - |x|^2)(1 - |y|^2)}{|\mathcal{R}^2 x - y|^2} \right) + \\ &- \frac{1}{4\pi} \log \left(1 - \frac{(1 - |x|^2)(1 - |y|^2)}{|\mathcal{S}x - y|^2} \right) - \frac{1}{4\pi} \log \left(1 - \frac{(1 - |x|^2)(1 - |y|^2)}{|\mathcal{R}\mathcal{S}x - y|^2} \right) + \\ &- \frac{1}{4\pi} \log \left(1 - \frac{(1 - |x|^2)(1 - |y|^2)}{|\mathcal{R}^2 \mathcal{S}x - y|^2} \right).\end{aligned} \quad (7)$$

4 The case that $x, y \in \Gamma_1$

In the first step for the computation of (1) we shift the limits inside the integral as follows:

$$\lim_{x_2 \downarrow 0, y_2 \downarrow 0} \frac{\int_{z \in \Omega} G_\Omega(x, z) G_\Omega(z, y) dz}{G_\Omega(x, y)} = \frac{\int_S \lim_{x_2 \downarrow 0} \frac{G_S(x, z)}{x_2} \lim_{y_2 \downarrow 0} \frac{G_S(z, y)}{y_2} dz}{\lim_{x_2 \downarrow 0, y_2 \downarrow 0} \frac{G_S(x, y)}{x_2 y_2}}. \quad (8)$$

For all three limits in the right hand side we can use representation (6).

4.1 Limit of the Green function

From the expression in (6) we find

$$\begin{aligned} \lim_{x_2 \downarrow 0} \frac{G_S(x, y)}{x_2} &= \frac{1}{\pi} \left(\frac{y_2}{|x-y|^2} - \frac{y_2}{[XY]^2} - \frac{\frac{1}{2}y_2 + \frac{1}{2}\sqrt{3}y_1}{|x - \mathcal{R}y|^2} + \right. \\ &\quad \left. + \frac{\frac{1}{2}y_2 + \frac{1}{2}\sqrt{3}y_1}{[XRY]^2} - \frac{\frac{1}{2}y_2 - \frac{1}{2}\sqrt{3}y_1}{|x - \mathcal{R}^2y|^2} + \frac{\frac{1}{2}y_2 - \frac{1}{2}\sqrt{3}y_1}{[X\mathcal{R}^2Y]^2} \right). \end{aligned} \quad (9)$$

Replacing y by $z = (r \cos \theta, r \sin \theta)$ we obtain

$$\begin{aligned} \lim_{x_2 \downarrow 0} \frac{G_S(x, z)}{x_2} &= \frac{1}{\pi} \left(\frac{z_2}{|x|^2 - 2x_1z_1 + |z|^2} - \frac{z_2}{1 - 2x_1z_1 + |x|^2|z|^2} \right) + \\ &+ \frac{1}{\pi} \left(\frac{-\frac{1}{2}\sqrt{3}z_1 - \frac{1}{2}z_2}{|x|^2 - 2x_1(-\frac{1}{2}z_1 + \frac{1}{2}\sqrt{3}z_2) + |z|^2} - \frac{-\frac{1}{2}\sqrt{3}z_1 - \frac{1}{2}z_2}{1 - 2x_1(-\frac{1}{2}z_1 + \frac{1}{2}\sqrt{3}z_2) + |x|^2|z|^2} \right) + \\ &+ \frac{1}{\pi} \left(\frac{\frac{1}{2}\sqrt{3}z_1 - \frac{1}{2}z_2}{|x|^2 - 2x_1(-\frac{1}{2}z_1 - \frac{1}{2}\sqrt{3}z_2) + |z|^2} - \frac{\frac{1}{2}\sqrt{3}z_1 - \frac{1}{2}z_2}{1 - 2x_1(-\frac{1}{2}z_1 - \frac{1}{2}\sqrt{3}z_2) + |x|^2|z|^2} \right) \\ &= \frac{1}{\pi} \sum_{k=0}^2 \left(\frac{r \sin(\theta + \frac{2}{3}k\pi)}{x_1^2 - 2x_1r \cos(\theta + \frac{2}{3}k\pi) + r^2} - \frac{r \sin(\theta + \frac{2}{3}k\pi)}{1 - 2x_1r \cos(\theta + \frac{2}{3}k\pi) + x_1^2r^2} \right). \end{aligned} \quad (10)$$

Using the symmetry $G_S(x, z) = G_S(z, x)$ and replacing x by y we find a similar formula for

$$\begin{aligned} \lim_{y_2 \downarrow 0} \frac{G_S(z, y)}{y_2} &= \frac{1}{\pi} \left(\frac{z_2}{|y|^2 - 2y_1z_1 + |z|^2} - \frac{z_2}{1 - 2y_1z_1 + |y|^2|z|^2} \right) + \\ &+ \frac{1}{\pi} \left(\frac{-\frac{1}{2}\sqrt{3}z_1 - \frac{1}{2}z_2}{|y|^2 - 2y_1(-\frac{1}{2}z_1 + \frac{1}{2}\sqrt{3}z_2) + |z|^2} - \frac{-\frac{1}{2}\sqrt{3}z_1 - \frac{1}{2}z_2}{1 - 2y_1(-\frac{1}{2}z_1 + \frac{1}{2}\sqrt{3}z_2) + |y|^2|z|^2} \right) + \\ &+ \frac{1}{\pi} \left(\frac{\frac{1}{2}\sqrt{3}z_1 - \frac{1}{2}z_2}{|y|^2 - 2y_1(-\frac{1}{2}z_1 - \frac{1}{2}\sqrt{3}z_2) + |z|^2} - \frac{\frac{1}{2}\sqrt{3}z_1 - \frac{1}{2}z_2}{1 - 2y_1(-\frac{1}{2}z_1 - \frac{1}{2}\sqrt{3}z_2) + |y|^2|z|^2} \right) = \\ &= \frac{1}{\pi} \left(\frac{r \sin \theta}{y_1^2 - 2y_1r \cos \theta + r^2} - \frac{r \sin \theta}{1 - 2y_1r \cos \theta + y_1^2r^2} \right) + \\ &+ \frac{1}{\pi} \left(\frac{r \sin(\theta - \frac{2}{3}\pi)}{y_1^2 - 2y_1r \cos(\theta - \frac{2}{3}\pi) + r^2} - \frac{r \sin(\theta - \frac{2}{3}\pi)}{1 - 2y_1r \cos(\theta - \frac{2}{3}\pi) + y_1^2r^2} \right) + \\ &+ \frac{1}{\pi} \left(\frac{r \sin(\theta + \frac{2}{3}\pi)}{y_1^2 - 2y_1r \cos(\theta + \frac{2}{3}\pi) + r^2} - \frac{r \sin(\theta + \frac{2}{3}\pi)}{1 - 2y_1r \cos(\theta + \frac{2}{3}\pi) + y_1^2r^2} \right). \end{aligned} \quad (11)$$

The last limit in this subsection is the denominator

$$\begin{aligned} \lim_{y_2 \downarrow 0} \lim_{x_2 \downarrow 0} \frac{G_S(x, y)}{x_2 y_2} &= \lim_{y_2 \downarrow 0} \frac{1}{y_2} \left(\frac{1}{\pi} \left(\frac{y_2}{|x|^2 - 2x_1y_1 + |y|^2} - \frac{y_2}{1 - 2x_1y_1 + |x|^2|y|^2} \right) + \right. \\ &+ \frac{1}{\pi} \left(\frac{-\frac{1}{2}\sqrt{3}y_1 - \frac{1}{2}y_2}{|x|^2 - 2x_1(-\frac{1}{2}y_1 + \frac{1}{2}\sqrt{3}y_2) + |y|^2} - \frac{-\frac{1}{2}\sqrt{3}y_1 - \frac{1}{2}y_2}{1 - 2x_1(-\frac{1}{2}y_1 + \frac{1}{2}\sqrt{3}y_2) + |x|^2|y|^2} \right) + \\ &\left. + \frac{1}{\pi} \left(\frac{\frac{1}{2}\sqrt{3}y_1 - \frac{1}{2}y_2}{|x|^2 - 2x_1(-\frac{1}{2}y_1 - \frac{1}{2}\sqrt{3}y_2) + |y|^2} - \frac{\frac{1}{2}\sqrt{3}y_1 - \frac{1}{2}y_2}{1 - 2x_1(-\frac{1}{2}y_1 - \frac{1}{2}\sqrt{3}y_2) + |x|^2|y|^2} \right) \right) \end{aligned}$$

$$\begin{aligned}
&= \frac{1}{\pi} \left(\frac{1}{(x_1 - y_1)^2} - \frac{1}{(1 - x_1 y_1)^2} \right) + \\
&+ \frac{1}{\pi} \left(-\frac{1}{x_1^2 + x_1 y_1 + y_1^2} + \frac{1}{1 + x_1 y_1 + x_1^2 y_1^2} \right) + \\
&+ \lim_{y_2 \downarrow 0} \frac{1}{\pi y_2} \left(-\frac{\frac{1}{2}\sqrt{3}y_1}{|x|^2 - 2x_1 \left(-\frac{1}{2}y_1 + \frac{1}{2}\sqrt{3}y_2\right) + |y|^2} + \frac{\frac{1}{2}\sqrt{3}y_1}{1 - 2x_1 \left(-\frac{1}{2}y_1 + \frac{1}{2}\sqrt{3}y_2\right) + |x|^2 |y|^2} \right) + \\
&+ \lim_{y_2 \downarrow 0} \frac{1}{\pi y_2} \left(\frac{\frac{1}{2}\sqrt{3}y_1}{|x|^2 + 2x_1 \left(\frac{1}{2}y_1 + \frac{1}{2}\sqrt{3}y_2\right) + |y|^2} + \frac{-\frac{1}{2}\sqrt{3}y_1}{1 + 2x_1 \left(\frac{1}{2}y_1 + \frac{1}{2}\sqrt{3}y_2\right) + |x|^2 |y|^2} \right) \\
&= \frac{1}{\pi} \left(\frac{1}{(x_1 - y_1)^2} - \frac{1}{(1 - x_1 y_1)^2} \right) + \\
&+ \frac{1}{\pi} \left(-\frac{1}{x_1^2 + x_1 y_1 + y_1^2} + \frac{1}{1 + x_1 y_1 + x_1^2 y_1^2} \right) + \\
&+ \lim_{y_2 \downarrow 0} \frac{\frac{1}{2}\sqrt{3}y_1}{\pi y_2} \left(\frac{1}{1 - 2x_1 \left(-\frac{1}{2}y_1 + \frac{1}{2}\sqrt{3}y_2\right) + |x|^2 |y|^2} - \frac{1}{|x|^2 - 2x_1 \left(-\frac{1}{2}y_1 + \frac{1}{2}\sqrt{3}y_2\right) + |y|^2} \right) + \\
&+ \lim_{y_2 \downarrow 0} \frac{\frac{1}{2}\sqrt{3}y_1}{\pi y_2} \left(\frac{1}{|x|^2 + 2x_1 \left(\frac{1}{2}y_1 + \frac{1}{2}\sqrt{3}y_2\right) + |y|^2} - \frac{1}{1 + 2x_1 \left(\frac{1}{2}y_1 + \frac{1}{2}\sqrt{3}y_2\right) + |x|^2 |y|^2} \right) \\
&= \frac{1}{\pi} \left(\frac{1}{(x_1 - y_1)^2} - \frac{1}{(1 - x_1 y_1)^2} \right) + \\
&+ \frac{1}{\pi} \left(-\frac{1}{x_1^2 + x_1 y_1 + y_1^2} + \frac{1}{1 + x_1 y_1 + x_1^2 y_1^2} \right) + \\
&+ \lim_{y_2 \downarrow 0} \frac{\frac{1}{2}\sqrt{3}y_1}{\pi y_2} \left(\frac{1}{|x|^2 + 2x_1 \left(\frac{1}{2}y_1 + \frac{1}{2}\sqrt{3}y_2\right) + |y|^2} - \frac{1}{|x|^2 - 2x_1 \left(-\frac{1}{2}y_1 + \frac{1}{2}\sqrt{3}y_2\right) + |y|^2} \right) + \\
&+ \lim_{y_2 \downarrow 0} \frac{\frac{1}{2}\sqrt{3}y_1}{\pi y_2} \left(\frac{1}{1 - 2x_1 \left(-\frac{1}{2}y_1 + \frac{1}{2}\sqrt{3}y_2\right) + |x|^2 |y|^2} - \frac{1}{1 + 2x_1 \left(\frac{1}{2}y_1 + \frac{1}{2}\sqrt{3}y_2\right) + |x|^2 |y|^2} \right) \\
&= \frac{1}{\pi} \left(\frac{1}{(x_1 - y_1)^2} - \frac{1}{(1 - x_1 y_1)^2} \right) + \\
&+ \frac{1}{\pi} \left(-\frac{1}{x_1^2 + x_1 y_1 + y_1^2} + \frac{1}{1 + x_1 y_1 + x_1^2 y_1^2} \right) + \\
&+ \lim_{y_2 \downarrow 0} \frac{\frac{1}{2}\sqrt{3}y_1}{\pi y_2} \left(\frac{-4x_1 \frac{1}{2}\sqrt{3}y_2}{\left(|x|^2 + 2x_1 \left(\frac{1}{2}y_1 + \frac{1}{2}\sqrt{3}y_2\right) + |y|^2\right) \left(|x|^2 + 2x_1 \left(\frac{1}{2}y_1 - \frac{1}{2}\sqrt{3}y_2\right) + |y|^2\right)} \right) + \\
&+ \lim_{y_2 \downarrow 0} \frac{\frac{1}{2}\sqrt{3}y_1}{\pi y_2} \left(\frac{4x_1 \frac{1}{2}\sqrt{3}y_2}{\left(1 + 2x_1 \left(\frac{1}{2}y_1 - \frac{1}{2}\sqrt{3}y_2\right) + |x|^2 |y|^2\right) \left(1 + 2x_1 \left(\frac{1}{2}y_1 + \frac{1}{2}\sqrt{3}y_2\right) + |x|^2 |y|^2\right)} \right)
\end{aligned}$$

$$\begin{aligned}
&= \frac{1}{\pi} \left(\frac{1}{(x_1 - y_1)^2} - \frac{1}{(1 - x_1 y_1)^2} \right) + \\
&+ \frac{1}{\pi} \left(\frac{1}{1 + x_1 y_1 + x_1^2 y_1^2} - \frac{1}{x_1^2 + x_1 y_1 + y_1^2} \right) + \\
&+ \frac{1}{\pi} \left(\frac{3x_1 y_1}{(1 + x_1 y_1 + x_1^2 y_1^2)^2} - \frac{3x_1 y_1}{(x_1^2 + x_1 y_1 + y_1^2)^2} \right). \tag{12}
\end{aligned}$$

4.2 Derivation of a contour integral

We introduce the notation

$$h(\theta) = \frac{r \sin \theta}{x_1^2 - 2x_1 r \cos \theta + r^2} - \frac{r \sin \theta}{1 - 2x_1 r \cos \theta + x_1^2 r^2} \tag{13}$$

$$g(\theta) = \frac{r \sin \theta}{y_1^2 - 2y_1 r \cos \theta + r^2} - \frac{r \sin \theta}{1 - 2y_1 r \cos \theta + y_1^2 r^2} \tag{14}$$

and the integral to be evaluated becomes

$$\begin{aligned}
&\pi^2 \int_{\theta=0}^{\frac{1}{3}\pi} \lim_{x_2 \downarrow 0} \frac{G_S(x, z)}{x_2} \lim_{y_2 \downarrow 0} \frac{G_S(z, y)}{y_2} d\theta = \\
&= \int_{\theta=0}^{\frac{\pi}{3}} (h(\theta) + h(\theta + \frac{2}{3}\pi) + h(\theta + \frac{4}{3}\pi)) (g(\theta) + g(\theta + \frac{2}{3}\pi) + g(\theta + \frac{4}{3}\pi)) d\theta \\
&= \frac{1}{2} \int_{\theta=-\frac{\pi}{3}}^{\frac{\pi}{3}} (h(\theta) + h(\theta + \frac{2}{3}\pi) + h(\theta + \frac{4}{3}\pi)) (g(\theta) + g(\theta + \frac{2}{3}\pi) + g(\theta + \frac{4}{3}\pi)) d\theta \\
&= \frac{1}{6} \int_{\theta=0}^{2\pi} (h(\theta) + h(\theta + \frac{2}{3}\pi) + h(\theta + \frac{4}{3}\pi)) (g(\theta) + g(\theta + \frac{2}{3}\pi) + g(\theta + \frac{4}{3}\pi)) d\theta \tag{15}
\end{aligned}$$

$$= \frac{1}{2} \int_{\theta=0}^{2\pi} (h(\theta) + h(\theta + \frac{2}{3}\pi) + h(\theta + \frac{4}{3}\pi)) g(\theta) d\theta. \tag{16}$$

Here we have used that h and g are odd and 2π -periodic. Note that the integrand contains singularities due to h at $\theta \in \{0, \frac{2}{3}\pi, \frac{4}{3}\pi\}$ when $r = x_1$, and due to g in $\theta = 0$ when $r = y_1$. The singularities are integrable whenever $x_1 \neq y_1$. To apply Cauchy's residue theorem, we introduce complex notation. We call $w = re^{i\theta}$. Then $r \sin \theta = \frac{w - \bar{w}}{2i}$, $r \cos \theta = \frac{w + \bar{w}}{2}$, $id\theta = dw/w$, $\bar{w} = \frac{r^2}{w}$, and after some straightforward computations we arrive at:

$$\begin{aligned}
h(\theta) &= \frac{1}{2i} \left(\frac{w - \bar{w}}{x_1^2 - x_1(w + \bar{w}) + w\bar{w}} - \frac{w - \bar{w}}{1 - x_1(w + \bar{w}) + x_1^2 w\bar{w}} \right) \\
&= \frac{1}{2i} \left(\frac{1}{\bar{w} - x_1} - \frac{1}{w - x_1} - \frac{x_1^{-2}}{\bar{w} - x_1^{-1}} + \frac{x_1^{-2}}{w - x_1^{-1}} \right) \\
&= \frac{1}{2i} \left(\frac{w}{r^2 - x_1 w} - \frac{1}{w - x_1} - \frac{x_1^{-2} w}{r^2 - x_1^{-1} w} + \frac{x_1^{-2}}{w - x_1^{-1}} \right) \\
&= \frac{1}{2i} \left(\frac{r^2}{w - x_1 r^2} + \frac{x_1^{-2}}{w - x_1^{-1}} - \frac{x_1^{-2} r^2}{w - x_1^{-1} r^2} - \frac{1}{w - x_1} \right), \tag{17}
\end{aligned}$$

$$\begin{aligned}
h(\theta + \frac{2}{3}\pi) &= \frac{e^{-\frac{2}{3}\pi i}}{2i} \left(\frac{r^2}{w - x_1 r^2 e^{-\frac{2}{3}\pi i}} + \frac{x_1^{-2}}{w - x_1^{-1} e^{-\frac{2}{3}\pi i}} + \right. \\
&\quad \left. - \frac{x_1^{-2} r^2}{w - x_1^{-1} r^2 e^{-\frac{2}{3}\pi i}} - \frac{1}{w - x_1 e^{-\frac{2}{3}\pi i}} \right), \tag{18}
\end{aligned}$$

$$h\left(\theta + \frac{4}{3}\pi\right) = \frac{e^{\frac{2}{3}\pi i}}{2i} \left(\frac{r^2}{w - x_1 r^2 e^{\frac{2}{3}\pi i}} + \frac{x_1^{-2}}{w - x_1^{-1} e^{\frac{2}{3}\pi i}} + \frac{x_1^{-2} r^2}{w - x_1^{-1} r^2 e^{\frac{2}{3}\pi i}} - \frac{1}{w - x_1 e^{\frac{2}{3}\pi i}} \right). \quad (19)$$

We find

$$\begin{aligned} \#_{(16)} &= \frac{i}{8} \oint_{|w|=r} \left(\Psi_{x_1,0,r}(w) + \Psi_{x_1,1,r}(w) + \Psi_{x_1,2,r}(w) \right) \Psi_{y_1,0,r}(w) \frac{dw}{w} \\ &= \frac{-\pi}{4} \sum_{\substack{\text{poles } w_*: \\ |w_*| < r}} \text{Res} \left(\frac{1}{w} \Psi_{y_1,0,r}(w) \sum_{m=0}^2 \Psi_{x_1,m,r}(w) \right)_{w=w_*}, \end{aligned} \quad (20)$$

where we define

$$\begin{aligned} \Psi_{t,m,r}(w) := & \frac{r^2 e^{-\frac{2}{3}m\pi i}}{w - t r^2 e^{-\frac{2}{3}m\pi i}} + \frac{t^{-2} e^{-\frac{2}{3}m\pi i}}{w - t^{-1} e^{-\frac{2}{3}m\pi i}} + \\ & - \frac{t^{-2} r^2 e^{-\frac{2}{3}m\pi i}}{w - t^{-1} r^2 e^{-\frac{2}{3}m\pi i}} - \frac{e^{-\frac{2}{3}m\pi i}}{w - t e^{-\frac{2}{3}m\pi i}}. \end{aligned} \quad (21)$$

For later use we also define

$$J_1(x_1, y_1, r; w) := \frac{1}{w} \Psi_{y_1,0,r}(w) \sum_{m=0}^2 \Psi_{x_1,m,r}(w) \quad (22)$$

Let us remark that the factor $w^{-1} \Psi_{y_1,0,r}(w)$ in (20), (22) has a removable singularity at $w = 0$.

4.3 Computation of the contour integral

Without loss of generality we may assume that $x_1 < y_1$. Then, according to the size of r , the integrand has different sets of poles. In the following table we give a scheme in which we denote how we split the integral (and which range of r corresponds to which poles).

poles due to:		$\Psi_{x_1,0,r}$		$\Psi_{x_1,1,r}$		$\Psi_{x_1,2,r}$		$\Psi_{y_1,0,r}$	
range:		$a_1.$	$a_2.$	$b_1.$	$b_2.$	$c_1.$	$c_2.$	$d_1.$	$d_2.$
$r \in (0, x_1)$	I.	$x_1 r^2$	$\frac{r^2}{x_1}$	$x_1 r^2 e^{-\frac{2}{3}\pi i}$	$\frac{r^2 e^{-\frac{2}{3}\pi i}}{x_1}$	$x_1 r^2 e^{\frac{2}{3}\pi i}$	$\frac{r^2 e^{\frac{2}{3}\pi i}}{x_1}$	$y_1 r^2$	$\frac{r^2}{y_1}$
$r \in (x_1, y_1)$	II.	$x_1 r^2$	x_1	$x_1 r^2 e^{-\frac{2}{3}\pi i}$	$x_1 e^{-\frac{2}{3}\pi i}$	$x_1 r^2 e^{\frac{2}{3}\pi i}$	$x_1 e^{\frac{2}{3}\pi i}$	$y_1 r^2$	$\frac{r^2}{y_1}$
$r \in (y_1, 1)$	III.	$x_1 r^2$	x_1	$x_1 r^2 e^{-\frac{2}{3}\pi i}$	$x_1 e^{-\frac{2}{3}\pi i}$	$x_1 r^2 e^{\frac{2}{3}\pi i}$	$x_1 e^{\frac{2}{3}\pi i}$	$y_1 r^2$	y_1

For ease of writing we define for $z \in \mathbb{C}$ the function $\text{Ln}(z) := \ln|z| + i \arg(z)$ with $\arg(z) \in (-\pi, \pi]$. The function $z \mapsto \text{Ln}(z)$ is a primitive of $z \mapsto z^{-1}$ on $\mathbb{C} \setminus (-\infty, 0]$.

Next we calculate the residues, one by one, as listed in the table.

I, II and III, a_1 : pole at $w = x_1 r^2$.

$$\begin{aligned} & \text{Res} \left(J_1(x_1, y_1, r; w) \right)_{w=x_1 r^2} = \\ &= \frac{1}{x_1} \left(\frac{r^2}{x_1 r^2 - y_1 r^2} + \frac{y_1^{-2}}{x_1 r^2 - y_1^{-1}} - \frac{y_1^{-2} r^2}{x_1 r^2 - y_1^{-1} r^2} - \frac{1}{x_1 r^2 - y_1} \right) \\ &= \frac{1}{x_1 y_1} \left(\frac{1}{1 - x_1 y_1} - \frac{1}{1 - \frac{x_1}{y_1}} \right) + \frac{1}{x_1^2 y_1^2} \frac{1}{r^2} - \frac{1}{x_1 y_1} - \frac{1}{x_1^2} \frac{1}{r^2} - \frac{y_1}{x_1} \end{aligned} \quad (23)$$

I, a_2 : pole at $w = x_1^{-1}r^2$.

$$\begin{aligned}
& \text{Res} \left(J_1(x_1, y_1, r; w) \right)_{w=x_1^{-1}r^2} = \\
&= \frac{-x_1^{-2}r^2}{x_1^{-1}r^2} \left(\frac{r^2}{x_1^{-1}r^2 - y_1r^2} + \frac{y_1^{-2}}{x_1^{-1}r^2 - y_1^{-1}} - \frac{y_1^{-2}r^2}{x_1^{-1}r^2 - y_1^{-1}r^2} - \frac{1}{x_1^{-1}r^2 - y_1} \right) \\
&= \frac{1}{y_1^2} \frac{1}{1 - \frac{x_1}{y_1}} - \frac{1}{1 - x_1y_1} + \frac{1}{r^2 - x_1y_1} - \frac{1}{y_1^2} \frac{1}{r^2 - \frac{x_1}{y_1}}
\end{aligned} \tag{24}$$

II and III, a_2 : pole at $w = x_1$.

$$\begin{aligned}
& \text{Res} \left(J_1(x_1, y_1, r; w) \right)_{w=x_1} = \\
&= -\frac{1}{x_1} \left(\frac{r^2}{x_1 - y_1r^2} + \frac{y_1^{-2}}{x_1 - y_1^{-1}} - \frac{y_1^{-2}r^2}{x_1 - y_1^{-1}r^2} - \frac{1}{x_1 - y_1} \right) \\
&= -\frac{1}{x_1y_1} \left(\frac{1}{1 - \frac{x_1}{y_1}} - \frac{1}{1 - x_1y_1} \right) - \frac{1}{r^2 - x_1y_1} + \frac{1}{y_1^2} \frac{1}{r^2 - \frac{x_1}{y_1}}
\end{aligned} \tag{25}$$

I, II and III, b_1 : pole at $w = x_1r^2e^{-\frac{2}{3}\pi i}$.

$$\begin{aligned}
& \text{Res} \left(J_1(x_1, y_1, r; w) \right)_{w=x_1r^2e^{-\frac{2}{3}\pi i}} = \\
&= \frac{1}{x_1} \left(\frac{r^2}{x_1r^2e^{-\frac{2}{3}\pi i} - y_1r^2} + \frac{y_1^{-2}}{x_1r^2e^{-\frac{2}{3}\pi i} - y_1^{-1}} - \frac{y_1^{-2}r^2}{x_1r^2e^{-\frac{2}{3}\pi i} - y_1^{-1}r^2} - \frac{1}{x_1r^2e^{-\frac{2}{3}\pi i} - y_1} \right) \\
&= \frac{e^{\frac{2}{3}\pi i}}{x_1y_1} \left(\frac{1}{e^{\frac{2}{3}\pi i} - x_1y_1} - \frac{1}{e^{\frac{2}{3}\pi i} - \frac{x_1}{y_1}} \right) + \frac{e^{\frac{2}{3}\pi i}}{x_1^2y_1^2} \frac{1}{r^2 - \frac{1}{x_1y_1}e^{\frac{2}{3}\pi i}} - \frac{e^{\frac{2}{3}\pi i}}{x_1^2} \frac{1}{r^2 - \frac{y_1}{x_1}e^{\frac{2}{3}\pi i}}
\end{aligned} \tag{26}$$

I, b_2 : pole at $w = x_1^{-1}r^2e^{-\frac{2}{3}\pi i}$.

$$\begin{aligned}
& \text{Res} \left(J_1(x_1, y_1, r; w) \right)_{w=x_1^{-1}r^2e^{-\frac{2}{3}\pi i}} = \\
&= -\frac{x_1^{-2}r^2e^{-\frac{2}{3}\pi i}}{x_1^{-1}r^2e^{-\frac{2}{3}\pi i}} \left(\frac{r^2}{x_1^{-1}r^2e^{-\frac{2}{3}\pi i} - y_1r^2} + \frac{y_1^{-2}}{x_1^{-1}r^2e^{-\frac{2}{3}\pi i} - y_1^{-1}} - \frac{y_1^{-2}r^2}{x_1^{-1}r^2e^{-\frac{2}{3}\pi i} - y_1^{-1}r^2} - \frac{1}{x_1^{-1}r^2e^{-\frac{2}{3}\pi i} - y_1} \right) \\
&= \frac{e^{\frac{2}{3}\pi i}}{y_1^2} \frac{1}{1 - \frac{x_1}{y_1}e^{\frac{2}{3}\pi i}} - e^{\frac{2}{3}\pi i} \frac{1}{1 - x_1y_1e^{\frac{2}{3}\pi i}} + e^{\frac{2}{3}\pi i} \frac{1}{r^2 - x_1y_1e^{\frac{2}{3}\pi i}} - \frac{e^{\frac{2}{3}\pi i}}{y_1^2} \frac{1}{r^2 - \frac{x_1}{y_1}e^{\frac{2}{3}\pi i}}
\end{aligned} \tag{27}$$

II and III, b_2 : pole at $w = x_1 e^{-\frac{2}{3}\pi i}$.

$$\begin{aligned}
& \text{Res} \left(J_1(x_1, y_1, r; w) \right)_{w=x_1 e^{-\frac{2}{3}\pi i}} = \\
&= \frac{-e^{-\frac{2}{3}\pi i}}{x_1 e^{-\frac{2}{3}\pi i}} \left(\frac{r^2}{x_1 e^{-\frac{2}{3}\pi i} - y_1 r^2} + \frac{y_1^{-2}}{x_1 e^{-\frac{2}{3}\pi i} - y_1^{-1}} - \frac{y_1^{-2} r^2}{x_1 e^{-\frac{2}{3}\pi i} - y_1^{-1} r^2} - \frac{1}{x_1 e^{-\frac{2}{3}\pi i} - y_1} \right) \\
&= \frac{1}{x_1 y_1} \left(\frac{1}{1 - x_1 y_1 e^{-\frac{2}{3}\pi i}} - \frac{1}{1 - \frac{x_1}{y_1} e^{-\frac{2}{3}\pi i}} \right) - e^{-\frac{2}{3}\pi i} \frac{1}{r^2 - x_1 y_1 e^{-\frac{2}{3}\pi i}} + \frac{e^{-\frac{2}{3}\pi i}}{y_1^2} \frac{1}{r^2 - \frac{x_1}{y_1} e^{-\frac{2}{3}\pi i}} \quad (28)
\end{aligned}$$

I, II and III, c_1 : pole at $w = x_1 r^2 e^{\frac{2}{3}\pi i}$.

$$\begin{aligned}
& \text{Res} \left(J_1(x_1, y_1, r; w) \right)_{w=x_1 r^2 e^{\frac{2}{3}\pi i}} = \\
&= \frac{1}{x_1} \left(\frac{r^2}{x_1 r^2 e^{\frac{2}{3}\pi i} - y_1 r^2} + \frac{y_1^{-2}}{x_1 r^2 e^{\frac{2}{3}\pi i} - y_1^{-1}} - \frac{y_1^{-2} r^2}{x_1 r^2 e^{\frac{2}{3}\pi i} - y_1^{-1} r^2} - \frac{1}{x_1 r^2 e^{\frac{2}{3}\pi i} - y_1} \right) \\
&= \frac{e^{-\frac{2}{3}\pi i}}{x_1 y_1} \left(\frac{1}{e^{-\frac{2}{3}\pi i} - x_1 y_1} - \frac{1}{e^{-\frac{2}{3}\pi i} - \frac{x_1}{y_1}} \right) + \frac{e^{-\frac{2}{3}\pi i}}{x_1^2 y_1^2} \frac{1}{r^2 - \frac{1}{x_1 y_1} e^{-\frac{2}{3}\pi i}} - \frac{e^{-\frac{2}{3}\pi i}}{x_1^2} \frac{1}{r^2 - \frac{y_1}{x_1} e^{-\frac{2}{3}\pi i}} \quad (29)
\end{aligned}$$

I, c_2 : pole at $w = x_1^{-1} r^2 e^{\frac{2}{3}\pi i}$.

$$\begin{aligned}
& \text{Res} \left(J_1(x_1, y_1, r; w) \right)_{w=x_1^{-1} r^2 e^{\frac{2}{3}\pi i}} = \\
&= -\frac{1}{x_1} \left(\frac{r^2}{x_1^{-1} r^2 e^{\frac{2}{3}\pi i} - y_1 r^2} + \frac{y_1^{-2}}{x_1^{-1} r^2 e^{\frac{2}{3}\pi i} - y_1^{-1}} - \frac{y_1^{-2} r^2}{x_1^{-1} r^2 e^{\frac{2}{3}\pi i} - y_1^{-1} r^2} - \frac{1}{x_1^{-1} r^2 e^{\frac{2}{3}\pi i} - y_1} \right) \\
&= \frac{e^{-\frac{2}{3}\pi i}}{y_1^2} \frac{1}{1 - \frac{x_1}{y_1} e^{-\frac{2}{3}\pi i}} - e^{-\frac{2}{3}\pi i} \frac{1}{1 - x_1 y_1 e^{-\frac{2}{3}\pi i}} + e^{-\frac{2}{3}\pi i} \frac{1}{r^2 - x_1 y_1 e^{-\frac{2}{3}\pi i}} - \frac{e^{-\frac{2}{3}\pi i}}{y_1^2} \frac{1}{r^2 - \frac{x_1}{y_1} e^{-\frac{2}{3}\pi i}} \quad (30)
\end{aligned}$$

II and III, c_2 : pole at $w = x_1 e^{\frac{2}{3}\pi i}$.

$$\begin{aligned}
& \text{Res} \left(J_1(x_1, y_1, r; w) \right)_{w=x_1 e^{\frac{2}{3}\pi i}} = \\
&= -\frac{1}{x_1} \left(\frac{r^2}{x_1 e^{\frac{2}{3}\pi i} - y_1 r^2} + \frac{y_1^{-2}}{x_1 e^{\frac{2}{3}\pi i} - y_1^{-1}} - \frac{y_1^{-2} r^2}{x_1 e^{\frac{2}{3}\pi i} - y_1^{-1} r^2} - \frac{1}{x_1 e^{\frac{2}{3}\pi i} - y_1} \right) \\
&= \frac{1}{x_1 y_1} \left(\frac{1}{\frac{x_1}{y_1} e^{\frac{2}{3}\pi i} - 1} - \frac{1}{x_1 y_1 e^{\frac{2}{3}\pi i} - 1} \right) - e^{\frac{2}{3}\pi i} \frac{1}{r^2 - x_1 y_1 e^{\frac{2}{3}\pi i}} + \frac{e^{\frac{2}{3}\pi i}}{y_1^2} \frac{1}{r^2 - \frac{x_1}{y_1} e^{\frac{2}{3}\pi i}} \quad (31)
\end{aligned}$$

I, II and III, d_1 : pole at $w = y_1 r^2$.

$$\begin{aligned}
& \text{Res} \left(J_1(x_1, y_1, r; w) \right)_{w=y_1 r^2} = \\
&= \frac{1}{y_1} \left(\frac{1}{y_1 - x_1} + \frac{x_1^{-2}}{y_1 r^2 - x_1^{-1}} - \frac{x_1^{-2}}{y_1 - x_1^{-1}} - \frac{1}{y_1 r^2 - x_1} + \right. \\
&+ \frac{e^{-\frac{2}{3}\pi i}}{y_1 - x_1 e^{-\frac{2}{3}\pi i}} + \frac{x_1^{-2} e^{-\frac{2}{3}\pi i}}{y_1 r^2 - x_1^{-1} e^{-\frac{2}{3}\pi i}} - \frac{x_1^{-2} e^{-\frac{2}{3}\pi i}}{y_1 - x_1^{-1} e^{-\frac{2}{3}\pi i}} - \frac{e^{-\frac{2}{3}\pi i}}{y_1 r^2 - x_1 e^{-\frac{2}{3}\pi i}} \\
&+ \left. \frac{e^{\frac{2}{3}\pi i}}{y_1 - x_1 e^{\frac{2}{3}\pi i}} + \frac{x_1^{-2} e^{\frac{2}{3}\pi i}}{y_1 r^2 - x_1^{-1} e^{\frac{2}{3}\pi i}} - \frac{x_1^{-2} e^{\frac{2}{3}\pi i}}{y_1 - x_1^{-1} e^{\frac{2}{3}\pi i}} - \frac{e^{\frac{2}{3}\pi i}}{y_1 r^2 - x_1 e^{\frac{2}{3}\pi i}} \right) \\
&= \sum_{k=0}^2 \left(\frac{1}{y_1^2} \frac{e^{\frac{2}{3}k\pi i}}{1 - \frac{x_1}{y_1} e^{\frac{2}{3}k\pi i}} + \frac{1}{x_1 y_1} \frac{e^{\frac{2}{3}k\pi i}}{e^{\frac{2}{3}k\pi i} - x_1 y_1} \right) + \\
&+ \sum_{k=0}^2 \left(\frac{1}{x_1^2 y_1^2} \frac{e^{\frac{2}{3}k\pi i}}{r^2 - \frac{1}{x_1 y_1} e^{\frac{2}{3}k\pi i}} - \frac{1}{y_1^2} \frac{e^{\frac{2}{3}k\pi i}}{r^2 - \frac{x_1}{y_1} e^{\frac{2}{3}k\pi i}} \right) \tag{32}
\end{aligned}$$

I and II, d_2 : pole at $w = y_1^{-1} r^2$.

$$\begin{aligned}
& \text{Res} \left(J_1(x_1, y_1, r; w) \right)_{w=y_1^{-1} r^2} = \\
&= -\frac{1}{y_1} \left(\frac{1}{y_1^{-1} - x_1} + \frac{x_1^{-2}}{y_1^{-1} r^2 - x_1^{-1}} - \frac{x_1^{-2}}{y_1^{-1} - x_1^{-1}} - \frac{1}{y_1^{-1} r^2 - x_1} + \right. \\
&+ \frac{e^{-\frac{2}{3}\pi i}}{y_1^{-1} - x_1 e^{-\frac{2}{3}\pi i}} + \frac{x_1^{-2} e^{-\frac{2}{3}\pi i}}{y_1^{-1} r^2 - x_1^{-1} e^{-\frac{2}{3}\pi i}} - \frac{x_1^{-2} e^{-\frac{2}{3}\pi i}}{y_1^{-1} - x_1^{-1} e^{-\frac{2}{3}\pi i}} - \frac{e^{-\frac{2}{3}\pi i}}{y_1^{-1} r^2 - x_1 e^{-\frac{2}{3}\pi i}} \\
&+ \left. \frac{e^{\frac{2}{3}\pi i}}{y_1^{-1} - x_1 e^{\frac{2}{3}\pi i}} + \frac{x_1^{-2} e^{\frac{2}{3}\pi i}}{y_1^{-1} r^2 - x_1^{-1} e^{\frac{2}{3}\pi i}} - \frac{x_1^{-2} e^{\frac{2}{3}\pi i}}{y_1^{-1} - x_1^{-1} e^{\frac{2}{3}\pi i}} - \frac{e^{\frac{2}{3}\pi i}}{y_1^{-1} r^2 - x_1 e^{\frac{2}{3}\pi i}} \right) \\
&= -\sum_{k=0}^2 \left(\frac{1}{x_1 y_1} \frac{1}{1 - \frac{x_1}{y_1} e^{\frac{2}{3}k\pi i}} + \frac{e^{\frac{2}{3}k\pi i}}{1 - x_1 y_1 e^{\frac{2}{3}k\pi i}} \right) + \\
&+ \sum_{k=0}^2 \left(\frac{e^{\frac{2}{3}k\pi i}}{r^2 - x_1 y_1 e^{\frac{2}{3}k\pi i}} - \frac{1}{x_1^2} \frac{e^{\frac{2}{3}k\pi i}}{r^2 - \frac{y_1}{x_1} e^{\frac{2}{3}k\pi i}} \right) \tag{33}
\end{aligned}$$

III, d_2 : pole at $w = y_1$.

$$\begin{aligned}
& \text{Res} \left(J_1(x_1, y_1, r; w) \right)_{w=y_1} = \\
& = -\frac{1}{y_1} \left(\frac{r^2}{y_1 - x_1 r^2} + \frac{x_1^{-2}}{y_1 - x_1^{-1}} - \frac{x_1^{-2} r^2}{y_1 - x_1^{-1} r^2} - \frac{1}{y_1 - x_1} + \right. \\
& \quad + \frac{r^2 e^{-\frac{2}{3}\pi i}}{y_1 - x_1 r^2 e^{-\frac{2}{3}\pi i}} + \frac{x_1^{-2} e^{-\frac{2}{3}\pi i}}{y_1 - x_1^{-1} e^{-\frac{2}{3}\pi i}} - \frac{x_1^{-2} r^2 e^{-\frac{2}{3}\pi i}}{y_1 - x_1^{-1} r^2 e^{-\frac{2}{3}\pi i}} - \frac{e^{-\frac{2}{3}\pi i}}{y_1 - x_1 e^{-\frac{2}{3}\pi i}} \\
& \quad \left. + \frac{r^2 e^{\frac{2}{3}\pi i}}{y_1 - x_1 r^2 e^{\frac{2}{3}\pi i}} + \frac{x_1^{-2} e^{\frac{2}{3}\pi i}}{y_1 - x_1^{-1} e^{\frac{2}{3}\pi i}} - \frac{x_1^{-2} r^2 e^{\frac{2}{3}\pi i}}{y_1 - x_1^{-1} r^2 e^{\frac{2}{3}\pi i}} - \frac{e^{\frac{2}{3}\pi i}}{y_1 - x_1 e^{\frac{2}{3}\pi i}} \right) \\
& = \sum_{k=0}^2 \left(\frac{1}{x_1 y_1} \frac{1}{1 - x_1 y_1 e^{\frac{2}{3}k\pi i}} + \frac{1}{y_1^2} \frac{e^{\frac{2}{3}k\pi i}}{1 - \frac{x_1}{y_1} e^{\frac{2}{3}k\pi i}} \right) + \\
& \quad + \sum_{k=0}^2 \left(-\frac{e^{\frac{2}{3}k\pi i}}{r^2 - x_1 y_1 e^{\frac{2}{3}k\pi i}} + \frac{1}{x_1^2} \frac{e^{\frac{2}{3}k\pi i}}{r^2 - \frac{y_1}{x_1} e^{\frac{2}{3}k\pi i}} \right) \tag{34}
\end{aligned}$$

4.4 Integration in the radial direction

It remains to combine the appropriate residues and to integrate them with respect to r . In doing so, we integrate r in three steps; from 0 to x_1 , from x_1 to y_1 and from y_1 to 1. In each step we list the appropriate residues and we rewrite them as partial fractions with each denominator linear in r^2 .

- For $r \in (0, x_1)$ this procedure leads to the integrand

$$\begin{aligned}
I_1(x_1, y_1; r) & = \#(23) + \#(24) + \#(26) + \#(27) + \#(29) + \#(30) + \#(32) + \#(33) = \\
& = \frac{1}{x_1 y_1} \left(\frac{1}{1 - x_1 y_1} - \frac{1}{1 - \frac{x_1}{y_1}} \right) + \frac{1}{x_1^2 y_1^2} \frac{1}{r^2 - \frac{1}{x_1 y_1}} - \frac{1}{x_1^2} \frac{1}{r^2 - \frac{y_1}{x_1}} + \\
& \quad + \frac{1}{y_1^2} \frac{1}{1 - \frac{x_1}{y_1}} - \frac{1}{1 - x_1 y_1} + \frac{1}{r^2 - x_1 y_1} - \frac{1}{y_1^2} \frac{1}{r^2 - \frac{x_1}{y_1}} + \\
& \quad + \frac{e^{\frac{2}{3}\pi i}}{x_1 y_1} \left(\frac{1}{e^{\frac{2}{3}\pi i} - x_1 y_1} - \frac{1}{e^{\frac{2}{3}\pi i} - \frac{x_1}{y_1}} \right) + \frac{e^{\frac{2}{3}\pi i}}{x_1^2 y_1^2} \frac{1}{r^2 - \frac{1}{x_1 y_1} e^{\frac{2}{3}\pi i}} - \frac{e^{\frac{2}{3}\pi i}}{x_1^2} \frac{1}{r^2 - \frac{y_1}{x_1} e^{\frac{2}{3}\pi i}} + \\
& \quad + \frac{e^{\frac{2}{3}\pi i}}{y_1^2} \frac{1}{1 - \frac{x_1}{y_1} e^{\frac{2}{3}\pi i}} - \frac{e^{\frac{2}{3}\pi i}}{1 - x_1 y_1 e^{\frac{2}{3}\pi i}} + \frac{e^{\frac{2}{3}\pi i}}{r^2 - x_1 y_1 e^{\frac{2}{3}\pi i}} - \frac{e^{\frac{2}{3}\pi i}}{y_1^2} \frac{1}{r^2 - \frac{x_1}{y_1} e^{\frac{2}{3}\pi i}} + \\
& \quad + \frac{e^{-\frac{2}{3}\pi i}}{x_1 y_1} \left(\frac{1}{e^{-\frac{2}{3}\pi i} - x_1 y_1} - \frac{1}{e^{-\frac{2}{3}\pi i} - \frac{x_1}{y_1}} \right) + \frac{e^{-\frac{2}{3}\pi i}}{x_1^2 y_1^2} \frac{1}{r^2 - \frac{1}{x_1 y_1} e^{-\frac{2}{3}\pi i}} - \frac{e^{-\frac{2}{3}\pi i}}{x_1^2} \frac{1}{r^2 - \frac{y_1}{x_1} e^{-\frac{2}{3}\pi i}} + \\
& \quad + \frac{e^{-\frac{2}{3}\pi i}}{y_1^2} \frac{1}{1 - \frac{x_1}{y_1} e^{-\frac{2}{3}\pi i}} - \frac{e^{-\frac{2}{3}\pi i}}{1 - x_1 y_1 e^{-\frac{2}{3}\pi i}} + \frac{e^{-\frac{2}{3}\pi i}}{r^2 - x_1 y_1 e^{-\frac{2}{3}\pi i}} - \frac{e^{-\frac{2}{3}\pi i}}{y_1^2} \frac{1}{r^2 - \frac{x_1}{y_1} e^{-\frac{2}{3}\pi i}} +
\end{aligned}$$

$$\begin{aligned}
& + \frac{1}{y_1^2} \left(\frac{1}{1 - \frac{x_1}{y_1}} + \frac{e^{-\frac{2}{3}\pi i}}{1 - \frac{x_1}{y_1} e^{-\frac{2}{3}\pi i}} + \frac{e^{\frac{2}{3}\pi i}}{1 - \frac{x_1}{y_1} e^{\frac{2}{3}\pi i}} \right) + \frac{1}{x_1 y_1} \left(\frac{1}{1 - x_1 y_1} + \frac{e^{-\frac{2}{3}\pi i}}{e^{-\frac{2}{3}\pi i} - x_1 y_1} + \frac{e^{\frac{2}{3}\pi i}}{e^{\frac{2}{3}\pi i} - x_1 y_1} \right) + \\
& + \frac{1}{x_1^2 y_1^2} \left(\frac{1}{r^2 - \frac{1}{x_1 y_1}} + \frac{e^{-\frac{2}{3}\pi i}}{r^2 - \frac{1}{x_1 y_1} e^{-\frac{2}{3}\pi i}} + \frac{e^{\frac{2}{3}\pi i}}{r^2 - \frac{1}{x_1 y_1} e^{\frac{2}{3}\pi i}} \right) + \\
& - \frac{1}{y_1^2} \left(\frac{1}{r^2 - \frac{x_1}{y_1}} + \frac{e^{-\frac{2}{3}\pi i}}{r^2 - \frac{x_1}{y_1} e^{-\frac{2}{3}\pi i}} + \frac{e^{\frac{2}{3}\pi i}}{r^2 - \frac{x_1}{y_1} e^{\frac{2}{3}\pi i}} \right) + \\
& - \frac{1}{x_1 y_1} \left(\frac{1}{1 - \frac{x_1}{y_1}} + \frac{1}{1 - \frac{x_1}{y_1} e^{\frac{2}{3}\pi i}} + \frac{1}{1 - \frac{x_1}{y_1} e^{-\frac{2}{3}\pi i}} \right) - \frac{1}{1 - x_1 y_1} - \frac{e^{-\frac{2}{3}\pi i}}{1 - x_1 y_1 e^{-\frac{2}{3}\pi i}} - \frac{e^{\frac{2}{3}\pi i}}{1 - x_1 y_1 e^{\frac{2}{3}\pi i}} + \\
& + \frac{1}{r^2 - x_1 y_1} + \frac{e^{\frac{2}{3}\pi i}}{r^2 - x_1 y_1 e^{\frac{2}{3}\pi i}} + \frac{e^{-\frac{2}{3}\pi i}}{r^2 - x_1 y_1 e^{-\frac{2}{3}\pi i}} - \frac{1}{x_1^2} \left(\frac{1}{r^2 - \frac{y_1}{x_1}} + \frac{e^{\frac{2}{3}\pi i}}{r^2 - \frac{y_1}{x_1} e^{\frac{2}{3}\pi i}} + \frac{e^{-\frac{2}{3}\pi i}}{r^2 - \frac{y_1}{x_1} e^{-\frac{2}{3}\pi i}} \right) \\
& = \frac{2}{x_1 y_1} \frac{1}{1 - x_1 y_1} - \frac{2}{1 - x_1 y_1} - \frac{2}{x_1 y_1} \frac{1}{1 - \frac{x_1}{y_1}} + \frac{2}{y_1^2} \frac{1}{1 - \frac{x_1}{y_1}} + \\
& + \frac{2}{r^2 - x_1 y_1} - \frac{2}{x_1^2} \frac{1}{r^2 - \frac{y_1}{x_1}} - \frac{2}{y_1^2} \frac{1}{r^2 - \frac{x_1}{y_1}} + \frac{2}{x_1^2 y_1^2} \frac{1}{r^2 - \frac{1}{x_1 y_1}} + \\
& + \frac{2}{x_1 y_1} \frac{e^{\frac{2}{3}\pi i}}{e^{\frac{2}{3}\pi i} - x_1 y_1} - \frac{2}{x_1 y_1} \frac{e^{\frac{2}{3}\pi i}}{e^{\frac{2}{3}\pi i} - \frac{x_1}{y_1}} + \frac{2}{y_1^2} \frac{e^{\frac{2}{3}\pi i}}{1 - \frac{x_1}{y_1} e^{\frac{2}{3}\pi i}} - \frac{2e^{\frac{2}{3}\pi i}}{1 - x_1 y_1 e^{\frac{2}{3}\pi i}} + \\
& + \frac{2e^{\frac{2}{3}\pi i}}{r^2 - x_1 y_1 e^{\frac{2}{3}\pi i}} - \frac{2}{x_1^2} \frac{e^{\frac{2}{3}\pi i}}{r^2 - \frac{y_1}{x_1} e^{\frac{2}{3}\pi i}} - \frac{2}{y_1^2} \frac{e^{\frac{2}{3}\pi i}}{r^2 - \frac{x_1}{y_1} e^{\frac{2}{3}\pi i}} + \frac{2}{x_1^2 y_1^2} \frac{e^{\frac{2}{3}\pi i}}{r^2 - \frac{1}{x_1 y_1} e^{\frac{2}{3}\pi i}} + \\
& + \frac{2}{x_1 y_1} \frac{e^{-\frac{2}{3}\pi i}}{e^{-\frac{2}{3}\pi i} - x_1 y_1} - \frac{2}{x_1 y_1} \frac{e^{-\frac{2}{3}\pi i}}{e^{-\frac{2}{3}\pi i} - \frac{x_1}{y_1}} - \frac{2e^{-\frac{2}{3}\pi i}}{1 - x_1 y_1 e^{-\frac{2}{3}\pi i}} + \frac{2}{y_1^2} \frac{e^{-\frac{2}{3}\pi i}}{1 - \frac{x_1}{y_1} e^{-\frac{2}{3}\pi i}} + \\
& + \frac{2e^{-\frac{2}{3}\pi i}}{r^2 - x_1 y_1 e^{-\frac{2}{3}\pi i}} - \frac{2}{x_1^2} \frac{e^{-\frac{2}{3}\pi i}}{r^2 - \frac{y_1}{x_1} e^{-\frac{2}{3}\pi i}} - \frac{2}{y_1^2} \frac{e^{-\frac{2}{3}\pi i}}{r^2 - \frac{x_1}{y_1} e^{-\frac{2}{3}\pi i}} + \frac{2}{x_1^2 y_1^2} \frac{e^{-\frac{2}{3}\pi i}}{r^2 - \frac{1}{x_1 y_1} e^{-\frac{2}{3}\pi i}}. \\
& = \sum_{k=0}^2 \left(\frac{2e^{\frac{2}{3}k\pi i}}{r^2 - x_1 y_1 e^{\frac{2}{3}k\pi i}} + \frac{2}{x_1^2 y_1^2} \frac{e^{\frac{2}{3}k\pi i}}{r^2 - \frac{1}{x_1 y_1} e^{\frac{2}{3}k\pi i}} - \frac{2}{x_1^2} \frac{e^{\frac{2}{3}k\pi i}}{r^2 - \frac{y_1}{x_1} e^{\frac{2}{3}k\pi i}} + \frac{2}{y_1^2} \frac{e^{\frac{2}{3}k\pi i}}{r^2 - \frac{x_1}{y_1} e^{\frac{2}{3}k\pi i}} \right),
\end{aligned}$$

so that, after a tedious calculation

$$\int_{r=0}^{x_1} I_1(x_1, y_1; r) r dr =$$

$$\begin{aligned}
&= \left[\ln(x_1 y_1 - r^2) - \frac{1}{x_1^2} \ln\left(\frac{y_1}{x_1} - r^2\right) - \frac{1}{y_1^2} \ln\left(\frac{x_1}{y_1} - r^2\right) + \frac{1}{x_1^2 y_1^2} \ln\left(\frac{1}{x_1 y_1} - r^2\right) \right]_{r=0}^{x_1} + \\
&+ \left[e^{\frac{2}{3}\pi i} \text{Ln}\left(x_1 y_1 - r^2 e^{-\frac{2}{3}\pi i}\right) - \frac{e^{\frac{2}{3}\pi i}}{x_1^2} \text{Ln}\left(\frac{y_1}{x_1} - r^2 e^{-\frac{2}{3}\pi i}\right) \right]_{r=0}^{x_1} + \\
&+ \left[-\frac{e^{\frac{2}{3}\pi i}}{y_1^2} \text{Ln}\left(\frac{x_1}{y_1} - r^2 e^{-\frac{2}{3}\pi i}\right) + \frac{e^{\frac{2}{3}\pi i}}{x_1^2 y_1^2} \text{Ln}\left(\frac{1}{x_1 y_1} - r^2 e^{-\frac{2}{3}\pi i}\right) \right]_{r=0}^{x_1} + \\
&+ \left[e^{-\frac{2}{3}\pi i} \text{Ln}\left(x_1 y_1 - r^2 e^{\frac{2}{3}\pi i}\right) - \frac{e^{-\frac{2}{3}\pi i}}{x_1^2} \text{Ln}\left(\frac{y_1}{x_1} - r^2 e^{\frac{2}{3}\pi i}\right) \right]_{r=0}^{x_1} + \\
&+ \left[-\frac{e^{-\frac{2}{3}\pi i}}{y_1^2} \text{Ln}\left(\frac{x_1}{y_1} - r^2 e^{\frac{2}{3}\pi i}\right) + \frac{e^{-\frac{2}{3}\pi i}}{x_1^2 y_1^2} \text{Ln}\left(\frac{1}{x_1 y_1} - r^2 e^{\frac{2}{3}\pi i}\right) \right]_{r=0}^{x_1} \\
&= \ln\left(1 - \frac{x_1}{y_1}\right) - \frac{1}{x_1^2} \ln\left(1 - \frac{x_1^3}{y_1}\right) - \frac{1}{y_1^2} \ln(1 - x_1 y_1) + \frac{1}{x_1^2 y_1^2} \ln(1 - x_1^3 y_1) + \\
&+ e^{\frac{2}{3}\pi i} \text{Ln}\left(1 - \frac{x_1 e^{-\frac{2}{3}\pi i}}{y_1}\right) - \frac{e^{\frac{2}{3}\pi i}}{x_1^2} \text{Ln}\left(1 - \frac{x_1^3 e^{-\frac{2}{3}\pi i}}{y_1}\right) + \\
&- \frac{e^{\frac{2}{3}\pi i}}{y_1^2} \text{Ln}\left(1 - x_1 y_1 e^{-\frac{2}{3}\pi i}\right) + \frac{e^{\frac{2}{3}\pi i}}{x_1^2 y_1^2} \text{Ln}\left(1 - x_1^3 y_1 e^{-\frac{2}{3}\pi i}\right) + \\
&+ e^{-\frac{2}{3}\pi i} \text{Ln}\left(1 - \frac{x_1 e^{\frac{2}{3}\pi i}}{y_1}\right) - \frac{e^{-\frac{2}{3}\pi i}}{x_1^2} \text{Ln}\left(1 - \frac{x_1^3 e^{\frac{2}{3}\pi i}}{y_1}\right) + \\
&- \frac{e^{-\frac{2}{3}\pi i}}{y_1^2} \text{Ln}\left(1 - x_1 y_1 e^{\frac{2}{3}\pi i}\right) + \frac{e^{-\frac{2}{3}\pi i}}{x_1^2 y_1^2} \text{Ln}\left(1 - x_1^3 y_1 e^{\frac{2}{3}\pi i}\right) \\
&= \ln\left(1 - \frac{x_1}{y_1}\right) - \frac{1}{x_1^2} \ln\left(1 - \frac{x_1^3}{y_1}\right) - \frac{1}{y_1^2} \ln(1 - x_1 y_1) + \frac{1}{x_1^2 y_1^2} \ln(1 - x_1^3 y_1) + \\
&+ e^{\frac{2}{3}\pi i} \text{Ln}\left(1 + \frac{x_1}{2y_1} + i \frac{x_1 \sqrt{3}}{2y_1}\right) - \frac{e^{\frac{2}{3}\pi i}}{x_1^2} \text{Ln}\left(1 + \frac{x_1^3}{2y_1} + i \frac{x_1^3 \sqrt{3}}{2y_1}\right) + \\
&- \frac{e^{\frac{2}{3}\pi i}}{y_1^2} \text{Ln}\left(1 + \frac{x_1 y_1}{2} + i \frac{x_1 y_1 \sqrt{3}}{2}\right) + \frac{e^{\frac{2}{3}\pi i}}{x_1^2 y_1^2} \text{Ln}\left(1 + \frac{x_1^3 y_1}{2} + i \frac{x_1^3 y_1 \sqrt{3}}{2}\right) + \\
&+ e^{-\frac{2}{3}\pi i} \text{Ln}\left(1 + \frac{x_1}{2y_1} - i \frac{x_1 \sqrt{3}}{2y_1}\right) - \frac{e^{-\frac{2}{3}\pi i}}{x_1^2} \text{Ln}\left(1 + \frac{x_1^3}{2y_1} - i \frac{x_1^3 \sqrt{3}}{2y_1}\right) + \\
&- \frac{e^{-\frac{2}{3}\pi i}}{y_1^2} \text{Ln}\left(1 + \frac{x_1 y_1}{2} - i \frac{x_1 y_1 \sqrt{3}}{2}\right) + \frac{e^{-\frac{2}{3}\pi i}}{x_1^2 y_1^2} \text{Ln}\left(1 + \frac{x_1^3 y_1}{2} - i \frac{x_1^3 y_1 \sqrt{3}}{2}\right)
\end{aligned}$$

$$\begin{aligned}
&= \ln\left(1 - \frac{x_1}{y_1}\right) - \frac{1}{x_1^2} \ln\left(1 - \frac{x_1^3}{y_1}\right) - \frac{1}{y_1^2} \ln(1 - x_1 y_1) + \frac{1}{x_1^2 y_1^2} \ln(1 - x_1^3 y_1) + \\
&\quad + e^{\frac{2}{3}\pi i} \ln \sqrt{\left(1 + \frac{x_1}{2y_1}\right)^2 + \left(\frac{x_1\sqrt{3}}{2y_1}\right)^2} + i e^{\frac{2}{3}\pi i} \arctan\left(\frac{\frac{x_1\sqrt{3}}{2y_1}}{1 + \frac{x_1}{2y_1}}\right) + \\
&\quad - \frac{e^{\frac{2}{3}\pi i}}{x_1^2} \ln \sqrt{\left(1 + \frac{x_1^3}{2y_1}\right)^2 + \left(\frac{x_1^3\sqrt{3}}{2y_1}\right)^2} - i \frac{e^{\frac{2}{3}\pi i}}{x_1^2} \arctan\left(\frac{\frac{x_1^3\sqrt{3}}{2y_1}}{1 + \frac{x_1^3}{2y_1}}\right) + \\
&\quad - \frac{e^{\frac{2}{3}\pi i}}{y_1^2} \ln \sqrt{\left(1 + \frac{x_1 y_1}{2}\right)^2 + \left(\frac{x_1 y_1 \sqrt{3}}{2}\right)^2} - i \frac{e^{\frac{2}{3}\pi i}}{y_1^2} \arctan\left(\frac{\frac{x_1 y_1 \sqrt{3}}{2}}{1 + \frac{x_1 y_1}{2}}\right) + \\
&\quad + \frac{e^{\frac{2}{3}\pi i}}{x_1^2 y_1^2} \ln \sqrt{\left(1 + \frac{x_1^3 y_1}{2}\right)^2 + \left(\frac{x_1^3 y_1 \sqrt{3}}{2}\right)^2} + i \frac{e^{\frac{2}{3}\pi i}}{x_1^2 y_1^2} \arctan\left(\frac{\frac{x_1^3 y_1 \sqrt{3}}{2}}{1 + \frac{x_1^3 y_1}{2}}\right) + \\
&\quad + e^{-\frac{2}{3}\pi i} \ln \sqrt{\left(1 + \frac{x_1}{2y_1}\right)^2 + \left(-\frac{x_1\sqrt{3}}{2y_1}\right)^2} + i e^{-\frac{2}{3}\pi i} \arctan\left(\frac{-\frac{x_1\sqrt{3}}{2y_1}}{1 + \frac{x_1}{2y_1}}\right) + \\
&\quad - \frac{e^{-\frac{2}{3}\pi i}}{x_1^2} \ln \sqrt{\left(1 + \frac{x_1^3}{2y_1}\right)^2 + \left(-\frac{x_1^3\sqrt{3}}{2y_1}\right)^2} - i \frac{e^{-\frac{2}{3}\pi i}}{x_1^2} \arctan\left(\frac{-\frac{x_1^3\sqrt{3}}{2y_1}}{1 + \frac{x_1^3}{2y_1}}\right) + \\
&\quad - \frac{e^{-\frac{2}{3}\pi i}}{y_1^2} \ln \sqrt{\left(1 + \frac{x_1 y_1}{2}\right)^2 + \left(-\frac{x_1 y_1 \sqrt{3}}{2}\right)^2} - i \frac{e^{-\frac{2}{3}\pi i}}{y_1^2} \arctan\left(\frac{-\frac{x_1 y_1 \sqrt{3}}{2}}{1 + \frac{x_1 y_1}{2}}\right) + \\
&\quad + \frac{e^{-\frac{2}{3}\pi i}}{x_1^2 y_1^2} \ln \sqrt{\left(1 + \frac{x_1^3 y_1}{2}\right)^2 + \left(-\frac{x_1^3 y_1 \sqrt{3}}{2}\right)^2} + i \frac{e^{-\frac{2}{3}\pi i}}{x_1^2 y_1^2} \arctan\left(\frac{-\frac{x_1^3 y_1 \sqrt{3}}{2}}{1 + \frac{x_1^3 y_1}{2}}\right) \\
&= \ln\left(1 - \frac{x_1}{y_1}\right) - \frac{1}{x_1^2} \ln\left(1 - \frac{x_1^3}{y_1}\right) - \frac{1}{y_1^2} \ln(1 - x_1 y_1) + \frac{1}{x_1^2 y_1^2} \ln(1 - x_1^3 y_1) + \\
&\quad - \ln \sqrt{\frac{y_1^2 + x_1 y_1 + x_1^2}{y_1^2}} - \sqrt{3} \arctan\left(\frac{x_1 \sqrt{3}}{2y_1 + x_1}\right) + \\
&\quad + \frac{1}{x_1^2} \ln \sqrt{\frac{y_1^2 + y_1 x_1^3 + x_1^6}{y_1^2}} + \frac{\sqrt{3}}{x_1^2} \arctan\left(\frac{x_1^3 \sqrt{3}}{2y_1 + x_1^3}\right) + \\
&\quad + \frac{1}{y_1^2} \ln \sqrt{1 + x_1 y_1 + x_1^2 y_1^2} + \frac{\sqrt{3}}{y_1^2} \arctan\left(\frac{x_1 y_1 \sqrt{3}}{2 + x_1 y_1}\right) + \\
&\quad - \frac{1}{x_1^2 y_1^2} \ln \sqrt{1 + y_1 x_1^3 + x_1^6 y_1^2} - \frac{\sqrt{3}}{x_1^2 y_1^2} \arctan\left(\frac{x_1^3 y_1 \sqrt{3}}{2 + y_1 x_1^3}\right)
\end{aligned}$$

$$\begin{aligned}
&= \ln\left(1 - \frac{x_1}{y_1}\right) - \ln\sqrt{\frac{y_1^2 + x_1y_1 + x_1^2}{y_1^2}} + \\
&\quad + \frac{1}{x_1^2} \left(\ln\sqrt{\frac{y_1^2 + y_1x_1^3 + x_1^6}{y_1^2}} - \ln\left(1 - \frac{x_1^3}{y_1}\right) \right) + \\
&\quad + \frac{1}{y_1^2} \left(\ln\sqrt{1 + x_1y_1 + x_1^2y_1^2} - \ln(1 - x_1y_1) \right) + \\
&\quad + \frac{1}{x_1^2y_1^2} \left(\ln(1 - x_1^3y_1) - \ln\sqrt{1 + y_1x_1^3 + x_1^6y_1^2} \right) + \\
&\quad - \sqrt{3} \arctan\left(\frac{x_1\sqrt{3}}{2y_1 + x_1}\right) + \frac{\sqrt{3}}{x_1^2} \arctan\left(\frac{x_1^3\sqrt{3}}{2y_1 + x_1^3}\right) + \\
&\quad + \frac{\sqrt{3}}{y_1^2} \arctan\left(\frac{x_1y_1\sqrt{3}}{2 + x_1y_1}\right) - \frac{\sqrt{3}}{x_1^2y_1^2} \arctan\left(\frac{x_1^3y_1\sqrt{3}}{2 + y_1x_1^3}\right) \\
&= \frac{1}{2} \ln\left(\frac{x_1^2 - 2x_1y_1 + y_1^2}{x_1^2 + x_1y_1 + y_1^2}\right) - \frac{1}{2y_1^2} \ln\left(\frac{1 - 2x_1y_1 + x_1^2y_1^2}{1 + x_1y_1 + x_1^2y_1^2}\right) + \\
&\quad - \frac{1}{2x_1^2} \ln\left(\frac{x_1^6 - 2x_1^3y_1 + y_1^2}{x_1^6 + x_1^3y_1 + y_1^2}\right) + \frac{1}{2x_1^2y_1^2} \ln\left(\frac{1 - 2x_1^3y_1 + y_1^2x_1^6}{1 + x_1^3y_1 + x_1^6y_1^2}\right) + \\
&\quad - \sqrt{3} \arctan\left(\frac{x_1\sqrt{3}}{2y_1 + x_1}\right) + \frac{\sqrt{3}}{x_1^2} \arctan\left(\frac{x_1^3\sqrt{3}}{2y_1 + x_1^3}\right) + \\
&\quad + \frac{\sqrt{3}}{y_1^2} \arctan\left(\frac{x_1y_1\sqrt{3}}{2 + x_1y_1}\right) - \frac{\sqrt{3}}{x_1^2y_1^2} \arctan\left(\frac{x_1^3y_1\sqrt{3}}{2 + y_1x_1^3}\right). \tag{35}
\end{aligned}$$

• For $r \in (x_1, y_1)$ the integrand becomes

$$\begin{aligned}
&I_2(x_1, y_1; r) = \#(23) + \#(25) + \#(26) + \#(28) + \#(29) + \#(31) + \#(32) + \#(33) \\
&= \frac{1}{x_1y_1} \left(\frac{1}{1 - x_1y_1} - \frac{1}{1 - \frac{x_1}{y_1}} \right) + \frac{1}{x_1^2y_1^2} \frac{1}{r^2 - \frac{1}{x_1y_1}} - \frac{1}{x_1^2} \frac{1}{r^2 - \frac{y_1}{x_1}} + \\
&\quad - \frac{1}{x_1y_1} \left(\frac{1}{1 - \frac{x_1}{y_1}} - \frac{1}{1 - x_1y_1} \right) - \frac{1}{r^2 - x_1y_1} + \frac{1}{y_1^2} \frac{1}{r^2 - \frac{x_1}{y_1}} + \\
&\quad + \frac{e^{\frac{2}{3}\pi i}}{x_1y_1} \left(\frac{1}{e^{\frac{2}{3}\pi i} - x_1y_1} - \frac{1}{e^{\frac{2}{3}\pi i} - \frac{x_1}{y_1}} \right) + \frac{e^{\frac{2}{3}\pi i}}{x_1^2y_1^2} \frac{1}{r^2 - \frac{1}{x_1y_1}e^{\frac{2}{3}\pi i}} - \frac{e^{\frac{2}{3}\pi i}}{x_1^2} \frac{1}{r^2 - \frac{y_1}{x_1}e^{\frac{2}{3}\pi i}} + \\
&\quad + \frac{1}{x_1y_1} \left(\frac{1}{1 - x_1y_1e^{-\frac{2}{3}\pi i}} - \frac{1}{1 - \frac{x_1}{y_1}e^{-\frac{2}{3}\pi i}} \right) - \frac{e^{-\frac{2}{3}\pi i}}{r^2 - x_1y_1e^{-\frac{2}{3}\pi i}} + \frac{e^{-\frac{2}{3}\pi i}}{y_1^2} \frac{1}{r^2 - \frac{x_1}{y_1}e^{-\frac{2}{3}\pi i}} + \\
&\quad + \frac{e^{-\frac{2}{3}\pi i}}{x_1y_1} \left(\frac{1}{e^{-\frac{2}{3}\pi i} - x_1y_1} - \frac{1}{e^{-\frac{2}{3}\pi i} - \frac{x_1}{y_1}} \right) + \frac{e^{-\frac{2}{3}\pi i}}{x_1^2y_1^2} \frac{1}{r^2 - \frac{1}{x_1y_1}e^{-\frac{2}{3}\pi i}} - \frac{e^{-\frac{2}{3}\pi i}}{x_1^2} \frac{1}{r^2 - \frac{y_1}{x_1}e^{-\frac{2}{3}\pi i}} +
\end{aligned}$$

$$\begin{aligned}
& + \frac{1}{x_1 y_1} \left(\frac{1}{\frac{x_1}{y_1} e^{\frac{2}{3}\pi i} - 1} - \frac{1}{x_1 y_1 e^{\frac{2}{3}\pi i} - 1} \right) - \frac{e^{\frac{2}{3}\pi i}}{r^2 - x_1 y_1 e^{\frac{2}{3}\pi i}} + \frac{e^{\frac{2}{3}\pi i}}{y_1^2} \frac{1}{r^2 - \frac{x_1}{y_1} e^{\frac{2}{3}\pi i}} + \\
& + \frac{1}{y_1^2} \left(\frac{1}{1 - \frac{x_1}{y_1}} + \frac{e^{-\frac{2}{3}\pi i}}{1 - \frac{x_1}{y_1} e^{-\frac{2}{3}\pi i}} + \frac{e^{\frac{2}{3}\pi i}}{1 - \frac{x_1}{y_1} e^{\frac{2}{3}\pi i}} \right) + \frac{1}{x_1 y_1} \left(\frac{1}{1 - x_1 y_1} + \frac{e^{-\frac{2}{3}\pi i}}{e^{-\frac{2}{3}\pi i} - x_1 y_1} + \frac{e^{\frac{2}{3}\pi i}}{e^{\frac{2}{3}\pi i} - x_1 y_1} \right) + \\
& + \frac{1}{x_1^2 y_1^2} \left(\frac{1}{r^2 - \frac{1}{x_1 y_1}} + \frac{e^{-\frac{2}{3}\pi i}}{r^2 - \frac{1}{x_1 y_1} e^{-\frac{2}{3}\pi i}} + \frac{e^{\frac{2}{3}\pi i}}{r^2 - \frac{1}{x_1 y_1} e^{\frac{2}{3}\pi i}} \right) - \frac{1}{y_1^2} \left(\frac{1}{r^2 - \frac{x_1}{y_1}} + \frac{e^{-\frac{2}{3}\pi i}}{r^2 - \frac{x_1}{y_1} e^{-\frac{2}{3}\pi i}} + \frac{e^{\frac{2}{3}\pi i}}{r^2 - \frac{x_1}{y_1} e^{\frac{2}{3}\pi i}} \right) + \\
& - \frac{1}{x_1 y_1} \left(\frac{1}{1 - \frac{x_1}{y_1}} + \frac{1}{1 - \frac{x_1}{y_1} e^{\frac{2}{3}\pi i}} + \frac{1}{1 - \frac{x_1}{y_1} e^{-\frac{2}{3}\pi i}} \right) - \frac{1}{1 - x_1 y_1} - \frac{e^{-\frac{2}{3}\pi i}}{1 - x_1 y_1 e^{-\frac{2}{3}\pi i}} - \frac{e^{\frac{2}{3}\pi i}}{1 - x_1 y_1 e^{\frac{2}{3}\pi i}} + \\
& + \frac{1}{r^2 - x_1 y_1} + \frac{e^{\frac{2}{3}\pi i}}{r^2 - x_1 y_1 e^{\frac{2}{3}\pi i}} + \frac{e^{-\frac{2}{3}\pi i}}{r^2 - x_1 y_1 e^{-\frac{2}{3}\pi i}} - \frac{1}{x_1^2} \left(\frac{1}{r^2 - \frac{y_1}{x_1}} + \frac{e^{\frac{2}{3}\pi i}}{r^2 - \frac{y_1}{x_1} e^{\frac{2}{3}\pi i}} + \frac{e^{-\frac{2}{3}\pi i}}{r^2 - \frac{y_1}{x_1} e^{-\frac{2}{3}\pi i}} \right) \\
& = \frac{1}{x_1 y_1} \left(\frac{1}{1 - x_1 y_1} - \frac{1}{1 - \frac{x_1}{y_1}} \right) + \frac{1}{x_1^2 y_1^2} \frac{1}{r^2 - \frac{1}{x_1 y_1}} - \frac{1}{x_1^2} \frac{1}{r^2 - \frac{y_1}{x_1}} + \\
& - \frac{1}{x_1 y_1} \left(\frac{1}{1 - \frac{x_1}{y_1}} - \frac{1}{1 - x_1 y_1} \right) + \\
& + \frac{e^{\frac{2}{3}\pi i}}{x_1 y_1} \left(\frac{1}{e^{\frac{2}{3}\pi i} - x_1 y_1} - \frac{1}{e^{\frac{2}{3}\pi i} - \frac{x_1}{y_1}} \right) + \frac{e^{\frac{2}{3}\pi i}}{x_1^2 y_1^2} \frac{1}{r^2 - \frac{1}{x_1 y_1} e^{\frac{2}{3}\pi i}} - \frac{e^{\frac{2}{3}\pi i}}{x_1^2} \frac{1}{r^2 - \frac{y_1}{x_1} e^{\frac{2}{3}\pi i}} + \\
& + \frac{1}{x_1 y_1} \left(\frac{1}{1 - x_1 y_1 e^{-\frac{2}{3}\pi i}} - \frac{1}{1 - \frac{x_1}{y_1} e^{-\frac{2}{3}\pi i}} \right) + \\
& + \frac{e^{-\frac{2}{3}\pi i}}{x_1 y_1} \left(\frac{1}{e^{-\frac{2}{3}\pi i} - x_1 y_1} - \frac{1}{e^{-\frac{2}{3}\pi i} - \frac{x_1}{y_1}} \right) + \frac{e^{-\frac{2}{3}\pi i}}{x_1^2 y_1^2} \frac{1}{r^2 - \frac{1}{x_1 y_1} e^{-\frac{2}{3}\pi i}} - \frac{e^{-\frac{2}{3}\pi i}}{x_1^2} \frac{1}{r^2 - \frac{y_1}{x_1} e^{-\frac{2}{3}\pi i}} + \\
& + \frac{1}{x_1 y_1} \left(\frac{1}{\frac{x_1}{y_1} e^{\frac{2}{3}\pi i} - 1} - \frac{1}{x_1 y_1 e^{\frac{2}{3}\pi i} - 1} \right) + \\
& + \frac{1}{y_1^2} \left(\frac{1}{1 - \frac{x_1}{y_1}} + \frac{e^{-\frac{2}{3}\pi i}}{1 - \frac{x_1}{y_1} e^{-\frac{2}{3}\pi i}} + \frac{e^{\frac{2}{3}\pi i}}{1 - \frac{x_1}{y_1} e^{\frac{2}{3}\pi i}} \right) + \frac{1}{x_1 y_1} \left(\frac{1}{1 - x_1 y_1} + \frac{e^{-\frac{2}{3}\pi i}}{e^{-\frac{2}{3}\pi i} - x_1 y_1} + \frac{e^{\frac{2}{3}\pi i}}{e^{\frac{2}{3}\pi i} - x_1 y_1} \right) + \\
& + \frac{1}{x_1^2 y_1^2} \left(\frac{1}{r^2 - \frac{1}{x_1 y_1}} + \frac{e^{-\frac{2}{3}\pi i}}{r^2 - \frac{1}{x_1 y_1} e^{-\frac{2}{3}\pi i}} + \frac{e^{\frac{2}{3}\pi i}}{r^2 - \frac{1}{x_1 y_1} e^{\frac{2}{3}\pi i}} \right) + \\
& - \frac{1}{x_1 y_1} \left(\frac{1}{1 - \frac{x_1}{y_1}} + \frac{1}{1 - \frac{x_1}{y_1} e^{\frac{2}{3}\pi i}} + \frac{1}{1 - \frac{x_1}{y_1} e^{-\frac{2}{3}\pi i}} \right) - \frac{1}{1 - x_1 y_1} - \frac{e^{-\frac{2}{3}\pi i}}{1 - x_1 y_1 e^{-\frac{2}{3}\pi i}} - \frac{e^{\frac{2}{3}\pi i}}{1 - x_1 y_1 e^{\frac{2}{3}\pi i}} + \\
& - \frac{1}{x_1^2} \left(\frac{1}{r^2 - \frac{y_1}{x_1}} + \frac{e^{\frac{2}{3}\pi i}}{r^2 - \frac{y_1}{x_1} e^{\frac{2}{3}\pi i}} + \frac{e^{-\frac{2}{3}\pi i}}{r^2 - \frac{y_1}{x_1} e^{-\frac{2}{3}\pi i}} \right)
\end{aligned}$$

$$\begin{aligned}
&= \frac{3}{x_1 y_1} \left(\frac{1}{1-x_1 y_1} - \frac{1}{1-\frac{x_1}{y_1}} \right) + \frac{1}{y_1^2} \frac{1}{1-\frac{x_1}{y_1}} - \frac{1}{1-x_1 y_1} + \\
&+ \frac{3}{x_1 y_1} \left(\frac{e^{\frac{2}{3}\pi i}}{e^{\frac{2}{3}\pi i} - x_1 y_1} - \frac{e^{\frac{2}{3}\pi i}}{e^{\frac{2}{3}\pi i} - \frac{x_1}{y_1}} + \frac{e^{-\frac{2}{3}\pi i}}{e^{-\frac{2}{3}\pi i} - x_1 y_1} - \frac{e^{-\frac{2}{3}\pi i}}{e^{-\frac{2}{3}\pi i} - \frac{x_1}{y_1}} \right) + \\
&+ \frac{1}{y_1^2} \frac{e^{-\frac{2}{3}\pi i}}{1-\frac{x_1}{y_1} e^{-\frac{2}{3}\pi i}} + \frac{1}{y_1^2} \frac{e^{\frac{2}{3}\pi i}}{1-\frac{x_1}{y_1} e^{\frac{2}{3}\pi i}} - \frac{e^{-\frac{2}{3}\pi i}}{1-x_1 y_1 e^{-\frac{2}{3}\pi i}} - \frac{e^{\frac{2}{3}\pi i}}{1-x_1 y_1 e^{\frac{2}{3}\pi i}} + \\
&+ \frac{2}{x_1^2} \left(\frac{1}{y_1^2} \frac{1}{r^2 - \frac{1}{x_1 y_1}} - \frac{1}{r^2 - \frac{y_1}{x_1}} \right) + \\
&+ \frac{2}{x_1^2} \left(\frac{1}{y_1^2} \frac{e^{\frac{2}{3}\pi i}}{r^2 - \frac{1}{x_1 y_1} e^{\frac{2}{3}\pi i}} - \frac{e^{\frac{2}{3}\pi i}}{r^2 - \frac{y_1}{x_1} e^{\frac{2}{3}\pi i}} \right) + \\
&+ \frac{2}{x_1^2} \left(\frac{1}{y_1^2} \frac{e^{-\frac{2}{3}\pi i}}{r^2 - \frac{1}{x_1 y_1} e^{-\frac{2}{3}\pi i}} - \frac{e^{-\frac{2}{3}\pi i}}{r^2 - \frac{y_1}{x_1} e^{-\frac{2}{3}\pi i}} \right). \\
&= \frac{3}{x_1 y_1} \left(\frac{1}{1-x_1 y_1} - \frac{1}{1-\frac{x_1}{y_1}} \right) + \frac{1}{y_1^2} \frac{1}{1-\frac{x_1}{y_1}} - \frac{1}{1-x_1 y_1} + \\
&+ \frac{3}{x_1 y_1} \left(\frac{e^{\frac{2}{3}\pi i}}{e^{\frac{2}{3}\pi i} - x_1 y_1} - \frac{e^{\frac{2}{3}\pi i}}{e^{\frac{2}{3}\pi i} - \frac{x_1}{y_1}} + \frac{e^{-\frac{2}{3}\pi i}}{e^{-\frac{2}{3}\pi i} - x_1 y_1} - \frac{e^{-\frac{2}{3}\pi i}}{e^{-\frac{2}{3}\pi i} - \frac{x_1}{y_1}} \right) + \\
&+ \frac{1}{y_1^2} \frac{e^{-\frac{2}{3}\pi i}}{1-\frac{x_1}{y_1} e^{-\frac{2}{3}\pi i}} + \frac{1}{y_1^2} \frac{e^{\frac{2}{3}\pi i}}{1-\frac{x_1}{y_1} e^{\frac{2}{3}\pi i}} - \frac{e^{-\frac{2}{3}\pi i}}{1-x_1 y_1 e^{-\frac{2}{3}\pi i}} - \frac{e^{\frac{2}{3}\pi i}}{1-x_1 y_1 e^{\frac{2}{3}\pi i}} + \\
&+ \frac{2}{x_1^2} \left(\frac{1}{y_1^2} \frac{1}{r^2 - \frac{1}{x_1 y_1}} - \frac{1}{r^2 - \frac{y_1}{x_1}} \right) + \\
&+ \frac{2}{x_1^2} \left(\frac{1}{y_1^2} \frac{e^{\frac{2}{3}\pi i}}{r^2 - \frac{1}{x_1 y_1} e^{\frac{2}{3}\pi i}} - \frac{e^{\frac{2}{3}\pi i}}{r^2 - \frac{y_1}{x_1} e^{\frac{2}{3}\pi i}} \right) + \\
&+ \frac{2}{x_1^2} \left(\frac{1}{y_1^2} \frac{e^{-\frac{2}{3}\pi i}}{r^2 - \frac{1}{x_1 y_1} e^{-\frac{2}{3}\pi i}} - \frac{e^{-\frac{2}{3}\pi i}}{r^2 - \frac{y_1}{x_1} e^{-\frac{2}{3}\pi i}} \right) \\
&= \frac{3}{x_1 y_1} \sum_{k=0}^2 \left(\frac{e^{\frac{2}{3}k\pi i}}{e^{\frac{2}{3}k\pi i} - x_1 y_1} - \frac{e^{\frac{2}{3}k\pi i}}{e^{\frac{2}{3}k\pi i} - \frac{x_1}{y_1}} \right) + \\
&+ \sum_{k=0}^2 \left(\frac{1}{y_1^2} \frac{e^{\frac{2}{3}k\pi i}}{1-\frac{x_1}{y_1} e^{\frac{2}{3}k\pi i}} - \frac{e^{\frac{2}{3}k\pi i}}{1-x_1 y_1 e^{\frac{2}{3}k\pi i}} \right) + \\
&+ \frac{2}{x_1^2} \sum_{k=0}^2 \left(\frac{1}{y_1^2} \frac{e^{\frac{2}{3}k\pi i}}{r^2 - \frac{1}{x_1 y_1} e^{\frac{2}{3}k\pi i}} - \frac{e^{\frac{2}{3}k\pi i}}{r^2 - \frac{y_1}{x_1} e^{\frac{2}{3}k\pi i}} \right),
\end{aligned}$$

and, after another lengthy calculation

$$\int_{r=x_1}^{y_1} I_2(x_1, y_1; r) r dr =$$

$$\begin{aligned}
&= \left[\left(\frac{3}{x_1 y_1} \left(\frac{1}{1-x_1 y_1} - \frac{1}{1-\frac{x_1}{y_1}} \right) + \frac{1}{y_1^2} \frac{1}{1-\frac{x_1}{y_1}} - \frac{1}{1-x_1 y_1} \right) \frac{1}{2} r^2 \right]_{r=x_1}^{y_1} + \\
&+ \left[\frac{3}{x_1 y_1} \left(\frac{e^{\frac{2}{3}\pi i}}{e^{\frac{2}{3}\pi i} - x_1 y_1} - \frac{e^{\frac{2}{3}\pi i}}{e^{\frac{2}{3}\pi i} - \frac{x_1}{y_1}} + \frac{e^{-\frac{2}{3}\pi i}}{e^{-\frac{2}{3}\pi i} - x_1 y_1} - \frac{e^{-\frac{2}{3}\pi i}}{e^{-\frac{2}{3}\pi i} - \frac{x_1}{y_1}} \right) \frac{1}{2} r^2 \right]_{r=x_1}^{y_1} + \\
&+ \left[\left(\frac{1}{y_1^2} \frac{e^{-\frac{2}{3}\pi i}}{1-\frac{x_1}{y_1} e^{-\frac{2}{3}\pi i}} + \frac{1}{y_1^2} \frac{e^{\frac{2}{3}\pi i}}{1-\frac{x_1}{y_1} e^{\frac{2}{3}\pi i}} - \frac{e^{-\frac{2}{3}\pi i}}{1-x_1 y_1 e^{-\frac{2}{3}\pi i}} - \frac{e^{\frac{2}{3}\pi i}}{1-x_1 y_1 e^{\frac{2}{3}\pi i}} \right) \frac{1}{2} r^2 \right]_{r=x_1}^{y_1} + \\
&+ \left[\frac{1}{x_1^2} \left(\frac{1}{y_1^2} \ln \left(\frac{1}{x_1 y_1} - r^2 \right) - \ln \left(\frac{y_1}{x_1} - r^2 \right) \right) \right]_{r=x_1}^{y_1} + \\
&+ \left[\frac{1}{x_1^2} \left(\frac{e^{\frac{2}{3}\pi i}}{y_1^2} \text{Ln} \left(\frac{1}{x_1 y_1} e^{\frac{2}{3}\pi i} - r^2 \right) - e^{\frac{2}{3}\pi i} \text{Ln} \left(\frac{y_1}{x_1} e^{\frac{2}{3}\pi i} - r^2 \right) \right) \right]_{r=x_1}^{y_1} + \\
&+ \left[\frac{1}{x_1^2} \left(\frac{e^{-\frac{2}{3}\pi i}}{y_1^2} \text{Ln} \left(\frac{1}{x_1 y_1} e^{-\frac{2}{3}\pi i} - r^2 \right) - e^{-\frac{2}{3}\pi i} \text{Ln} \left(\frac{y_1}{x_1} e^{-\frac{2}{3}\pi i} - r^2 \right) \right) \right]_{r=x_1}^{y_1} \\
&= \left(\frac{3}{x_1 y_1} \left(\frac{1}{1-x_1 y_1} - \frac{1}{1-\frac{x_1}{y_1}} \right) + \frac{1}{y_1^2} \frac{1}{1-\frac{x_1}{y_1}} - \frac{1}{1-x_1 y_1} \right) \frac{1}{2} (y_1^2 - x_1^2) + \\
&+ \frac{3}{x_1 y_1} \left(\frac{e^{\frac{2}{3}\pi i}}{e^{\frac{2}{3}\pi i} - x_1 y_1} - \frac{e^{\frac{2}{3}\pi i}}{e^{\frac{2}{3}\pi i} - \frac{x_1}{y_1}} + \frac{e^{-\frac{2}{3}\pi i}}{e^{-\frac{2}{3}\pi i} - x_1 y_1} - \frac{e^{-\frac{2}{3}\pi i}}{e^{-\frac{2}{3}\pi i} - \frac{x_1}{y_1}} \right) \frac{1}{2} (y_1^2 - x_1^2) + \\
&+ \left(\frac{1}{y_1^2} \frac{e^{-\frac{2}{3}\pi i}}{1-\frac{x_1}{y_1} e^{-\frac{2}{3}\pi i}} + \frac{1}{y_1^2} \frac{e^{\frac{2}{3}\pi i}}{1-\frac{x_1}{y_1} e^{\frac{2}{3}\pi i}} - \frac{e^{-\frac{2}{3}\pi i}}{1-x_1 y_1 e^{-\frac{2}{3}\pi i}} - \frac{e^{\frac{2}{3}\pi i}}{1-x_1 y_1 e^{\frac{2}{3}\pi i}} \right) \frac{1}{2} (y_1^2 - x_1^2) + \\
&+ \frac{1}{x_1^2} \left(\frac{1}{y_1^2} \ln \left(\frac{\frac{1}{x_1 y_1} - y_1^2}{\frac{1}{x_1 y_1} - x_1^2} \right) - \ln \left(\frac{\frac{y_1}{x_1} - y_1^2}{\frac{y_1}{x_1} - x_1^2} \right) \right) + \\
&+ \frac{1}{x_1^2} \left(\frac{e^{\frac{2}{3}\pi i}}{y_1^2} \text{Ln} \left(\frac{\frac{1}{x_1 y_1} e^{\frac{2}{3}\pi i} - y_1^2}{\frac{1}{x_1 y_1} e^{\frac{2}{3}\pi i} - x_1^2} \right) - e^{\frac{2}{3}\pi i} \text{Ln} \left(\frac{\frac{y_1}{x_1} e^{\frac{2}{3}\pi i} - y_1^2}{\frac{y_1}{x_1} e^{\frac{2}{3}\pi i} - x_1^2} \right) \right) + \\
&+ \frac{1}{x_1^2} \left(\frac{e^{-\frac{2}{3}\pi i}}{y_1^2} \text{Ln} \left(\frac{\frac{1}{x_1 y_1} e^{-\frac{2}{3}\pi i} - y_1^2}{\frac{1}{x_1 y_1} e^{-\frac{2}{3}\pi i} - x_1^2} \right) - e^{-\frac{2}{3}\pi i} \text{Ln} \left(\frac{\frac{y_1}{x_1} e^{-\frac{2}{3}\pi i} - y_1^2}{\frac{y_1}{x_1} e^{-\frac{2}{3}\pi i} - x_1^2} \right) \right)
\end{aligned}$$

$$\begin{aligned}
&= -\frac{(x_1 + y_1)(1 - y_1^2)}{y_1(1 - x_1 y_1)} + \\
&+ \frac{3}{x_1 y_1} \left(\frac{e^{\frac{2}{3}\pi i}}{e^{\frac{2}{3}\pi i} - x_1 y_1} + \frac{e^{-\frac{2}{3}\pi i}}{e^{-\frac{2}{3}\pi i} - x_1 y_1} \right) \frac{1}{2} (y_1^2 - x_1^2) + \\
&- \frac{3}{x_1 y_1} \left(\frac{e^{\frac{2}{3}\pi i}}{e^{\frac{2}{3}\pi i} - \frac{x_1}{y_1}} + \frac{e^{-\frac{2}{3}\pi i}}{e^{-\frac{2}{3}\pi i} - \frac{x_1}{y_1}} \right) \frac{1}{2} (y_1^2 - x_1^2) + \\
&+ \frac{1}{y_1^2} \left(\frac{e^{-\frac{2}{3}\pi i}}{1 - \frac{x_1}{y_1} e^{-\frac{2}{3}\pi i}} + \frac{e^{\frac{2}{3}\pi i}}{1 - \frac{x_1}{y_1} e^{\frac{2}{3}\pi i}} \right) \frac{1}{2} (y_1^2 - x_1^2) + \\
&- \left(\frac{e^{-\frac{2}{3}\pi i}}{1 - x_1 y_1 e^{-\frac{2}{3}\pi i}} + \frac{e^{\frac{2}{3}\pi i}}{1 - x_1 y_1 e^{\frac{2}{3}\pi i}} \right) \frac{1}{2} (y_1^2 - x_1^2) + \\
&+ \frac{1}{x_1^2} \left(\frac{1}{y_1^2} \ln \left(\frac{1}{x_1 y_1} \frac{1 - y_1^2}{x_1 y_1 - x_1^2} \right) - \ln \left(\frac{y_1 - y_1^2}{x_1 - x_1^2} \right) \right) + \\
&+ \frac{1}{x_1^2} \left(\frac{e^{\frac{2}{3}\pi i}}{y_1^2} \text{Ln} \left(\frac{1}{x_1 y_1} e^{\frac{2}{3}\pi i} \frac{1 - y_1^2}{x_1 y_1 - x_1^2} \right) - e^{\frac{2}{3}\pi i} \text{Ln} \left(\frac{y_1 e^{\frac{2}{3}\pi i} - y_1^2}{x_1 e^{\frac{2}{3}\pi i} - x_1^2} \right) \right) + \\
&+ \frac{1}{x_1^2} \left(\frac{e^{-\frac{2}{3}\pi i}}{y_1^2} \text{Ln} \left(\frac{1}{x_1 y_1} e^{-\frac{2}{3}\pi i} \frac{1 - y_1^2}{x_1 y_1 - x_1^2} \right) - e^{-\frac{2}{3}\pi i} \text{Ln} \left(\frac{y_1 e^{-\frac{2}{3}\pi i} - y_1^2}{x_1 e^{-\frac{2}{3}\pi i} - x_1^2} \right) \right) \\
&= -\frac{(x_1 + y_1)(1 - y_1^2)}{y_1(1 - x_1 y_1)} + \\
&+ \frac{3}{x_1 y_1} \left(\frac{2 + x_1 y_1}{1 + x_1 y_1 + x_1^2 y_1^2} \right) \frac{1}{2} (y_1^2 - x_1^2) + \\
&- \frac{3}{x_1 y_1} \left(\frac{2y_1^2 + x_1 y_1}{y_1^2 + x_1 y_1 + x_1^2} \right) \frac{1}{2} (y_1^2 - x_1^2) + \\
&- \frac{1}{y_1^2} \left(\frac{y_1^2 + 2x_1 y_1}{y_1^2 + x_1 y_1 + x_1^2} \right) \frac{1}{2} (y_1^2 - x_1^2) + \\
&+ \left(\frac{1 + 2x_1 y_1}{1 + x_1 y_1 + x_1^2 y_1^2} \right) \frac{1}{2} (y_1^2 - x_1^2) + \\
&+ \frac{1}{x_1^2 y_1^2} \ln \left(\frac{1 - x_1 y_1^3}{1 - x_1^3 y_1} \right) - \frac{1}{x_1^2} \ln \left(\frac{1 - x_1 y_1}{1 - x_1^2 \frac{x_1}{y_1}} \right) + \\
&+ \frac{e^{\frac{2}{3}\pi i}}{x_1^2 y_1^2} \text{Ln} \left(\frac{1 - x_1 y_1^3 e^{-\frac{2}{3}\pi i}}{1 - x_1^3 y_1 e^{-\frac{2}{3}\pi i}} \right) - \frac{e^{\frac{2}{3}\pi i}}{x_1^2} \text{Ln} \left(\frac{1 - x_1 y_1 e^{-\frac{2}{3}\pi i}}{1 - x_1^2 \frac{x_1}{y_1} e^{-\frac{2}{3}\pi i}} \right) + \\
&+ \frac{e^{-\frac{2}{3}\pi i}}{x_1^2 y_1^2} \text{Ln} \left(\frac{1 - x_1 y_1^3 e^{\frac{2}{3}\pi i}}{1 - x_1^3 y_1 e^{\frac{2}{3}\pi i}} \right) - \frac{e^{-\frac{2}{3}\pi i}}{x_1^2} \text{Ln} \left(\frac{1 - x_1 y_1 e^{\frac{2}{3}\pi i}}{1 - x_1^2 \frac{x_1}{y_1} e^{\frac{2}{3}\pi i}} \right)
\end{aligned}$$

$$\begin{aligned}
&= -\frac{(x_1 + y_1)(1 - y_1^2)}{y_1(1 - x_1 y_1)} + \\
&\quad + \frac{y_1^2 - x_1^2}{x_1 y_1} \frac{3 + 2x_1 y_1 + x_1^2 y_1^2}{1 + x_1 y_1 + x_1^2 y_1^2} + \\
&\quad - \frac{y_1^2 - x_1^2}{x_1 y_1} \frac{3y_1^2 + 2x_1 y_1 + x_1^2}{y_1^2 + x_1 y_1 + x_1^2} + \\
&\quad + \frac{1}{x_1^2 y_1^2} \ln \left(\frac{1 - x_1 y_1^3}{1 - x_1^3 y_1} \right) - \frac{1}{x_1^2} \ln \left(\frac{1 - x_1 y_1}{1 - x_1^2 \frac{x_1}{y_1}} \right) + \\
&\quad + \frac{e^{\frac{2}{3}\pi i}}{x_1^2 y_1^2} \text{Ln} \left(2 + y_1^3 x_1 + i y_1^3 x_1 \sqrt{3} \right) - \frac{e^{\frac{2}{3}\pi i}}{x_1^2 y_1^2} \text{Ln} \left(2 + x_1^3 y_1 + i x_1^3 y_1 \sqrt{3} \right) + \\
&\quad - \frac{e^{\frac{2}{3}\pi i}}{x_1^2} \text{Ln} \left(y_1 \left(2 + x_1 y_1 + i x_1 y_1 \sqrt{3} \right) \right) + \frac{e^{\frac{2}{3}\pi i}}{x_1^2} \text{Ln} \left(2y_1 + x_1^3 + i x_1^3 \sqrt{3} \right) + \\
&\quad + \frac{e^{-\frac{2}{3}\pi i}}{x_1^2 y_1^2} \text{Ln} \left(2 + y_1^3 x_1 - i y_1^3 x_1 \sqrt{3} \right) - \frac{e^{-\frac{2}{3}\pi i}}{x_1^2 y_1^2} \text{Ln} \left(2 + x_1^3 y_1 - i x_1^3 y_1 \sqrt{3} \right) + \\
&\quad - \frac{e^{-\frac{2}{3}\pi i}}{x_1^2} \text{Ln} \left(y_1 \left(2 + x_1 y_1 - i x_1 y_1 \sqrt{3} \right) \right) + \frac{e^{-\frac{2}{3}\pi i}}{x_1^2} \text{Ln} \left(2y_1 + x_1^3 - i x_1^3 \sqrt{3} \right) \\
&= -\frac{(x_1 + y_1)(1 - y_1^2)}{y_1(1 - x_1 y_1)} + \\
&\quad + \frac{y_1^2 - x_1^2}{x_1 y_1} \frac{3 + 2x_1 y_1 + x_1^2 y_1^2}{1 + x_1 y_1 + x_1^2 y_1^2} + \\
&\quad - \frac{y_1^2 - x_1^2}{x_1 y_1} \frac{3y_1^2 + 2x_1 y_1 + x_1^2}{y_1^2 + x_1 y_1 + x_1^2} + \\
&\quad + \frac{1}{x_1^2 y_1^2} \ln \left(\frac{1 - x_1 y_1^3}{1 - x_1^3 y_1} \right) - \frac{1}{x_1^2} \ln \left(\frac{1 - x_1 y_1}{1 - x_1^2 \frac{x_1}{y_1}} \right) + \\
&\quad + \frac{e^{\frac{2}{3}\pi i} + e^{-\frac{2}{3}\pi i}}{x_1^2 y_1^2} \ln \sqrt{(2 + y_1^3 x_1)^2 + (y_1^3 x_1 \sqrt{3})^2} + i \frac{e^{\frac{2}{3}\pi i} - e^{-\frac{2}{3}\pi i}}{x_1^2 y_1^2} \arctan \left(\frac{y_1^3 x_1 \sqrt{3}}{2 + y_1^3 x_1} \right) + \\
&\quad + \frac{e^{\frac{2}{3}\pi i} + e^{-\frac{2}{3}\pi i}}{x_1^2} \ln \sqrt{(2y_1 + x_1^3)^2 + (x_1^3 \sqrt{3})^2} + i \frac{e^{\frac{2}{3}\pi i} - e^{-\frac{2}{3}\pi i}}{x_1^2} \arctan \left(\frac{x_1^3 \sqrt{3}}{2y_1 + x_1^3} \right) + \\
&\quad - \frac{e^{\frac{2}{3}\pi i} + e^{-\frac{2}{3}\pi i}}{x_1^2 y_1^2} \ln \sqrt{(2 + x_1^3 y_1)^2 + (x_1^3 y_1 \sqrt{3})^2} - i \frac{e^{\frac{2}{3}\pi i} - e^{-\frac{2}{3}\pi i}}{x_1^2 y_1^2} \arctan \left(\frac{x_1^3 y_1 \sqrt{3}}{2 + x_1^3 y_1} \right) + \\
&\quad - \frac{e^{\frac{2}{3}\pi i} + e^{-\frac{2}{3}\pi i}}{x_1^2} \ln \sqrt{y_1^2 (2 + x_1 y_1)^2 + y_1^2 (x_1 y_1 \sqrt{3})^2} - i \frac{e^{\frac{2}{3}\pi i} - e^{-\frac{2}{3}\pi i}}{x_1^2} \arctan \left(\frac{x_1 y_1 \sqrt{3}}{2 + x_1 y_1} \right)
\end{aligned}$$

$$\begin{aligned}
&= -\frac{(x_1 + y_1)(1 - y_1^2)}{y_1(1 - x_1 y_1)} + \\
&\quad + \frac{y_1^2 - x_1^2}{x_1 y_1} \frac{3 + 2x_1 y_1 + x_1^2 y_1^2}{1 + x_1 y_1 + x_1^2 y_1^2} + \\
&\quad - \frac{y_1^2 - x_1^2}{x_1 y_1} \frac{3y_1^2 + 2x_1 y_1 + x_1^2}{y_1^2 + x_1 y_1 + x_1^2} + \\
&\quad + \frac{1}{x_1^2 y_1^2} \ln\left(\frac{1 - x_1 y_1^3}{1 - x_1^3 y_1}\right) - \frac{1}{x_1^2} \ln\left(\frac{1 - x_1 y_1}{1 - x_1^2 \frac{x_1}{y_1}}\right) + \\
&\quad - \frac{1}{x_1^2 y_1^2} \ln\left(2\sqrt{(1 + y_1^3 x_1 + y_1^6 x_1^2)}\right) - \frac{\sqrt{3}}{x_1^2 y_1^2} \arctan\left(\frac{y_1^3 x_1 \sqrt{3}}{2 + y_1^3 x_1}\right) + \\
&\quad - \frac{1}{x_1^2} \ln\left(2\sqrt{(y_1^2 + x_1^3 y_1 + x_1^6)}\right) - \frac{\sqrt{3}}{x_1^2} \arctan\left(\frac{x_1^3 \sqrt{3}}{2y_1 + x_1^3}\right) + \\
&\quad + \frac{1}{x_1^2 y_1^2} \ln\left(2\sqrt{(1 + x_1^3 y_1 + x_1^6 y_1^2)}\right) + \frac{\sqrt{3}}{x_1^2 y_1^2} \arctan\left(\frac{x_1^3 y_1 \sqrt{3}}{2 + x_1^3 y_1}\right) + \\
&\quad + \frac{1}{x_1^2} \ln\left(2y_1 \sqrt{(1 + x_1 y_1 + x_1^2 y_1^2)}\right) + \frac{\sqrt{3}}{x_1^2} \arctan\left(\frac{x_1 y_1 \sqrt{3}}{2 + x_1 y_1}\right) \\
&= -\frac{(x_1 + y_1)(1 - y_1^2)}{y_1(1 - x_1 y_1)} + \\
&\quad + \frac{y_1^2 - x_1^2}{x_1 y_1} \frac{3 + 2x_1 y_1 + x_1^2 y_1^2}{1 + x_1 y_1 + x_1^2 y_1^2} + \\
&\quad - \frac{y_1^2 - x_1^2}{x_1 y_1} \frac{3y_1^2 + 2x_1 y_1 + x_1^2}{y_1^2 + x_1 y_1 + x_1^2} + \\
&\quad + \frac{1}{x_1^2 y_1^2} \ln\left(\frac{1 - x_1 y_1^3}{1 - x_1^3 y_1}\right) - \frac{1}{x_1^2} \ln\left(\frac{1 - x_1 y_1}{1 - x_1^2 \frac{x_1}{y_1}}\right) + \\
&\quad - \frac{1}{2x_1^2 y_1^2} \ln(1 + y_1^3 x_1 + y_1^6 x_1^2) - \frac{\sqrt{3}}{x_1^2 y_1^2} \arctan\left(\frac{y_1^3 x_1 \sqrt{3}}{2 + y_1^3 x_1}\right) + \\
&\quad - \frac{1}{2x_1^2} \ln(y_1^2 + x_1^3 y_1 + x_1^6) - \frac{\sqrt{3}}{x_1^2} \arctan\left(\frac{x_1^3 \sqrt{3}}{2y_1 + x_1^3}\right) + \\
&\quad + \frac{1}{2x_1^2 y_1^2} \ln(1 + x_1^3 y_1 + x_1^6 y_1^2) + \frac{\sqrt{3}}{x_1^2 y_1^2} \arctan\left(\frac{x_1^3 y_1 \sqrt{3}}{2 + x_1^3 y_1}\right) + \\
&\quad + \frac{1}{2x_1^2} \ln(y_1^2 (1 + x_1 y_1 + x_1^2 y_1^2)) + \frac{\sqrt{3}}{x_1^2} \arctan\left(\frac{x_1 y_1 \sqrt{3}}{2 + x_1 y_1}\right)
\end{aligned}$$

$$\begin{aligned}
&= -\frac{(x_1 + y_1)(1 - y_1^2)}{y_1(1 - x_1 y_1)} + \\
&\quad + \frac{y_1^2 - x_1^2}{x_1 y_1} \left(\frac{3 + 2x_1 y_1 + x_1^2 y_1^2}{1 + x_1 y_1 + x_1^2 y_1^2} - \frac{3y_1^2 + 2x_1 y_1 + x_1^2}{y_1^2 + x_1 y_1 + x_1^2} \right) + \\
&\quad + \frac{1}{x_1^2 y_1^2} \left(\ln \left(\frac{1 - x_1 y_1^3}{1 - x_1^3 y_1} \right) - \frac{1}{2} \ln \left(\frac{1 + x_1 y_1^3 + y_1^6 x_1^2}{1 + x_1^3 y_1 + x_1^6 y_1^2} \right) \right) + \\
&\quad + \frac{1}{x_1^2} \left(\frac{1}{2} \ln \left(\frac{1 + x_1 y_1 + x_1^2 y_1^2}{y_1^2 + x_1^3 y_1 + x_1^6} \right) - \ln \left(\frac{1 - x_1 y_1}{y_1 - x_1^3} \right) \right) + \\
&\quad + \frac{\sqrt{3}}{x_1^2 y_1^2} \left(\arctan \left(\frac{x_1^3 y_1 \sqrt{3}}{2 + x_1^3 y_1} \right) - \arctan \left(\frac{y_1^3 x_1 \sqrt{3}}{2 + y_1^3 x_1} \right) \right) + \\
&\quad + \frac{\sqrt{3}}{x_1^2} \left(\arctan \left(\frac{x_1 y_1 \sqrt{3}}{2 + x_1 y_1} \right) - \arctan \left(\frac{x_1^3 \sqrt{3}}{2y_1 + x_1^3} \right) \right). \tag{36}
\end{aligned}$$

• For $r \in (y_1, 1)$ the integrand turns out to be

$$\begin{aligned}
&I_3(x_1, y_1; r) = \#(23) + \#(25) + \#(26) + \#(28) + \#(29) + \#(31) + \#(32) + \#(34) = \\
&= \frac{1}{x_1 y_1} \left(\frac{1}{1 - x_1 y_1} - \frac{1}{1 - \frac{x_1}{y_1}} \right) + \frac{1}{x_1^2 y_1^2} \frac{1}{r^2 - \frac{1}{x_1 y_1}} - \frac{1}{x_1^2} \frac{1}{r^2 - \frac{y_1}{x_1}} + \\
&- \frac{1}{x_1 y_1} \left(\frac{1}{1 - \frac{x_1}{y_1}} - \frac{1}{1 - x_1 y_1} \right) - \frac{1}{r^2 - x_1 y_1} + \frac{1}{y_1^2} \frac{1}{r^2 - \frac{x_1}{y_1}} + \\
&+ \frac{e^{\frac{2}{3}\pi i}}{x_1 y_1} \left(\frac{1}{e^{\frac{2}{3}\pi i} - x_1 y_1} - \frac{1}{e^{\frac{2}{3}\pi i} - \frac{x_1}{y_1}} \right) + \frac{e^{\frac{2}{3}\pi i}}{x_1^2 y_1^2} \frac{1}{r^2 - \frac{1}{x_1 y_1} e^{\frac{2}{3}\pi i}} - \frac{e^{\frac{2}{3}\pi i}}{x_1^2} \frac{1}{r^2 - \frac{y_1}{x_1} e^{\frac{2}{3}\pi i}} + \\
&+ \frac{1}{x_1 y_1} \left(\frac{1}{1 - x_1 y_1 e^{-\frac{2}{3}\pi i}} - \frac{1}{1 - \frac{x_1}{y_1} e^{-\frac{2}{3}\pi i}} \right) - \frac{e^{-\frac{2}{3}\pi i}}{r^2 - x_1 y_1 e^{-\frac{2}{3}\pi i}} + \frac{e^{-\frac{2}{3}\pi i}}{y_1^2} \frac{1}{r^2 - \frac{x_1}{y_1} e^{-\frac{2}{3}\pi i}} + \\
&+ \frac{e^{-\frac{2}{3}\pi i}}{x_1 y_1} \left(\frac{1}{e^{-\frac{2}{3}\pi i} - x_1 y_1} - \frac{1}{e^{-\frac{2}{3}\pi i} - \frac{x_1}{y_1}} \right) + \frac{e^{-\frac{2}{3}\pi i}}{x_1^2 y_1^2} \frac{1}{r^2 - \frac{1}{x_1 y_1} e^{-\frac{2}{3}\pi i}} - \frac{e^{-\frac{2}{3}\pi i}}{x_1^2} \frac{1}{r^2 - \frac{y_1}{x_1} e^{-\frac{2}{3}\pi i}} + \\
&+ \frac{1}{x_1 y_1} \left(\frac{1}{\frac{x_1}{y_1} e^{\frac{2}{3}\pi i} - 1} - \frac{1}{x_1 y_1 e^{\frac{2}{3}\pi i} - 1} \right) - e^{\frac{2}{3}\pi i} \frac{1}{r^2 - x_1 y_1 e^{\frac{2}{3}\pi i}} + \frac{e^{\frac{2}{3}\pi i}}{y_1^2} \frac{1}{r^2 - \frac{x_1}{y_1} e^{\frac{2}{3}\pi i}} + \\
&+ \frac{1}{y_1^2} \left(\frac{1}{1 - \frac{x_1}{y_1}} + \frac{e^{-\frac{2}{3}\pi i}}{1 - \frac{x_1}{y_1} e^{-\frac{2}{3}\pi i}} + \frac{e^{\frac{2}{3}\pi i}}{1 - \frac{x_1}{y_1} e^{\frac{2}{3}\pi i}} \right) + \frac{1}{x_1 y_1} \left(\frac{1}{1 - x_1 y_1} + \frac{e^{-\frac{2}{3}\pi i}}{e^{-\frac{2}{3}\pi i} - x_1 y_1} + \frac{e^{\frac{2}{3}\pi i}}{e^{\frac{2}{3}\pi i} - x_1 y_1} \right) + \\
&+ \frac{1}{x_1^2 y_1^2} \left(\frac{1}{r^2 - \frac{1}{x_1 y_1}} + \frac{e^{-\frac{2}{3}\pi i}}{r^2 - \frac{1}{x_1 y_1} e^{-\frac{2}{3}\pi i}} + \frac{e^{\frac{2}{3}\pi i}}{r^2 - \frac{1}{x_1 y_1} e^{\frac{2}{3}\pi i}} \right) - \frac{1}{y_1^2} \left(\frac{1}{r^2 - \frac{x_1}{y_1}} + \frac{e^{-\frac{2}{3}\pi i}}{r^2 - \frac{x_1}{y_1} e^{-\frac{2}{3}\pi i}} + \frac{e^{\frac{2}{3}\pi i}}{r^2 - \frac{x_1}{y_1} e^{\frac{2}{3}\pi i}} \right) + \\
&+ \frac{1}{x_1 y_1} \left(\frac{1}{1 - x_1 y_1} + \frac{1}{1 - x_1 y_1 e^{-\frac{2}{3}\pi i}} + \frac{1}{1 - x_1 y_1 e^{\frac{2}{3}\pi i}} \right) + \frac{1}{y_1^2} \left(\frac{1}{1 - \frac{x_1}{y_1}} + \frac{e^{-\frac{2}{3}\pi i}}{1 - \frac{x_1}{y_1} e^{-\frac{2}{3}\pi i}} + \frac{e^{\frac{2}{3}\pi i}}{1 - \frac{x_1}{y_1} e^{\frac{2}{3}\pi i}} \right) + \\
&- \frac{1}{r^2 - x_1 y_1} - \frac{e^{-\frac{2}{3}\pi i}}{r^2 - x_1 y_1 e^{-\frac{2}{3}\pi i}} - \frac{e^{\frac{2}{3}\pi i}}{r^2 - x_1 y_1 e^{\frac{2}{3}\pi i}} + \frac{1}{x_1^2} \left(\frac{1}{r^2 - \frac{y_1}{x_1}} + \frac{e^{-\frac{2}{3}\pi i}}{r^2 - \frac{y_1}{x_1} e^{-\frac{2}{3}\pi i}} + \frac{e^{\frac{2}{3}\pi i}}{r^2 - \frac{y_1}{x_1} e^{\frac{2}{3}\pi i}} \right)
\end{aligned}$$

$$\begin{aligned}
&= \sum_{k=0}^2 \left(\frac{2}{x_1 y_1} \left(\frac{2e^{\frac{2}{3}k\pi i}}{e^{\frac{2}{3}k\pi i} - x_1 y_1} - \frac{e^{\frac{2}{3}k\pi i}}{e^{\frac{2}{3}k\pi i} - \frac{x_1}{y_1}} \right) + \frac{2}{y_1^2} \left(\frac{e^{\frac{2}{3}k\pi i}}{1 - \frac{x_1}{y_1} e^{\frac{2}{3}k\pi i}} \right) \right) + \\
&\quad + \sum_{k=0}^2 \left(\frac{2}{x_1^2 y_1^2} \left(\frac{e^{\frac{2}{3}k\pi i}}{r^2 - \frac{1}{x_1 y_1} e^{\frac{2}{3}k\pi i}} \right) - \frac{2e^{\frac{2}{3}k\pi i}}{r^2 - x_1 y_1 e^{\frac{2}{3}k\pi i}} \right),
\end{aligned}$$

while

$$\begin{aligned}
&\int_{r=y_1}^1 I_3(x_1, y_1; r) r dr = \\
&= \left[\left(\frac{2}{x_1 y_1} \left(\frac{1}{1 - x_1 y_1} - \frac{1}{1 - \frac{x_1}{y_1}} \right) + \frac{2}{x_1 y_1} \frac{1}{1 - x_1 y_1} + \frac{2}{y_1^2} \frac{1}{1 - \frac{x_1}{y_1}} \right) \frac{1}{2} r^2 \right]_{r=y_1}^1 + \\
&\quad + \left[\frac{1}{x_1^2 y_1^2} \ln \left(\frac{1}{x_1 y_1} - r^2 \right) - \ln \left(r^2 - x_1 y_1 \right) \right]_{r=y_1}^1 + \\
&\quad + \left[\frac{4}{x_1 y_1} \left(\frac{e^{\frac{2}{3}\pi i}}{e^{\frac{2}{3}\pi i} - x_1 y_1} + \frac{e^{-\frac{2}{3}\pi i}}{e^{-\frac{2}{3}\pi i} - x_1 y_1} \right) \frac{1}{2} r^2 \right]_{r=y_1}^1 + \\
&\quad + \left[\left(-\frac{2}{x_1 y_1} \left(\frac{e^{-\frac{2}{3}\pi i}}{e^{-\frac{2}{3}\pi i} - \frac{x_1}{y_1}} + \frac{e^{\frac{2}{3}\pi i}}{e^{\frac{2}{3}\pi i} - \frac{x_1}{y_1}} \right) + \frac{2}{y_1^2} \left(\frac{e^{-\frac{2}{3}\pi i}}{1 - \frac{x_1}{y_1} e^{-\frac{2}{3}\pi i}} + \frac{e^{\frac{2}{3}\pi i}}{1 - \frac{x_1}{y_1} e^{\frac{2}{3}\pi i}} \right) \right) \frac{1}{2} r^2 \right]_{r=y_1}^1 + \\
&\quad + \left[\frac{1}{x_1^2 y_1^2} \left(e^{-\frac{2}{3}\pi i} \text{Ln} \left(\frac{1}{x_1 y_1} e^{-\frac{2}{3}\pi i} - r^2 \right) + e^{\frac{2}{3}\pi i} \text{Ln} \left(\frac{1}{x_1 y_1} e^{\frac{2}{3}\pi i} - r^2 \right) \right) \right]_{r=y_1}^1 + \\
&\quad + \left[-e^{-\frac{2}{3}\pi i} \text{Ln} \left(r^2 - x_1 y_1 e^{-\frac{2}{3}\pi i} \right) - e^{\frac{2}{3}\pi i} \text{Ln} \left(r^2 - x_1 y_1 e^{\frac{2}{3}\pi i} \right) \right]_{r=y_1}^1 \\
&= \left(\frac{2}{x_1 y_1} \left(\frac{1}{1 - x_1 y_1} - \frac{1}{1 - \frac{x_1}{y_1}} \right) + \frac{2}{x_1 y_1} \frac{1}{1 - x_1 y_1} + \frac{2}{y_1^2} \frac{1}{1 - \frac{x_1}{y_1}} \right) \frac{1}{2} (1 - y_1^2) + \\
&\quad + \frac{1}{x_1^2 y_1^2} \ln \left(\frac{\frac{1}{x_1 y_1} - 1}{\frac{1}{x_1 y_1} - y_1^2} \right) - \ln \left(\frac{x_1 y_1 - 1}{x_1 y_1 - y_1^2} \right) + \\
&\quad + \frac{4}{x_1 y_1} \left(\frac{e^{\frac{2}{3}\pi i}}{e^{\frac{2}{3}\pi i} - x_1 y_1} + \frac{e^{-\frac{2}{3}\pi i}}{e^{-\frac{2}{3}\pi i} - x_1 y_1} \right) \frac{1}{2} (1 - y_1^2) + \\
&\quad + \left(-\frac{2}{x_1 y_1} \left(\frac{e^{-\frac{2}{3}\pi i}}{e^{-\frac{2}{3}\pi i} - \frac{x_1}{y_1}} + \frac{e^{\frac{2}{3}\pi i}}{e^{\frac{2}{3}\pi i} - \frac{x_1}{y_1}} \right) + \frac{2}{y_1^2} \left(\frac{e^{-\frac{2}{3}\pi i}}{1 - \frac{x_1}{y_1} e^{-\frac{2}{3}\pi i}} + \frac{e^{\frac{2}{3}\pi i}}{1 - \frac{x_1}{y_1} e^{\frac{2}{3}\pi i}} \right) \right) \frac{1}{2} (1 - y_1^2) + \\
&\quad + \frac{1}{x_1^2 y_1^2} \left(e^{-\frac{2}{3}\pi i} \text{Ln} \left(\frac{\frac{1}{x_1 y_1} e^{-\frac{2}{3}\pi i} - 1}{\frac{1}{x_1 y_1} e^{-\frac{2}{3}\pi i} - y_1^2} \right) + e^{\frac{2}{3}\pi i} \text{Ln} \left(\frac{\frac{1}{x_1 y_1} e^{\frac{2}{3}\pi i} - 1}{\frac{1}{x_1 y_1} e^{\frac{2}{3}\pi i} - y_1^2} \right) \right) + \\
&\quad - e^{-\frac{2}{3}\pi i} \text{Ln} \left(\frac{1 - x_1 y_1 e^{-\frac{2}{3}\pi i}}{y_1^2 - x_1 y_1 e^{-\frac{2}{3}\pi i}} \right) - e^{\frac{2}{3}\pi i} \text{Ln} \left(\frac{1 - x_1 y_1 e^{\frac{2}{3}\pi i}}{y_1^2 - x_1 y_1 e^{\frac{2}{3}\pi i}} \right)
\end{aligned}$$

$$\begin{aligned}
&= \frac{1}{x_1 y_1} \frac{x_1 y_1 + 1}{(1 - x_1 y_1)} (1 - y_1^2) + \\
&\quad + \frac{1}{x_1^2 y_1^2} \ln \left(\frac{1 - x_1 y_1}{1 - x_1 y_1^3} \right) - \ln \left(\frac{x_1 y_1 - 1}{x_1 y_1 - y_1^2} \right) + \\
&\quad + \left(\frac{2}{x_1 y_1} \frac{2 + x_1 y_1}{1 + x_1 y_1 + x_1^2 y_1^2} - \frac{1}{x_1} \frac{2 y_1 + x_1}{y_1^2 + x_1 y_1 + x_1^2} - \frac{1}{y_1} \frac{y_1 + 2 x_1}{y_1^2 + x_1 y_1 + x_1^2} \right) (1 - y_1^2) + \\
&\quad + \frac{e^{-\frac{2}{3}\pi i}}{x_1^2 y_1^2} \operatorname{Ln} \left(\frac{1 + i\sqrt{3} + 2x_1 y_1}{1 + i\sqrt{3} + 2y_1^3 x_1} \right) + \frac{e^{\frac{2}{3}\pi i}}{x_1^2 y_1^2} \operatorname{Ln} \left(\frac{1 - i\sqrt{3} + 2x_1 y_1}{1 - i\sqrt{3} + 2y_1^3 x_1} \right) + \\
&\quad - e^{-\frac{2}{3}\pi i} \operatorname{Ln} \left(\frac{2 + x_1 y_1 + i x_1 y_1 \sqrt{3}}{y_1 (2y_1 + x_1 + i x_1 \sqrt{3})} \right) - e^{\frac{2}{3}\pi i} \operatorname{Ln} \left(\frac{-2 - x_1 y_1 + i x_1 y_1 \sqrt{3}}{y_1 (-2y_1 - x_1 + i x_1 \sqrt{3})} \right) \\
\\
&= \frac{1}{x_1 y_1} \frac{x_1 y_1 + 1}{(1 - x_1 y_1)} (1 - y_1^2) + \\
&\quad + \frac{1}{x_1^2 y_1^2} \ln \left(\frac{1 - x_1 y_1}{1 - x_1 y_1^3} \right) - \ln \left(\frac{x_1 y_1 - 1}{x_1 y_1 - y_1^2} \right) + \\
&\quad + \left(\frac{2}{x_1 y_1} \frac{2 + x_1 y_1}{1 + x_1 y_1 + x_1^2 y_1^2} - \frac{1}{x_1} \frac{2 y_1 + x_1}{y_1^2 + x_1 y_1 + x_1^2} - \frac{1}{y_1} \frac{y_1 + 2 x_1}{y_1^2 + x_1 y_1 + x_1^2} \right) (1 - y_1^2) + \\
&\quad + \frac{e^{-\frac{2}{3}\pi i}}{x_1^2 y_1^2} \operatorname{Ln} (1 + i\sqrt{3} + 2x_1 y_1) + \frac{e^{\frac{2}{3}\pi i}}{x_1^2 y_1^2} \operatorname{Ln} (1 - i\sqrt{3} + 2x_1 y_1) + \\
&\quad - \frac{e^{-\frac{2}{3}\pi i}}{x_1^2 y_1^2} \operatorname{Ln} (1 + i\sqrt{3} + 2y_1^3 x_1) - \frac{e^{\frac{2}{3}\pi i}}{x_1^2 y_1^2} \operatorname{Ln} (1 - i\sqrt{3} + 2y_1^3 x_1) + \\
&\quad + e^{-\frac{2}{3}\pi i} \operatorname{Ln} (y_1 (2y_1 + x_1 + i x_1 \sqrt{3})) + e^{\frac{2}{3}\pi i} \operatorname{Ln} (y_1 (2y_1 + x_1 - i x_1 \sqrt{3})) \\
&\quad - e^{-\frac{2}{3}\pi i} \operatorname{Ln} (2 + x_1 y_1 + i x_1 y_1 \sqrt{3}) - e^{\frac{2}{3}\pi i} \operatorname{Ln} (2 + x_1 y_1 - i x_1 y_1 \sqrt{3}) \\
\\
&= \frac{1}{x_1 y_1} \frac{x_1 y_1 + 1}{(1 - x_1 y_1)} (1 - y_1^2) + \\
&\quad + \frac{1}{x_1^2 y_1^2} \ln \left(\frac{1 - x_1 y_1}{1 - x_1 y_1^3} \right) - \ln \left(\frac{x_1 y_1 - 1}{x_1 y_1 - y_1^2} \right) + \\
&\quad + \left(\frac{2}{x_1 y_1} \frac{2 + x_1 y_1}{1 + x_1 y_1 + x_1^2 y_1^2} - \frac{1}{x_1} \frac{2 y_1 + x_1}{y_1^2 + x_1 y_1 + x_1^2} - \frac{1}{y_1} \frac{y_1 + 2 x_1}{y_1^2 + x_1 y_1 + x_1^2} \right) (1 - y_1^2) + \\
&\quad + \frac{e^{-\frac{2}{3}\pi i} + e^{\frac{2}{3}\pi i}}{x_1^2 y_1^2} \ln \sqrt{(1 + 2x_1 y_1)^2 + 3} + i \frac{e^{-\frac{2}{3}\pi i} - e^{\frac{2}{3}\pi i}}{x_1^2 y_1^2} \arctan \left(\frac{\sqrt{3}}{1 + 2x_1 y_1} \right) + \\
&\quad - \frac{e^{-\frac{2}{3}\pi i} + e^{\frac{2}{3}\pi i}}{x_1^2 y_1^2} \ln \sqrt{(1 + 2y_1^3 x_1)^2 + 3} - i \frac{e^{-\frac{2}{3}\pi i} - e^{\frac{2}{3}\pi i}}{x_1^2 y_1^2} \arctan \left(\frac{\sqrt{3}}{1 + 2y_1^3 x_1} \right) + \\
&\quad + \left(e^{-\frac{2}{3}\pi i} + e^{\frac{2}{3}\pi i} \right) \ln \sqrt{y_1^2 (2y_1 + x_1)^2 + 3x_1^2 y_1^2} + i \left(e^{-\frac{2}{3}\pi i} - e^{\frac{2}{3}\pi i} \right) \arctan \left(\frac{\sqrt{3} x_1}{2y_1 + x_1} \right) \\
&\quad - \left(e^{-\frac{2}{3}\pi i} + e^{\frac{2}{3}\pi i} \right) \ln \sqrt{(2 + x_1 y_1)^2 + 3x_1^2 y_1^2} - i \left(e^{-\frac{2}{3}\pi i} - e^{\frac{2}{3}\pi i} \right) \arctan \left(\frac{x_1 y_1 \sqrt{3}}{2 + x_1 y_1} \right)
\end{aligned}$$

$$\begin{aligned}
&= \frac{1}{x_1 y_1} \frac{x_1 y_1 + 1}{(1 - x_1 y_1)} (1 - y_1^2) + \\
&+ \frac{1}{x_1^2 y_1^2} \ln \left(\frac{1 - x_1 y_1}{1 - x_1 y_1^3} \right) - \ln \left(\frac{x_1 y_1 - 1}{x_1 y_1 - y_1^2} \right) + \\
&+ \left(\frac{2}{x_1 y_1} \frac{2 + x_1 y_1}{1 + x_1 y_1 + x_1^2 y_1^2} - \frac{1}{x_1} \frac{2 y_1 + x_1}{y_1^2 + x_1 y_1 + x_1^2} - \frac{1}{y_1} \frac{y_1 + 2 x_1}{y_1^2 + x_1 y_1 + x_1^2} \right) (1 - y_1^2) + \\
&- \frac{1}{x_1^2 y_1^2} \ln \sqrt{(1 + 2 x_1 y_1)^2 + 3} + \frac{\sqrt{3}}{x_1^2 y_1^2} \arctan \left(\frac{\sqrt{3}}{1 + 2 x_1 y_1} \right) + \\
&+ \frac{1}{x_1^2 y_1^2} \ln \sqrt{(1 + 2 y_1^3 x_1)^2 + 3} - \frac{\sqrt{3}}{x_1^2 y_1^2} \arctan \left(\frac{\sqrt{3}}{1 + 2 y_1^3 x_1} \right) + \\
&- \ln \sqrt{y_1^2 (2 y_1 + x_1)^2 + 3 x_1^2 y_1^2} + \sqrt{3} \arctan \left(\frac{\sqrt{3} x_1}{2 y_1 + x_1} \right) \\
&+ \ln \sqrt{(2 + x_1 y_1)^2 + 3 x_1^2 y_1^2} - \sqrt{3} \arctan \left(\frac{x_1 y_1 \sqrt{3}}{2 + x_1 y_1} \right) \\
&= \frac{1}{x_1 y_1} \frac{x_1 y_1 + 1}{(1 - x_1 y_1)} (1 - y_1^2) + \\
&+ \frac{1}{x_1^2 y_1^2} \ln \left(\frac{1 - x_1 y_1}{1 - x_1 y_1^3} \right) - \ln \left(\frac{x_1 y_1 - 1}{x_1 y_1 - y_1^2} \right) + \\
&+ \left(\frac{2}{x_1 y_1} \frac{2 + x_1 y_1}{1 + x_1 y_1 + x_1^2 y_1^2} - \frac{1}{x_1} \frac{2 y_1 + x_1}{y_1^2 + x_1 y_1 + x_1^2} - \frac{1}{y_1} \frac{y_1 + 2 x_1}{y_1^2 + x_1 y_1 + x_1^2} \right) (1 - y_1^2) + \\
&- \frac{1}{x_1^2 y_1^2} \ln \left(2 \sqrt{(1 + x_1 y_1 + x_1^2 y_1^2)} \right) + \frac{\sqrt{3}}{x_1^2 y_1^2} \arctan \left(\frac{\sqrt{3}}{1 + 2 x_1 y_1} \right) + \\
&+ \frac{1}{x_1^2 y_1^2} \ln \left(2 \sqrt{(1 + y_1^3 x_1 + y_1^6 x_1^2)} \right) - \frac{\sqrt{3}}{x_1^2 y_1^2} \arctan \left(\frac{\sqrt{3}}{1 + 2 y_1^3 x_1} \right) + \\
&- \ln \left(2 y_1 \sqrt{(y_1^2 + x_1 y_1 + x_1^2)} \right) + \sqrt{3} \arctan \left(\frac{\sqrt{3} x_1}{2 y_1 + x_1} \right) \\
&+ \ln \left(2 \sqrt{(1 + x_1 y_1 + x_1^2 y_1^2)} \right) - \sqrt{3} \arctan \left(\frac{x_1 y_1 \sqrt{3}}{2 + x_1 y_1} \right)
\end{aligned}$$

$$\begin{aligned}
&= \frac{1}{x_1 y_1} \frac{1+x_1 y_1}{1-x_1 y_1} (1-y_1^2) + \\
&+ \frac{1}{x_1^2 y_1^2} \ln \left(\frac{1-x_1 y_1}{1-x_1 y_1^3} \right) - \ln \left(\frac{x_1 y_1 - 1}{x_1 y_1 - y_1^2} \right) + \\
&+ \frac{2}{x_1 y_1} \frac{1-x_1^2 y_1^2}{1+x_1 y_1+x_1^2 y_1^2} (1-y_1^2) + \\
&- \frac{1}{x_1^2 y_1^2} \ln \left(\sqrt{(1+x_1 y_1+x_1^2 y_1^2)} \right) + \frac{\sqrt{3}}{x_1^2 y_1^2} \arctan \left(\frac{\sqrt{3}}{1+2x_1 y_1} \right) + \\
&+ \frac{1}{x_1^2 y_1^2} \ln \left(\sqrt{(1+y_1^3 x_1+y_1^6 x_1^2)} \right) - \frac{\sqrt{3}}{x_1^2 y_1^2} \arctan \left(\frac{\sqrt{3}}{1+2y_1^3 x_1} \right) + \\
&- \ln \left(y_1 \sqrt{(y_1^2+x_1 y_1+x_1^2)} \right) + \sqrt{3} \arctan \left(\frac{\sqrt{3} x_1}{2y_1+x_1} \right) \\
&+ \ln \left(\sqrt{(1+x_1 y_1+x_1^2 y_1^2)} \right) - \sqrt{3} \arctan \left(\frac{x_1 y_1 \sqrt{3}}{2+x_1 y_1} \right) \\
&= \frac{1-y_1^2}{x_1 y_1} \left(\frac{1+x_1 y_1}{1-x_1 y_1} + 2 \frac{1-x_1^2 y_1^2}{1+x_1 y_1+x_1^2 y_1^2} \right) + \\
&- \ln \left(\frac{1-x_1 y_1}{y_1-x_1} \right) + \frac{1}{2} \ln \left(\frac{1+x_1 y_1+x_1^2 y_1^2}{x_1^2+x_1 y_1+y_1^2} \right) + \\
&+ \frac{1}{x_1^2 y_1^2} \left(\ln \left(\frac{1-x_1 y_1}{1-x_1 y_1^3} \right) - \frac{1}{2} \ln \left(\frac{1+x_1 y_1+x_1^2 y_1^2}{1+x_1 y_1^3+x_1^2 y_1^6} \right) \right) + \\
&+ \frac{\sqrt{3}}{x_1^2 y_1^2} \left(\arctan \left(\frac{\sqrt{3}}{1+2x_1 y_1} \right) - \arctan \left(\frac{\sqrt{3}}{1+2y_1^3 x_1} \right) \right) + \\
&+ \sqrt{3} \left(\arctan \left(\frac{\sqrt{3} x_1}{2y_1+x_1} \right) - \arctan \left(\frac{x_1 y_1 \sqrt{3}}{2+x_1 y_1} \right) \right). \tag{37}
\end{aligned}$$

4.5 Conclusion of case 1)

Recall that we want to evaluate (8). And when adding the three integrals we find, separated according to a main expression:

- algebraic terms

$$\begin{aligned}
&- \frac{(x_1+y_1)(1-y_1^2)}{y_1(1-x_1 y_1)} + \frac{y_1^2-x_1^2}{x_1 y_1} \frac{3+2x_1 y_1+x_1^2 y_1^2}{1+x_1 y_1+x_1^2 y_1^2} - \frac{y_1^2-x_1^2}{x_1 y_1} \frac{3y_1^2+2x_1 y_1+x_1^2}{y_1^2+x_1 y_1+x_1^2} + \\
&+ \frac{1}{x_1 y_1} \frac{1+x_1 y_1}{1-x_1 y_1} (1-y_1^2) + \frac{2}{x_1 y_1} \frac{1-x_1^2 y_1^2}{1+x_1 y_1+x_1^2 y_1^2} (1-y_1^2) \\
&= \frac{3(1-x_1^2)(1-y_1^2)}{x_1 y_1(1+x_1 y_1+x_1^2 y_1^2)} \left(\frac{1}{1-x_1 y_1} + \frac{x_1^2 y_1^2}{y_1^2+x_1 y_1+x_1^2} \right)
\end{aligned}$$

- pure logarithmic terms

$$\frac{1}{2} \ln \left(\frac{x_1^2-2x_1 y_1+y_1^2}{x_1^2+x_1 y_1+y_1^2} \right) + \left(-\ln \left(\frac{1-x_1 y_1}{y_1-x_1} \right) + \frac{1}{2} \ln \left(\frac{1+x_1 y_1+x_1^2 y_1^2}{x_1^2+x_1 y_1+y_1^2} \right) \right)$$

$$\begin{aligned}
&= \frac{1}{2} \ln \left(\frac{x_1^2 - 2x_1y_1 + y_1^2}{x_1^2 + x_1y_1 + y_1^2} \left(\frac{y_1 - x_1}{1 - x_1y_1} \right)^2 \frac{1 + x_1y_1 + x_1^2y_1^2}{x_1^2 + x_1y_1 + y_1^2} \right) \\
&= \frac{1}{2} \ln \left(\frac{(1 - x_1y_1)^2}{1 + x_1y_1 + x_1^2y_1^2} \right) - 2 \ln \left(\frac{1 - x_1y_1}{y_1 - x_1} \right) + \ln \left(\frac{1 + x_1y_1 + x_1^2y_1^2}{x_1^2 + x_1y_1 + y_1^2} \right)
\end{aligned}$$

- logarithmic terms with x_1^{-2}

$$\begin{aligned}
&- \frac{1}{2x_1^2} \ln \left(\frac{x_1^6 - 2x_1^3y_1 + y_1^2}{x_1^6 + x_1^3y_1 + y_1^2} \right) + \frac{1}{2x_1^2} \ln \left(\frac{1 + x_1y_1 + x_1^2y_1^2}{y_1^2 + x_1^3y_1 + x_1^6} \right) - \frac{1}{x_1^2} \ln \left(\frac{1 - x_1y_1}{y_1 - x_1^3} \right) \\
&= \frac{1}{2x_1^2} \ln \left(\frac{1 + x_1y_1 + x_1^2y_1^2}{y_1^2 + x_1^3y_1 + x_1^6} \frac{x_1^6 + x_1^3y_1 + y_1^2}{x_1^6 - 2x_1^3y_1 + y_1^2} \left(\frac{y_1 - x_1^3}{1 - x_1y_1} \right)^2 \right) \\
&= - \frac{1}{2x_1^2} \ln \left(\frac{(1 - x_1y_1)^2}{1 + x_1y_1 + x_1^2y_1^2} \right)
\end{aligned}$$

- logarithmic terms with y_1^{-2}

$$- \frac{1}{2y_1^2} \ln \left(\frac{(1 - x_1y_1)^2}{1 + x_1y_1 + x_1^2y_1^2} \right)$$

- logarithmic terms with $x_1^{-2}y_1^{-2}$

$$\begin{aligned}
&\frac{1}{2x_1^2y_1^2} \ln \left(\frac{1 - 2x_1^3y_1 + y_1^2x_1^6}{1 + x_1^3y_1 + x_1^6y_1^2} \right) + \frac{1}{x_1^2y_1^2} \ln \left(\frac{1 - x_1y_1^3}{1 - x_1^3y_1} \right) - \frac{1}{2x_1^2y_1^2} \ln \left(\frac{1 + x_1y_1^3 + y_1^6x_1^2}{1 + x_1^3y_1 + x_1^6y_1^2} \right) + \\
&+ \frac{1}{x_1^2y_1^2} \ln \left(\frac{1 - x_1y_1}{1 - x_1y_1^3} \right) - \frac{1}{2x_1^2y_1^2} \ln \left(\frac{1 + x_1y_1 + x_1^2y_1^2}{1 + x_1y_1^3 + x_1^2y_1^6} \right) \\
&= \frac{1}{2x_1^2y_1^2} \ln \left(\frac{1 - 2x_1^3y_1 + y_1^2x_1^6}{1 + x_1^3y_1 + x_1^6y_1^2} \left(\frac{1 - x_1y_1^3}{1 - x_1^3y_1} \right)^2 \frac{1 + x_1^3y_1 + x_1^6y_1^2}{1 + x_1y_1^3 + y_1^6x_1^2} \left(\frac{1 - x_1y_1}{1 - x_1y_1^3} \right)^2 \frac{1 + x_1y_1^3 + x_1^2y_1^6}{1 + x_1y_1 + x_1^2y_1^2} \right) \\
&= \frac{1}{2x_1^2y_1^2} \ln \left(\frac{(1 - x_1y_1)^2}{1 + x_1y_1 + x_1^2y_1^2} \right)
\end{aligned}$$

- terms containing arctan:

$$\begin{aligned}
&\sqrt{3} \arctan \left(\frac{\sqrt{3}x_1}{2y_1 + x_1} \right) - \sqrt{3} \arctan \left(\frac{x_1\sqrt{3}}{2y_1 + x_1} \right) + \\
&+ \frac{\sqrt{3}}{x_1^2} \arctan \left(\frac{x_1^3\sqrt{3}}{2y_1 + x_1^3} \right) - \frac{\sqrt{3}}{x_1^2} \arctan \left(\frac{x_1^3\sqrt{3}}{2y_1 + x_1^3} \right) + \\
&+ \frac{\sqrt{3}}{x_1^2} \arctan \left(\frac{x_1y_1\sqrt{3}}{2 + x_1y_1} \right) + \frac{\sqrt{3}}{y_1^2} \arctan \left(\frac{x_1y_1\sqrt{3}}{2 + x_1y_1} \right) + \\
&+ \frac{\sqrt{3}}{x_1^2y_1^2} \arctan \left(\frac{x_1^3y_1\sqrt{3}}{2 + x_1^3y_1} \right) - \frac{\sqrt{3}}{x_1^2y_1^2} \arctan \left(\frac{x_1^3y_1\sqrt{3}}{2 + y_1x_1^3} \right) + \\
&- \frac{\sqrt{3}}{x_1^2y_1^2} \arctan \left(\frac{y_1^3x_1\sqrt{3}}{2 + y_1^3x_1} \right) - \sqrt{3} \arctan \left(\frac{x_1y_1\sqrt{3}}{2 + x_1y_1} \right) + \\
&+ \frac{\sqrt{3}}{x_1^2y_1^2} \arctan \left(\frac{\sqrt{3}}{1 + 2x_1y_1} \right) - \frac{\sqrt{3}}{x_1^2y_1^2} \arctan \left(\frac{\sqrt{3}}{1 + 2y_1^3x_1} \right)
\end{aligned}$$

$$\begin{aligned}
&= \sqrt{3} \left(\frac{1}{x_1^2} + \frac{1}{y_1^2} - 1 \right) \arctan \left(\frac{x_1 y_1 \sqrt{3}}{2 + x_1 y_1} \right) + \\
&\quad + \frac{\sqrt{3}}{x_1^2 y_1^2} \left(\arctan \left(\frac{\sqrt{3}}{1 + 2x_1 y_1} \right) - \arctan \left(\frac{\sqrt{3}}{1 + 2y_1^3 x_1} \right) - \arctan \left(\frac{y_1^3 x_1 \sqrt{3}}{2 + y_1^3 x_1} \right) \right) \\
&= -\sqrt{3} \left(1 - \frac{1}{x_1^2} - \frac{1}{y_1^2} + \frac{1}{x_1^2 y_1^2} \right) \arctan \left(\frac{x_1 y_1 \sqrt{3}}{2 + x_1 y_1} \right)
\end{aligned}$$

Combining the expressions in (35), (36) and (37), listed as above, the numerator of (8) becomes

$$\begin{aligned}
&\int_{|z|<1} \lim_{x_2 \downarrow 0} \frac{G(x, z)}{x_2} \lim_{y_2 \downarrow 0} \frac{G(z, y)}{y_2} dz = \\
&= \left(\frac{-\frac{1}{4}\pi}{\pi^2} \right) \left(\frac{3(1-x_1^2)(1-y_1^2)}{x_1 y_1 (1+x_1 y_1 + x_1^2 y_1^2)} \left(\frac{1}{1-x_1 y_1} + \frac{x_1^2 y_1^2}{y_1^2 + x_1 y_1 + x_1^2} \right) + \right. \\
&\quad - 2 \ln \left(\frac{1-x_1 y_1}{y_1 - x_1} \right) + \ln \left(\frac{1+x_1 y_1 + x_1^2 y_1^2}{x_1^2 + x_1 y_1 + y_1^2} \right) + \\
&\quad + \frac{1}{2} \left(1 - \frac{1}{x_1^2} - \frac{1}{y_1^2} + \frac{1}{x_1^2 y_1^2} \right) \ln \left(\frac{(1-x_1 y_1)^2}{1+x_1 y_1 + x_1^2 y_1^2} \right) + \\
&\quad \left. - \sqrt{3} \left(1 - \frac{1}{x_1^2} - \frac{1}{y_1^2} + \frac{1}{x_1^2 y_1^2} \right) \arctan \left(\frac{x_1 y_1 \sqrt{3}}{2 + x_1 y_1} \right) \right), \tag{38}
\end{aligned}$$

while the denominator is given by (12). If we define

$$\begin{aligned}
T_{11}(x_1, y_1) &:= \frac{\int_{|z|<1} \lim_{x_2 \downarrow 0} \frac{G(x, z)}{x_2} \lim_{y_2 \downarrow 0} \frac{G(z, y)}{y_2} dz}{\lim_{x_2 \downarrow 0} \lim_{y_2 \downarrow 0} \frac{G(x, y)}{x_2 y_2}} = \\
&= \left(\frac{-\frac{1}{4}\pi}{\pi^2} \right) \left(\frac{\frac{3(1-x_1^2)(1-y_1^2)}{x_1 y_1 (1+x_1 y_1 + x_1^2 y_1^2)} \left(\frac{1}{1-x_1 y_1} + \frac{x_1^2 y_1^2}{y_1^2 + x_1 y_1 + x_1^2} \right) - 2 \ln \left(\frac{1-x_1 y_1}{y_1 - x_1} \right) + \ln \left(\frac{1+x_1 y_1 + x_1^2 y_1^2}{x_1^2 + x_1 y_1 + y_1^2} \right)}{\frac{1}{\pi} \left(\frac{1}{(x_1 - y_1)^2} - \frac{1}{(1-x_1 y_1)^2} + \frac{1}{1+x_1 y_1 + x_1^2 y_1^2} - \frac{1}{x_1^2 + x_1 y_1 + y_1^2} + \frac{3x_1 y_1}{(1+x_1 y_1 + x_1^2 y_1^2)^2} - \frac{3x_1 y_1}{(x_1^2 + x_1 y_1 + y_1^2)^2} \right)} + \right. \\
&\quad \left. + \frac{\frac{1}{2} \left(1 - \frac{1}{x_1^2} - \frac{1}{y_1^2} + \frac{1}{x_1^2 y_1^2} \right) \ln \left(\frac{(1-x_1 y_1)^2}{1+x_1 y_1 + x_1^2 y_1^2} \right) - \sqrt{3} \left(1 - \frac{1}{x_1^2} - \frac{1}{y_1^2} + \frac{1}{x_1^2 y_1^2} \right) \arctan \left(\frac{x_1 y_1 \sqrt{3}}{2 + x_1 y_1} \right)}{\frac{1}{\pi} \left(\frac{1}{(x_1 - y_1)^2} - \frac{1}{(1-x_1 y_1)^2} + \frac{1}{1+x_1 y_1 + x_1^2 y_1^2} - \frac{1}{x_1^2 + x_1 y_1 + y_1^2} + \frac{3x_1 y_1}{(1+x_1 y_1 + x_1^2 y_1^2)^2} - \frac{3x_1 y_1}{(x_1^2 + x_1 y_1 + y_1^2)^2} \right)} \right)
\end{aligned}$$

then this function represents $E_x^y(\tau_S)$ for $x, y \in \Gamma_1$ and $x_1 < y_1$. By symmetry we obtain this function for $x_1 > y_1$ as $T_{11}(x_1, y_1) = T_{11}(y_1, x_1)$. A close inspection reveals that

$$\sup_{x_1, y_1 \in (0,1)} T_{11}(x_1, y_1) = \lim_{x_1 \downarrow 0, y_1 \uparrow 1} T_{11}(x_1, y_1) = \frac{1}{16}. \tag{39}$$

See Figure 1 on p. 31.

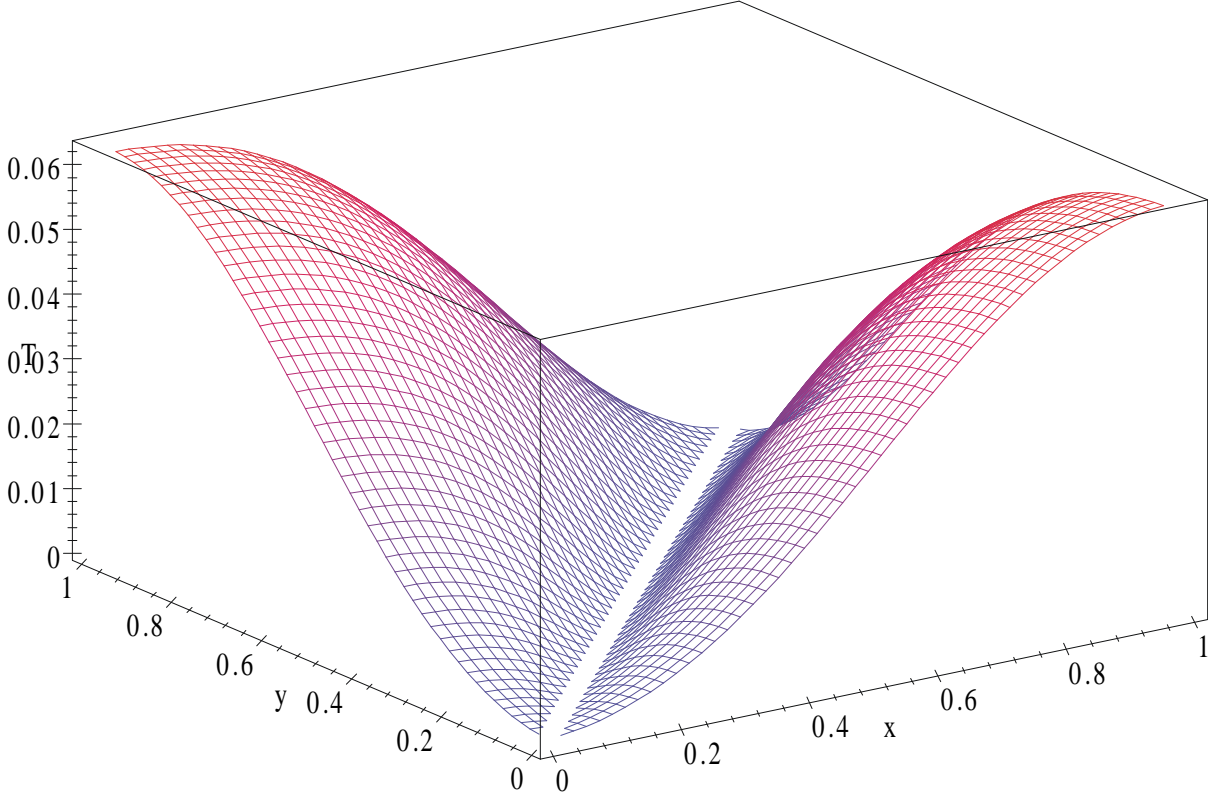


Figure 1: x and y both on Γ_1 : $T_{11}(x_1, y_1)$.

5 The case that $x \in \Gamma_1$, $y \in \Gamma_2$

In this section we let $y \rightarrow (\cos \psi, \sin \psi)$ and $x \rightarrow (x_1, 0)$ and we will consider

$$T_{12}(x_1, \psi) := \frac{\int_{z \in S} \lim_{x_2 \downarrow 0} \frac{G_S(x, z)}{x_2} \lim_{|y| \uparrow 1} \frac{G_S(z, y)}{1 - |y|^2} dz}{\lim_{|y| \uparrow 1 \text{ \& } x_2 \downarrow 0} \frac{G_S(x, y)}{x_2(1 - |y|^2)}}.$$

Again the limits are computed from the expression in (6).

5.1 Limit of the Green function

Next to the expression in (9), with y replaced by z ,

$$\begin{aligned} & \lim_{x_2 \downarrow 0} \frac{G_S(x, z)}{x_2} = \\ &= \frac{1}{\pi} \left(\frac{r \sin \theta}{x_1^2 - 2x_1 r \cos \theta + r^2} - \frac{r \sin \theta}{1 - 2x_1 r \cos \theta + x_1^2 r^2} \right) + \\ &+ \frac{1}{\pi} \left(\frac{r \sin(\theta - \frac{2}{3}\pi)}{x_1^2 - 2x_1 r \cos(\theta - \frac{2}{3}\pi) + r^2} - \frac{r \sin(\theta - \frac{2}{3}\pi)}{1 - 2x_1 r \cos(\theta - \frac{2}{3}\pi) + x_1^2 r^2} \right) + \\ &+ \frac{1}{\pi} \left(\frac{r \sin(\theta + \frac{2}{3}\pi)}{x_1^2 - 2x_1 r \cos(\theta + \frac{2}{3}\pi) + r^2} - \frac{r \sin(\theta + \frac{2}{3}\pi)}{1 - 2x_1 r \cos(\theta + \frac{2}{3}\pi) + x_1^2 r^2} \right), \end{aligned}$$

we need

$$\begin{aligned}
& \lim_{y \rightarrow (\cos \psi, \sin \psi)} \frac{G_S(z, y)}{1 - |y|^2} = \\
&= \frac{1 - |z|^2}{4\pi} \left(-\frac{1}{|z - y|^2} - \frac{1}{|\mathcal{R}z - y|^2} - \frac{1}{|\mathcal{R}^2z - y|^2} + \frac{1}{|\mathcal{S}z - y|^2} + \frac{1}{|\mathcal{R}\mathcal{S}z - y|^2} + \frac{1}{|\mathcal{R}^2\mathcal{S}z - y|^2} \right)_{y=(\cos \psi, \sin \psi)} \\
&= \frac{1 - r^2}{4\pi} \left(\frac{1}{r^2 - 2r \cos(\theta + \psi) + 1} - \frac{1}{r^2 - 2r \cos(\theta - \psi) + 1} \right) \\
&+ \frac{1 - r^2}{4\pi} \left(\frac{1}{r^2 - 2r \cos(\theta + \psi + \frac{2}{3}\pi) + 1} - \frac{1}{r^2 - 2r \cos(\theta - \psi + \frac{2}{3}\pi) + 1} \right) + \\
&+ \frac{1 - r^2}{4\pi} \left(\frac{1}{r^2 - 2r \cos(\theta + \psi - \frac{2}{3}\pi) + 1} - \frac{1}{r^2 - 2r \cos(\theta - \psi - \frac{2}{3}\pi) + 1} \right) = \\
&= \frac{1 - r^2}{4\pi} \sum_{k=0}^2 \sum_{\sigma=\pm 1} \frac{\sigma}{r^2 - 2r \cos(\theta + \sigma\psi + \frac{2}{3}k\pi) + 1}, \tag{40}
\end{aligned}$$

and the double limit of the denominator

$$\begin{aligned}
& \lim_{|y| \uparrow 1} \lim_{x_2 \downarrow 0} \frac{G_S(x, y)}{(1 - |y|^2) x_2} = \\
&= \lim_{|y| \uparrow 1} \left(\frac{1}{\pi(1 - r^2)} \left(\frac{r \sin \psi}{x_1^2 - 2x_1 r \cos \psi + r^2} - \frac{r \sin \psi}{1 - 2x_1 r \cos \psi + x_1^2 r^2} \right) + \right. \\
&+ \frac{1}{\pi} \left(\frac{r \sin(\psi + \frac{2}{3}\pi)}{x_1^2 - 2x_1 r \cos(\psi + \frac{2}{3}\pi) + r^2} - \frac{r \sin(\psi + \frac{2}{3}\pi)}{1 - 2x_1 r \cos(\psi + \frac{2}{3}\pi) + x_1^2 r^2} \right) + \\
&+ \left. \frac{1}{\pi} \left(\frac{r \sin(\psi - \frac{2}{3}\pi)}{x_1^2 - 2x_1 r \cos(\psi - \frac{2}{3}\pi) + r^2} - \frac{r \sin(\psi - \frac{2}{3}\pi)}{1 - 2x_1 r \cos(\psi - \frac{2}{3}\pi) + x_1^2 r^2} \right) \right) \\
&= \frac{(1 - x_1^2) \sin \psi}{\pi(x_1^2 - 2x_1 \cos \psi + 1)^2} + \frac{(1 - x_1^2) \sin(\psi + \frac{2}{3}\pi)}{\pi(x_1^2 - 2x_1 \cos(\psi + \frac{2}{3}\pi) + 1)^2} + \frac{(1 - x_1^2) \sin(\psi - \frac{2}{3}\pi)}{\pi(x_1^2 - 2x_1 \cos(\psi - \frac{2}{3}\pi) + 1)^2}. \tag{41}
\end{aligned}$$

5.2 Derivation of a contour integral

In addition to the h as in (17,18,19) we define

$$f(\theta) = \frac{1 - r^2}{4} \left(\frac{1}{r^2 - 2r \cos(\theta + \psi) + 1} - \frac{1}{r^2 - 2r \cos(\theta - \psi) + 1} \right), \tag{42}$$

$$\begin{aligned}
f_\psi(\theta) &= \frac{1 - r^2}{4} \left(\frac{1}{r^2 - 2r \cos(\theta + \psi) + 1} \right) \\
&= \frac{-e^{-i\psi}}{4} \left(\frac{1}{w - e^{-i\psi}} - \frac{r^2}{w - r^2 e^{-i\psi}} \right). \tag{43}
\end{aligned}$$

Note that $h(-\theta) = -h(\theta)$ and also $f(\theta) = f(-\theta)$. Hence $\theta \mapsto \sum_{k=0}^2 h(\theta + k\frac{2}{3}\pi)$ and also $\theta \mapsto \sum_{k=0}^2 f(\theta + k\frac{2}{3}\pi)$ are odd and also periodic with period $\frac{2}{3}\pi$. Using h and f (respectively f_ψ) we may rewrite the angular integral in

the enumerator of (5) as follows

$$\begin{aligned} & \pi^2 \int_{\theta=0}^{\frac{1}{3}\pi} \lim_{x \rightarrow (x_1, 0)} \frac{G_S(x, z)}{x_2} \lim_{y \rightarrow (\cos \psi, \sin \psi)} \frac{G_S(z, y)}{1 - |y|^2} d\theta = \\ & = \int_{\theta=0}^{\frac{1}{3}\pi} \sum_{k=0}^2 h\left(\theta + k\frac{2}{3}\pi\right) \sum_{m=0}^2 f\left(\theta + m\frac{2}{3}\pi\right) d\theta = \end{aligned} \quad (44)$$

$$\begin{aligned} & = \frac{1}{6} \int_{\theta=0}^{2\pi} \sum_{m=0}^2 h\left(\theta + m\frac{2}{3}\pi\right) \sum_{m=0}^2 f\left(\theta + m\frac{2}{3}\pi\right) d\theta = \\ & = \frac{1}{2} \int_{\theta=0}^{2\pi} \sum_{k=0}^2 h\left(\theta + k\frac{2}{3}\pi\right) f(\theta) d\theta = \\ & = \int_{\theta=0}^{2\pi} \sum_{k=0}^2 h\left(\theta + k\frac{2}{3}\pi\right) f_\psi(\theta) d\theta . \end{aligned} \quad (45)$$

It leaves us to compute:

$$\begin{aligned} & \pi^2 \int_{z \in S} \lim_{x \rightarrow (x_1, 0)} \frac{G_S(x, z)}{x_2} \lim_{y \rightarrow (\cos \psi, \sin \psi)} \frac{G_S(z, y)}{1 - |y|^2} dz = \\ & = \int_{r=0}^1 \int_{\theta=0}^{2\pi} \left(h(\theta) + h\left(\theta + \frac{2}{3}\pi\right) + h\left(\theta + \frac{4}{3}\pi\right) \right) f_\psi(\theta) d\theta r dr \\ & = \int_{r=0}^1 \oint_{|w|=r} \left(\frac{1}{2i} \left(\frac{r^2}{w - x_1 r^2} + \frac{x_1^{-2}}{w - x_1^{-1}} - \frac{x_1^{-2} r^2}{w - x_1^{-1} r^2} - \frac{1}{w - x_1} \right) + \right. \\ & \quad \left. + \frac{e^{-\frac{2}{3}\pi i}}{2i} \left(\frac{r^2}{w - x_1 r^2 e^{-\frac{2}{3}\pi i}} + \frac{x_1^{-2}}{w - x_1^{-1} e^{-\frac{2}{3}\pi i}} - \frac{x_1^{-2} r^2}{w - x_1^{-1} r^2 e^{-\frac{2}{3}\pi i}} - \frac{1}{w - x_1 e^{-\frac{2}{3}\pi i}} \right) + \right. \\ & \quad \left. + \frac{e^{\frac{2}{3}\pi i}}{2i} \left(\frac{r^2}{w - x_1 r^2 e^{\frac{2}{3}\pi i}} + \frac{x_1^{-2}}{w - x_1^{-1} e^{\frac{2}{3}\pi i}} - \frac{x_1^{-2} r^2}{w - x_1^{-1} r^2 e^{\frac{2}{3}\pi i}} - \frac{1}{w - x_1 e^{\frac{2}{3}\pi i}} \right) \right) \times \\ & \quad \left(\frac{-e^{-i\psi}}{4} \left(\frac{1}{w - e^{-i\psi}} - \frac{r^2}{w - r^2 e^{-i\psi}} \right) \right) \frac{dw}{iw} r dr. \\ & = \frac{1}{8} \int_{r=0}^1 \oint_{|w|=r} e^{-i\psi} \left(\left(\frac{r^2}{w - x_1 r^2} + \frac{x_1^{-2}}{w - x_1^{-1}} - \frac{x_1^{-2} r^2}{w - x_1^{-1} r^2} - \frac{1}{w - x_1} \right) + \right. \\ & \quad \left. + e^{-\frac{2}{3}\pi i} \left(\frac{r^2}{w - x_1 r^2 e^{-\frac{2}{3}\pi i}} + \frac{x_1^{-2}}{w - x_1^{-1} e^{-\frac{2}{3}\pi i}} - \frac{x_1^{-2} r^2}{w - x_1^{-1} r^2 e^{-\frac{2}{3}\pi i}} - \frac{1}{w - x_1 e^{-\frac{2}{3}\pi i}} \right) + \right. \\ & \quad \left. + e^{\frac{2}{3}\pi i} \left(\frac{r^2}{w - x_1 r^2 e^{\frac{2}{3}\pi i}} + \frac{x_1^{-2}}{w - x_1^{-1} e^{\frac{2}{3}\pi i}} - \frac{x_1^{-2} r^2}{w - x_1^{-1} r^2 e^{\frac{2}{3}\pi i}} - \frac{1}{w - x_1 e^{\frac{2}{3}\pi i}} \right) \right) \times \\ & \quad \left(\frac{1}{w - e^{-i\psi}} - \frac{r^2}{w - r^2 e^{-i\psi}} \right) \frac{dw}{w} r dr. \end{aligned}$$

Using the expression from (21) and a newly defined

$$\Phi_{\psi, r}(w) := \frac{1}{w - e^{-i\psi}} - \frac{r^2}{w - r^2 e^{-i\psi}} , \quad (46)$$

we obtain

$$\#_{(45)} = \frac{e^{-i\psi}}{8} \oint_{|w|=r} \left(\Psi_{x_1,0,r}(w) + \Psi_{x_1,1,r}(w) + \Psi_{x_1,2,r}(w) \right) \Phi_{\psi,r}(w) \frac{dw}{w}. \quad (47)$$

In this section we use

$$J_2(x_1, \psi, r; w) := \frac{1}{w} \Phi_{\psi,0,r}(w) \sum_{m=0}^2 \Psi_{x_1,m,r}(w) \quad (48)$$

5.3 Computation of the contour integral

Two cases have to be distinguished, namely $r < x_1$ and $r > x_1$. Again $w = 0$ does not contribute and the scheme for the residues is as follows:

poles due to:		$\Psi_{x_1,0,r}$		$\Psi_{x_1,1,r}$		$\Psi_{x_1,2,r}$		$\Phi_{\psi,r}$
		$a_1.$	$a_2.$	$b_1.$	$b_2.$	$c_1.$	$c_2.$	$d.$
$r \in (0, x_1)$	I.	$x_1 r^2$	$\frac{r^2}{x_1}$	$x_1 r^2 e^{-\frac{2}{3}\pi i}$	$\frac{r^2}{x_1} e^{-\frac{2}{3}\pi i}$	$x_1 r^2 e^{\frac{2}{3}\pi i}$	$\frac{r^2}{x_1} e^{\frac{2}{3}\pi i}$	$r^2 e^{-i\psi}$
$r \in (x_1, 1)$	II.	$x_1 r^2$	x_1	$x_1 r^2 e^{-\frac{2}{3}\pi i}$	$x_1 e^{-\frac{2}{3}\pi i}$	$x_1 r^2 e^{\frac{2}{3}\pi i}$	$x_1 e^{\frac{2}{3}\pi i}$	$r^2 e^{-i\psi}$

We proceed as before.

I and II, a_1 : pole at $w = x_1 r^2$.

$$\begin{aligned} & \text{Res} \left(J_2(x_1, \psi, r; w) \right)_{w=x_1 r^2} = \\ &= \frac{r^2}{w} \left(\frac{1}{w - e^{-i\psi}} - \frac{r^2}{w - r^2 e^{-i\psi}} \right)_{w=x_1 r^2} \\ &= \frac{1}{x_1} \left(\frac{1}{x_1 r^2 - e^{-i\psi}} - \frac{r^2}{x_1 r^2 - r^2 e^{-i\psi}} \right) \\ &= \frac{1}{x_1^2} \left(\frac{1}{r^2 - x_1^{-1} e^{-i\psi}} - \frac{1}{1 - x_1^{-1} e^{-i\psi}} \right). \end{aligned} \quad (49)$$

I, a_2 : pole at $w = x_1^{-1} r^2$.

$$\begin{aligned} & \text{Res} \left(J_2(x_1, \psi, r; w) \right)_{w=x_1^{-1} r^2} = \\ &= \frac{-x_1^{-2} r^2}{w} \left(\frac{1}{w - e^{-i\psi}} - \frac{r^2}{w - r^2 e^{-i\psi}} \right)_{w=x_1^{-1} r^2} \\ &= \frac{-x_1^{-2} r^2}{x_1^{-1} r^2} \left(\frac{1}{x_1^{-1} r^2 - e^{-i\psi}} - \frac{r^2}{x_1^{-1} r^2 - r^2 e^{-i\psi}} \right) \\ &= \frac{1}{1 - x_1 e^{-i\psi}} - \frac{1}{r^2 - x_1 e^{-i\psi}}. \end{aligned} \quad (50)$$

II, a_2 : pole at $w = x_1$.

$$\text{Res} \left(J_2(x_1, \psi, r; w) \right)_{w=x_1} =$$

$$\begin{aligned}
&= -\frac{1}{w} \left(\frac{1}{w - e^{-i\psi}} - \frac{r^2}{w - r^2 e^{-i\psi}} \right)_{w=x_1} \\
&= -\frac{1}{x_1} \left(\frac{1}{x_1 - e^{-i\psi}} - \frac{r^2}{x_1 - r^2 e^{-i\psi}} \right) \\
&= e^{2i\psi} \left(\frac{1}{1 - x_1 e^{i\psi}} - \frac{1}{r^2 - x_1 e^{i\psi}} \right)
\end{aligned} \tag{51}$$

I and II, b_1 : pole at $w = x_1 r^2 e^{-\frac{2}{3}\pi i}$.

$$\begin{aligned}
&\text{Res} \left(J_2(x_1, \psi, r; w) \right)_{w=x_1 r^2 e^{-\frac{2}{3}\pi i}} = \\
&= \frac{r^2 e^{-\frac{2}{3}\pi i}}{w} \left(\frac{1}{w - e^{-i\psi}} - \frac{r^2}{w - r^2 e^{-i\psi}} \right)_{w=x_1 r^2 e^{-\frac{2}{3}\pi i}} \\
&= \frac{1}{x_1} \left(\frac{1}{x_1 r^2 e^{-\frac{2}{3}\pi i} - e^{-i\psi}} - \frac{r^2}{x_1 r^2 e^{-\frac{2}{3}\pi i} - r^2 e^{-i\psi}} \right) \\
&= \frac{e^{\frac{2}{3}\pi i}}{x_1^2} \left(\frac{1}{r^2 - x_1^{-1} e^{\frac{2}{3}\pi i - i\psi}} - \frac{1}{1 - x_1^{-1} e^{\frac{2}{3}\pi i - i\psi}} \right).
\end{aligned} \tag{52}$$

I, b_2 : pole at $w = x_1^{-1} r^2 e^{-\frac{2}{3}\pi i}$.

$$\begin{aligned}
&\text{Res} \left(J_2(x_1, \psi, r; w) \right)_{w=x_1^{-1} r^2 e^{-\frac{2}{3}\pi i}} = \\
&= -\frac{x_1^{-2} r^2 e^{-\frac{2}{3}\pi i}}{w} \left(\frac{1}{w - e^{-i\psi}} - \frac{r^2}{w - r^2 e^{-i\psi}} \right)_{w=x_1^{-1} r^2 e^{-\frac{2}{3}\pi i}} \\
&= -\frac{x_1^{-2} r^2 e^{-\frac{2}{3}\pi i}}{x_1^{-1} r^2 e^{-\frac{2}{3}\pi i}} \left(\frac{1}{x_1^{-1} r^2 e^{-\frac{2}{3}\pi i} - e^{-i\psi}} - \frac{r^2}{x_1^{-1} r^2 e^{-\frac{2}{3}\pi i} - r^2 e^{-i\psi}} \right) \\
&= e^{\frac{2}{3}\pi i} \left(\frac{1}{1 - x_1 e^{\frac{2}{3}\pi i - i\psi}} - \frac{1}{r^2 - x_1 e^{\frac{2}{3}\pi i - i\psi}} \right).
\end{aligned} \tag{53}$$

II, b_2 : pole at $w = x_1 e^{-\frac{2}{3}\pi i}$.

$$\text{Res} \left(J_2(x_1, \psi, r; w) \right)_{w=x_1 e^{-\frac{2}{3}\pi i}} =$$

$$\begin{aligned}
&= \frac{-e^{-\frac{2}{3}\pi i}}{w} \left(\frac{1}{w - e^{-i\psi}} - \frac{r^2}{w - r^2 e^{-i\psi}} \right)_{w=x_1 e^{-\frac{2}{3}\pi i}} \\
&= \frac{-e^{-\frac{2}{3}\pi i}}{x_1 e^{-\frac{2}{3}\pi i}} \left(\frac{1}{x_1 e^{-\frac{2}{3}\pi i} - e^{-i\psi}} - \frac{r^2}{x_1 e^{-\frac{2}{3}\pi i} - r^2 e^{-i\psi}} \right) \\
&= \frac{-1}{x_1} \left(\frac{1}{x_1 e^{-\frac{2}{3}\pi i} - e^{-i\psi}} - e^{i\psi} \frac{r^2 e^{-i\psi}}{x_1 e^{-\frac{2}{3}\pi i} - r^2 e^{-i\psi}} \right) \\
&= \frac{-1}{x_1} \left(\frac{1}{x_1 e^{-\frac{2}{3}\pi i} - e^{-i\psi}} - e^{i\psi} \left(-1 + \frac{x_1 e^{-\frac{2}{3}\pi i}}{x_1 e^{-\frac{2}{3}\pi i} - r^2 e^{-i\psi}} \right) \right) \\
&= \frac{-1}{x_1} \left(\frac{x_1 e^{i\psi} e^{-\frac{2}{3}\pi i}}{x_1 e^{-\frac{2}{3}\pi i} - e^{-i\psi}} - e^{i\psi} \frac{x_1 e^{-\frac{2}{3}\pi i}}{x_1 e^{-\frac{2}{3}\pi i} - r^2 e^{-i\psi}} \right) \\
&= e^{-\frac{2}{3}\pi i + 2i\psi} \left(\frac{1}{1 - x_1 e^{-\frac{2}{3}\pi i + i\psi}} - \frac{1}{r^2 - x_1 e^{-\frac{2}{3}\pi i + i\psi}} \right)
\end{aligned} \tag{54}$$

I and II, c_1 : pole at $w = x_1 r^2 e^{\frac{2}{3}\pi i}$.

$$\begin{aligned}
&\text{Res} \left(J_2(x_1, \psi, r; w) \right)_{w=x_1 r^2 e^{\frac{2}{3}\pi i}} = \\
&= \frac{r^2 e^{\frac{2}{3}\pi i}}{w} \left(\frac{1}{w - e^{-i\psi}} - \frac{r^2}{w - r^2 e^{-i\psi}} \right)_{w=x_1 r^2 e^{\frac{2}{3}\pi i}} \\
&= \frac{1}{x_1} \left(\frac{1}{x_1 r^2 e^{\frac{2}{3}\pi i} - e^{-i\psi}} - \frac{r^2}{x_1 r^2 e^{\frac{2}{3}\pi i} - r^2 e^{-i\psi}} \right) \\
&= \frac{e^{-\frac{2}{3}\pi i}}{x_1^2} \left(\frac{1}{r^2 - x_1^{-1} e^{-\frac{2}{3}\pi i - i\psi}} - \frac{1}{1 - x_1^{-1} e^{-\frac{2}{3}\pi i - i\psi}} \right)
\end{aligned} \tag{55}$$

I, c_2 : pole at $w = x_1^{-1} r^2 e^{\frac{2}{3}\pi i}$.

$$\begin{aligned}
&\text{Res} \left(J_2(x_1, \psi, r; w) \right)_{w=x_1^{-1} r^2 e^{\frac{2}{3}\pi i}} = \\
&= - \frac{x_1^{-2} r^2 e^{\frac{2}{3}\pi i}}{w} \left(\frac{1}{w - e^{-i\psi}} - \frac{r^2}{w - r^2 e^{-i\psi}} \right)_{w=x_1^{-1} r^2 e^{\frac{2}{3}\pi i}} \\
&= - \frac{1}{x_1} \left(\frac{1}{x_1^{-1} r^2 e^{\frac{2}{3}\pi i} - e^{-i\psi}} - \frac{r^2}{x_1^{-1} r^2 e^{\frac{2}{3}\pi i} - r^2 e^{-i\psi}} \right) \\
&= e^{-\frac{2}{3}\pi i} \left(\frac{1}{1 - x_1 e^{-\frac{2}{3}\pi i - i\psi}} - \frac{1}{r^2 - x_1 e^{-\frac{2}{3}\pi i - i\psi}} \right).
\end{aligned} \tag{56}$$

II, c_2 : pole at $w = x_1 e^{\frac{2}{3}\pi i}$.

$$\text{Res} \left(J_2(x_1, \psi, r; w) \right)_{w=x_1 e^{\frac{2}{3}\pi i}} =$$

$$\begin{aligned}
&= -\frac{e^{\frac{2}{3}\pi i}}{w} \left(\frac{1}{w - e^{-i\psi}} - \frac{r^2}{w - r^2 e^{-i\psi}} \right)_{w=x_1 e^{\frac{2}{3}\pi i}} \\
&= -\frac{1}{x_1} \left(\frac{1}{x_1 e^{\frac{2}{3}\pi i} - e^{-i\psi}} - \frac{r^2}{x_1 e^{\frac{2}{3}\pi i} - r^2 e^{-i\psi}} \right) \\
&= -\frac{1}{x_1} \left(\frac{1}{x_1 e^{\frac{2}{3}\pi i} - e^{-i\psi}} + e^{i\psi} \left(1 - \frac{x_1 e^{\frac{2}{3}\pi i}}{x_1 e^{\frac{2}{3}\pi i} - r^2 e^{-i\psi}} \right) \right) \\
&= -\frac{1}{x_1} \left(\frac{x_1 e^{\frac{2}{3}\pi i} e^{i\psi}}{x_1 e^{\frac{2}{3}\pi i} - e^{-i\psi}} - \frac{x_1 e^{\frac{2}{3}\pi i} e^{i\psi}}{x_1 e^{\frac{2}{3}\pi i} - r^2 e^{-i\psi}} \right) \\
&= e^{\frac{2}{3}\pi i} e^{2i\psi} \left(\frac{1}{1 - x_1 e^{\frac{2}{3}\pi i} e^{i\psi}} - \frac{1}{r^2 - x_1 e^{\frac{2}{3}\pi i} e^{i\psi}} \right). \tag{57}
\end{aligned}$$

I and II, d: pole at $w = r^2 e^{-i\psi}$.

$$\begin{aligned}
&\text{Res} \left(J_2(x_1, y_1, r; w) \right)_{w=r^2 e^{-i\psi}} = \\
&= \frac{-r^2}{w} \left(\left(\frac{r^2}{w - x_1 r^2} + \frac{x_1^{-2}}{w - x_1^{-1}} - \frac{x_1^{-2} r^2}{w - x_1^{-1} r^2} - \frac{1}{w - x_1} \right) + \right. \\
&\quad \left. + e^{-\frac{2}{3}\pi i} \left(\frac{r^2}{w - x_1 r^2 e^{-\frac{2}{3}\pi i}} + \frac{x_1^{-2}}{w - x_1^{-1} e^{-\frac{2}{3}\pi i}} - \frac{x_1^{-2} r^2}{w - x_1^{-1} r^2 e^{-\frac{2}{3}\pi i}} - \frac{1}{w - x_1 e^{-\frac{2}{3}\pi i}} \right) + \right. \\
&\quad \left. + e^{\frac{2}{3}\pi i} \left(\frac{r^2}{w - x_1 r^2 e^{\frac{2}{3}\pi i}} + \frac{x_1^{-2}}{w - x_1^{-1} e^{\frac{2}{3}\pi i}} - \frac{x_1^{-2} r^2}{w - x_1^{-1} r^2 e^{\frac{2}{3}\pi i}} - \frac{1}{w - x_1 e^{\frac{2}{3}\pi i}} \right) \right)_{w=r^2 e^{-i\psi}} \\
&= \frac{-r^2}{r^2 e^{-i\psi}} \left(\left(\frac{r^2}{r^2 e^{-i\psi} - x_1 r^2} + \frac{x_1^{-2}}{r^2 e^{-i\psi} - x_1^{-1}} - \frac{x_1^{-2} r^2}{r^2 e^{-i\psi} - x_1^{-1} r^2} - \frac{1}{r^2 e^{-i\psi} - x_1} \right) + \right. \\
&\quad \left. + e^{-\frac{2}{3}\pi i} \left(\frac{r^2}{r^2 e^{-i\psi} - x_1 r^2 e^{-\frac{2}{3}\pi i}} + \frac{x_1^{-2}}{r^2 e^{-i\psi} - x_1^{-1} e^{-\frac{2}{3}\pi i}} - \frac{x_1^{-2} r^2}{r^2 e^{-i\psi} - x_1^{-1} r^2 e^{-\frac{2}{3}\pi i}} - \frac{1}{r^2 e^{-i\psi} - x_1 e^{-\frac{2}{3}\pi i}} \right) + \right. \\
&\quad \left. + e^{\frac{2}{3}\pi i} \left(\frac{r^2}{r^2 e^{-i\psi} - x_1 r^2 e^{\frac{2}{3}\pi i}} + \frac{x_1^{-2}}{r^2 e^{-i\psi} - x_1^{-1} e^{\frac{2}{3}\pi i}} - \frac{x_1^{-2} r^2}{r^2 e^{-i\psi} - x_1^{-1} r^2 e^{\frac{2}{3}\pi i}} - \frac{1}{r^2 e^{-i\psi} - x_1 e^{\frac{2}{3}\pi i}} \right) \right) \\
&= -e^{2i\psi} \left(\left(\frac{1}{1 - x_1 e^{i\psi}} + \frac{x_1^{-2}}{r^2 - x_1^{-1} e^{i\psi}} - \frac{x_1^{-2}}{1 - x_1^{-1} e^{i\psi}} - \frac{1}{r^2 - x_1 e^{i\psi}} \right) + \right. \\
&\quad \left. + e^{-\frac{2}{3}\pi i} \left(\frac{1}{1 - x_1 e^{-\frac{2}{3}\pi i} e^{i\psi}} + \frac{x_1^{-2}}{r^2 - x_1^{-1} e^{-\frac{2}{3}\pi i} e^{i\psi}} - \frac{x_1^{-2}}{1 - x_1^{-1} e^{-\frac{2}{3}\pi i} e^{i\psi}} - \frac{1}{r^2 - x_1 e^{-\frac{2}{3}\pi i} e^{i\psi}} \right) + \right. \\
&\quad \left. + e^{\frac{2}{3}\pi i} \left(\frac{1}{1 - x_1 e^{\frac{2}{3}\pi i} e^{i\psi}} + \frac{x_1^{-2}}{r^2 - x_1^{-1} e^{\frac{2}{3}\pi i} e^{i\psi}} - \frac{x_1^{-2}}{1 - x_1^{-1} e^{\frac{2}{3}\pi i} e^{i\psi}} - \frac{1}{r^2 - x_1 e^{\frac{2}{3}\pi i} e^{i\psi}} \right) \right)
\end{aligned}$$

$$\begin{aligned}
&= -e^{2i\psi} \left(\frac{1}{1-x_1 e^{i\psi}} + \frac{e^{-\frac{2}{3}\pi i}}{1-x_1 e^{-\frac{2}{3}\pi i} e^{i\psi}} + \frac{e^{\frac{2}{3}\pi i}}{1-x_1 e^{\frac{2}{3}\pi i} e^{i\psi}} \right) + \\
&+ \frac{e^{2i\psi}}{x_1^2} \left(\frac{1}{1-x_1^{-1} e^{i\psi}} + \frac{e^{-\frac{2}{3}\pi i}}{1-x_1^{-1} e^{-\frac{2}{3}\pi i} e^{i\psi}} + \frac{e^{\frac{2}{3}\pi i}}{1-x_1^{-1} e^{\frac{2}{3}\pi i} e^{i\psi}} \right) + \\
&+ e^{2i\psi} \left(\frac{1}{r^2-x_1 e^{i\psi}} + \frac{e^{-\frac{2}{3}\pi i}}{r^2-x_1 e^{-\frac{2}{3}\pi i} e^{i\psi}} + \frac{e^{\frac{2}{3}\pi i}}{r^2-x_1 e^{\frac{2}{3}\pi i} e^{i\psi}} \right) + \\
&- \frac{e^{2i\psi}}{x_1^2} \left(\frac{1}{r^2-x_1^{-1} e^{i\psi}} + \frac{e^{-\frac{2}{3}\pi i}}{r^2-x_1^{-1} e^{-\frac{2}{3}\pi i} e^{i\psi}} + \frac{e^{\frac{2}{3}\pi i}}{r^2-x_1^{-1} e^{\frac{2}{3}\pi i} e^{i\psi}} \right). \\
&= \sum_{k=0}^2 \left(-\frac{e^{\frac{2}{3}k\pi i+2i\psi}}{1-x_1 e^{\frac{2}{3}k\pi i} e^{i\psi}} + \frac{1}{x_1^2} \frac{e^{\frac{2}{3}k\pi i+2i\psi}}{1-x_1^{-1} e^{\frac{2}{3}k\pi i} e^{i\psi}} \right) + \\
&+ \sum_{k=0}^2 \left(\frac{e^{\frac{2}{3}k\pi i+2i\psi}}{r^2-x_1 e^{\frac{2}{3}k\pi i} e^{i\psi}} - \frac{1}{x_1^2} \frac{e^{\frac{2}{3}k\pi i+2i\psi}}{r^2-x_1^{-1} e^{\frac{2}{3}k\pi i} e^{i\psi}} \right) \tag{58}
\end{aligned}$$

5.4 Integration in the radial direction

- For $r \in (0, x_1)$ we have to compute

$$\int_{r=0}^{x_1} e^{-i\psi} I_4(x_1, \psi, r) r dr,$$

where

$$\frac{e^{-i\psi} I_4(x_1, \psi, r)}{2\pi i} = e^{-i\psi} (\#(49) + \#(50) + \#(52) + \#(53) + \#(55) + \#(56) + \#(58)) =$$

$$\begin{aligned}
&= e^{-i\psi} \left(\frac{1}{x_1^2} \left(\frac{1}{r^2 - x_1^{-1} e^{-i\psi}} - \frac{1}{1 - x_1^{-1} e^{-i\psi}} \right) + \right. \\
&\quad + \frac{1}{1 - x_1 e^{-i\psi}} - \frac{1}{r^2 - x_1 e^{-i\psi}} + \\
&\quad + \frac{e^{\frac{2}{3}\pi i}}{x_1^2} \left(\frac{1}{r^2 - x_1^{-1} e^{\frac{2}{3}\pi i - i\psi}} - \frac{1}{1 - x_1^{-1} e^{\frac{2}{3}\pi i - i\psi}} \right) + \\
&\quad + e^{\frac{2}{3}\pi i} \left(\frac{1}{1 - x_1 e^{\frac{2}{3}\pi i - i\psi}} - \frac{1}{r^2 - x_1 e^{\frac{2}{3}\pi i - i\psi}} \right) + \\
&\quad + \frac{e^{-\frac{2}{3}\pi i}}{x_1^2} \left(\frac{1}{r^2 - x_1^{-1} e^{-\frac{2}{3}\pi i - i\psi}} - \frac{1}{1 - x_1^{-1} e^{-\frac{2}{3}\pi i - i\psi}} \right) + \\
&\quad + e^{-\frac{2}{3}\pi i} \left(\frac{1}{1 - x_1 e^{-\frac{2}{3}\pi i - i\psi}} - \frac{1}{r^2 - x_1 e^{-\frac{2}{3}\pi i - i\psi}} \right) + \\
&\quad - e^{2i\psi} \left(\frac{1}{1 - x_1 e^{i\psi}} + \frac{e^{-\frac{2}{3}\pi i}}{1 - x_1 e^{-\frac{2}{3}\pi i} e^{i\psi}} + \frac{e^{\frac{2}{3}\pi i}}{1 - x_1 e^{\frac{2}{3}\pi i} e^{i\psi}} \right) + \\
&\quad + \frac{e^{2i\psi}}{x_1^2} \left(\frac{1}{1 - x_1^{-1} e^{i\psi}} + \frac{e^{-\frac{2}{3}\pi i}}{1 - x_1^{-1} e^{-\frac{2}{3}\pi i} e^{i\psi}} + \frac{e^{\frac{2}{3}\pi i}}{1 - x_1^{-1} e^{\frac{2}{3}\pi i} e^{i\psi}} \right) + \\
&\quad + e^{2i\psi} \left(\frac{1}{r^2 - x_1 e^{i\psi}} + \frac{e^{-\frac{2}{3}\pi i}}{r^2 - x_1 e^{-\frac{2}{3}\pi i} e^{i\psi}} + \frac{e^{\frac{2}{3}\pi i}}{r^2 - x_1 e^{\frac{2}{3}\pi i} e^{i\psi}} \right) + \\
&\quad \left. - \frac{e^{2i\psi}}{x_1^2} \left(\frac{1}{r^2 - x_1^{-1} e^{i\psi}} + \frac{e^{-\frac{2}{3}\pi i}}{r^2 - x_1^{-1} e^{-\frac{2}{3}\pi i} e^{i\psi}} + \frac{e^{\frac{2}{3}\pi i}}{r^2 - x_1^{-1} e^{\frac{2}{3}\pi i} e^{i\psi}} \right) \right) \\
&= e^{-i\psi} \left(\frac{1}{1 - x_1 e^{-i\psi}} + \frac{e^{\frac{2}{3}\pi i}}{1 - x_1 e^{\frac{2}{3}\pi i} e^{-i\psi}} + \frac{e^{-\frac{2}{3}\pi i}}{1 - x_1 e^{-\frac{2}{3}\pi i} e^{-i\psi}} \right) + \\
&\quad - e^{i\psi} \left(\frac{1}{1 - x_1 e^{i\psi}} + \frac{e^{\frac{2}{3}\pi i}}{1 - x_1 e^{\frac{2}{3}\pi i} e^{i\psi}} + \frac{e^{-\frac{2}{3}\pi i}}{1 - x_1 e^{-\frac{2}{3}\pi i} e^{i\psi}} \right) + \\
&\quad - \frac{e^{-i\psi}}{x_1^2} \left(\frac{1}{1 - x_1^{-1} e^{-i\psi}} + \frac{e^{\frac{2}{3}\pi i}}{1 - x_1^{-1} e^{\frac{2}{3}\pi i} e^{-i\psi}} + \frac{e^{-\frac{2}{3}\pi i}}{1 - x_1^{-1} e^{-\frac{2}{3}\pi i} e^{-i\psi}} \right) + \\
&\quad + \frac{e^{i\psi}}{x_1^2} \left(\frac{1}{1 - x_1^{-1} e^{i\psi}} + \frac{e^{-\frac{2}{3}\pi i}}{1 - x_1^{-1} e^{-\frac{2}{3}\pi i} e^{i\psi}} + \frac{e^{\frac{2}{3}\pi i}}{1 - x_1^{-1} e^{\frac{2}{3}\pi i} e^{i\psi}} \right) +
\end{aligned}$$

$$\begin{aligned}
& - e^{-i\psi} \left(\frac{1}{r^2 - x_1 e^{-i\psi}} + \frac{e^{-\frac{2}{3}\pi i}}{r^2 - x_1 e^{-\frac{2}{3}\pi i} e^{-i\psi}} + \frac{e^{\frac{2}{3}\pi i}}{r^2 - x_1 e^{\frac{2}{3}\pi i} e^{-i\psi}} \right) + \\
& + e^{i\psi} \left(\frac{1}{r^2 - x_1 e^{i\psi}} + \frac{e^{-\frac{2}{3}\pi i}}{r^2 - x_1 e^{-\frac{2}{3}\pi i} e^{i\psi}} + \frac{e^{\frac{2}{3}\pi i}}{r^2 - x_1 e^{\frac{2}{3}\pi i} e^{i\psi}} \right) + \\
& + \frac{e^{-i\psi}}{x_1^2} \left(\frac{1}{r^2 - x_1^{-1} e^{-i\psi}} + \frac{e^{\frac{2}{3}\pi i}}{r^2 - x_1^{-1} e^{\frac{2}{3}\pi i} e^{-i\psi}} + \frac{e^{-\frac{2}{3}\pi i}}{r^2 - x_1^{-1} e^{-\frac{2}{3}\pi i} e^{-i\psi}} \right) + \\
& - \frac{e^{i\psi}}{x_1^2} \left(\frac{1}{r^2 - x_1^{-1} e^{i\psi}} + \frac{e^{-\frac{2}{3}\pi i}}{r^2 - x_1^{-1} e^{-\frac{2}{3}\pi i} e^{i\psi}} + \frac{e^{\frac{2}{3}\pi i}}{r^2 - x_1^{-1} e^{\frac{2}{3}\pi i} e^{i\psi}} \right) \\
= & \left(\frac{e^{-i\psi}}{1 - x_1 e^{-i\psi}} + \frac{e^{\frac{2}{3}\pi i} e^{-i\psi}}{1 - x_1 e^{\frac{2}{3}\pi i} e^{-i\psi}} + \frac{e^{-\frac{2}{3}\pi i} e^{-i\psi}}{1 - x_1 e^{-\frac{2}{3}\pi i} e^{-i\psi}} \right) + \\
& - \left(\frac{e^{i\psi}}{1 - x_1 e^{i\psi}} + \frac{e^{\frac{2}{3}\pi i} e^{i\psi}}{1 - x_1 e^{\frac{2}{3}\pi i} e^{i\psi}} + \frac{e^{-\frac{2}{3}\pi i} e^{i\psi}}{1 - x_1 e^{-\frac{2}{3}\pi i} e^{i\psi}} \right) + \\
& - \frac{1}{x_1^2} \left(\frac{e^{-i\psi}}{1 - x_1^{-1} e^{-i\psi}} + \frac{e^{\frac{2}{3}\pi i} e^{-i\psi}}{1 - x_1^{-1} e^{\frac{2}{3}\pi i} e^{-i\psi}} + \frac{e^{-\frac{2}{3}\pi i} e^{-i\psi}}{1 - x_1^{-1} e^{-\frac{2}{3}\pi i} e^{-i\psi}} \right) + \\
& + \frac{1}{x_1^2} \left(\frac{e^{i\psi}}{1 - x_1^{-1} e^{i\psi}} + \frac{e^{-\frac{2}{3}\pi i} e^{i\psi}}{1 - x_1^{-1} e^{-\frac{2}{3}\pi i} e^{i\psi}} + \frac{e^{\frac{2}{3}\pi i} e^{i\psi}}{1 - x_1^{-1} e^{\frac{2}{3}\pi i} e^{i\psi}} \right) + \\
& - \left(\frac{e^{-i\psi}}{r^2 - x_1 e^{-i\psi}} + \frac{e^{-\frac{2}{3}\pi i} e^{-i\psi}}{r^2 - x_1 e^{-\frac{2}{3}\pi i} e^{-i\psi}} + \frac{e^{\frac{2}{3}\pi i} e^{-i\psi}}{r^2 - x_1 e^{\frac{2}{3}\pi i} e^{-i\psi}} \right) + \\
& + \left(\frac{e^{i\psi}}{r^2 - x_1 e^{i\psi}} + \frac{e^{-\frac{2}{3}\pi i} e^{i\psi}}{r^2 - x_1 e^{-\frac{2}{3}\pi i} e^{i\psi}} + \frac{e^{\frac{2}{3}\pi i} e^{i\psi}}{r^2 - x_1 e^{\frac{2}{3}\pi i} e^{i\psi}} \right) + \\
& + \frac{1}{x_1^2} \left(\frac{e^{-i\psi}}{r^2 - x_1^{-1} e^{-i\psi}} + \frac{e^{\frac{2}{3}\pi i} e^{-i\psi}}{r^2 - x_1^{-1} e^{\frac{2}{3}\pi i} e^{-i\psi}} + \frac{e^{-\frac{2}{3}\pi i} e^{-i\psi}}{r^2 - x_1^{-1} e^{-\frac{2}{3}\pi i} e^{-i\psi}} \right) + \\
& - \frac{1}{x_1^2} \left(\frac{1 e^{i\psi}}{r^2 - x_1^{-1} e^{i\psi}} + \frac{e^{-\frac{2}{3}\pi i} e^{i\psi}}{r^2 - x_1^{-1} e^{-\frac{2}{3}\pi i} e^{i\psi}} + \frac{e^{\frac{2}{3}\pi i} e^{i\psi}}{r^2 - x_1^{-1} e^{\frac{2}{3}\pi i} e^{i\psi}} \right)
\end{aligned}$$

$$\begin{aligned}
&= \frac{e^{-i\psi}}{1-x_1e^{-i\psi}} - \frac{e^{i\psi}}{1-x_1e^{i\psi}} + \frac{e^{\frac{2}{3}\pi i}e^{-i\psi}}{1-x_1e^{\frac{2}{3}\pi i}e^{-i\psi}} - \frac{e^{\frac{2}{3}\pi i}e^{i\psi}}{1-x_1e^{\frac{2}{3}\pi i}e^{i\psi}} + \frac{e^{-\frac{2}{3}\pi i}e^{-i\psi}}{1-x_1e^{-\frac{2}{3}\pi i}e^{-i\psi}} - \frac{e^{-\frac{2}{3}\pi i}e^{i\psi}}{1-x_1e^{-\frac{2}{3}\pi i}e^{i\psi}} + \\
&+ \frac{1}{x_1^2} \left(\frac{e^{i\psi}}{1-x_1^{-1}e^{i\psi}} - \frac{e^{-i\psi}}{1-x_1^{-1}e^{-i\psi}} + \frac{e^{-\frac{2}{3}\pi i}e^{i\psi}}{1-x_1^{-1}e^{-\frac{2}{3}\pi i}e^{i\psi}} + \right. \\
&\quad \left. - \frac{e^{-\frac{2}{3}\pi i}e^{-i\psi}}{1-x_1^{-1}e^{-\frac{2}{3}\pi i}e^{-i\psi}} + \frac{e^{\frac{2}{3}\pi i}e^{i\psi}}{1-x_1^{-1}e^{\frac{2}{3}\pi i}e^{i\psi}} - \frac{e^{\frac{2}{3}\pi i}e^{-i\psi}}{1-x_1^{-1}e^{\frac{2}{3}\pi i}e^{-i\psi}} \right) + \\
&+ \left(\frac{e^{i\psi}}{r^2-x_1e^{i\psi}} - \frac{e^{-i\psi}}{r^2-x_1e^{-i\psi}} + \frac{e^{-\frac{2}{3}\pi i}e^{i\psi}}{r^2-x_1e^{-\frac{2}{3}\pi i}e^{i\psi}} + \right. \\
&\quad \left. - \frac{e^{-\frac{2}{3}\pi i}e^{-i\psi}}{r^2-x_1e^{-\frac{2}{3}\pi i}e^{-i\psi}} + \frac{e^{\frac{2}{3}\pi i}e^{i\psi}}{r^2-x_1e^{\frac{2}{3}\pi i}e^{i\psi}} - \frac{e^{\frac{2}{3}\pi i}e^{-i\psi}}{r^2-x_1e^{\frac{2}{3}\pi i}e^{-i\psi}} \right) + \\
&+ \frac{1}{x_1^2} \left(\frac{e^{-i\psi}}{r^2-x_1^{-1}e^{-i\psi}} - \frac{e^{i\psi}}{r^2-x_1^{-1}e^{i\psi}} + \frac{e^{\frac{2}{3}\pi i}e^{-i\psi}}{r^2-x_1^{-1}e^{\frac{2}{3}\pi i}e^{-i\psi}} + \right. \\
&\quad \left. - \frac{e^{\frac{2}{3}\pi i}e^{i\psi}}{r^2-x_1^{-1}e^{\frac{2}{3}\pi i}e^{i\psi}} + \frac{e^{-\frac{2}{3}\pi i}e^{-i\psi}}{r^2-x_1^{-1}e^{-\frac{2}{3}\pi i}e^{-i\psi}} - \frac{e^{-\frac{2}{3}\pi i}e^{i\psi}}{r^2-x_1^{-1}e^{-\frac{2}{3}\pi i}e^{i\psi}} \right) \\
&= \frac{-e^{2i\psi}+1}{(-e^{i\psi}+x_1)(-1+x_1e^{i\psi})} + \frac{e^{\frac{2}{3}\pi i}e^{-i\psi}}{1-x_1e^{\frac{2}{3}\pi i}e^{-i\psi}} - \frac{e^{-\frac{2}{3}\pi i}e^{i\psi}}{1-x_1e^{-\frac{2}{3}\pi i}e^{i\psi}} + \frac{e^{-\frac{2}{3}\pi i}e^{-i\psi}}{1-x_1e^{-\frac{2}{3}\pi i}e^{-i\psi}} - \frac{e^{\frac{2}{3}\pi i}e^{i\psi}}{1-x_1e^{\frac{2}{3}\pi i}e^{i\psi}} + \\
&+ \frac{1}{x_1^2} \left(\frac{e^{i\psi}}{1-x_1^{-1}e^{i\psi}} - \frac{e^{-i\psi}}{1-x_1^{-1}e^{-i\psi}} + \frac{e^{-\frac{2}{3}\pi i}e^{i\psi}}{1-x_1^{-1}e^{-\frac{2}{3}\pi i}e^{i\psi}} + \right. \\
&\quad \left. - \frac{e^{\frac{2}{3}\pi i}e^{-i\psi}}{1-x_1^{-1}e^{\frac{2}{3}\pi i}e^{-i\psi}} + \frac{e^{\frac{2}{3}\pi i}e^{i\psi}}{1-x_1^{-1}e^{\frac{2}{3}\pi i}e^{i\psi}} - \frac{e^{-\frac{2}{3}\pi i}e^{-i\psi}}{1-x_1^{-1}e^{-\frac{2}{3}\pi i}e^{-i\psi}} \right) + \\
&+ \left(\frac{e^{i\psi}}{r^2-x_1e^{i\psi}} - \frac{e^{-i\psi}}{r^2-x_1e^{-i\psi}} + \frac{e^{-\frac{2}{3}\pi i}e^{i\psi}}{r^2-x_1e^{-\frac{2}{3}\pi i}e^{i\psi}} + \right. \\
&\quad \left. - \frac{e^{\frac{2}{3}\pi i}e^{-i\psi}}{r^2-x_1e^{\frac{2}{3}\pi i}e^{-i\psi}} + \frac{e^{\frac{2}{3}\pi i}e^{i\psi}}{r^2-x_1e^{\frac{2}{3}\pi i}e^{i\psi}} - \frac{e^{-\frac{2}{3}\pi i}e^{-i\psi}}{r^2-x_1e^{-\frac{2}{3}\pi i}e^{-i\psi}} \right) + \\
&+ \frac{1}{x_1^2} \left(\frac{e^{-i\psi}}{r^2-x_1^{-1}e^{-i\psi}} - \frac{e^{i\psi}}{r^2-x_1^{-1}e^{i\psi}} + \frac{e^{\frac{2}{3}\pi i}e^{-i\psi}}{r^2-x_1^{-1}e^{\frac{2}{3}\pi i}e^{-i\psi}} + \right. \\
&\quad \left. - \frac{e^{-\frac{2}{3}\pi i}e^{i\psi}}{r^2-x_1^{-1}e^{-\frac{2}{3}\pi i}e^{i\psi}} + \frac{e^{-\frac{2}{3}\pi i}e^{-i\psi}}{r^2-x_1^{-1}e^{-\frac{2}{3}\pi i}e^{-i\psi}} - \frac{e^{\frac{2}{3}\pi i}e^{i\psi}}{r^2-x_1^{-1}e^{\frac{2}{3}\pi i}e^{i\psi}} \right)
\end{aligned}$$

$$\begin{aligned}
&= \sum_{k=0}^2 \sum_{\sigma=\pm 1} \sigma \left(\frac{e^{(\frac{2}{3}k\pi-\sigma\psi)i}}{1-x_1 e^{(\frac{2}{3}k\pi-\sigma\psi)i}} - \frac{1}{x_1^2} \frac{e^{(\frac{2}{3}k\pi-\sigma\psi)i}}{1-x_1^{-1} e^{(\frac{2}{3}k\pi-\sigma\psi)i}} \right) + \\
&\quad - \sum_{k=0}^2 \sum_{\sigma=\pm 1} \sigma \left(\frac{e^{(\frac{2}{3}k\pi-\sigma\psi)i}}{r^2-x_1 e^{(\frac{2}{3}k\pi-\sigma\psi)i}} - \frac{1}{x_1^2} \frac{e^{(\frac{2}{3}k\pi-\sigma\psi)i}}{r^2-x_1^{-1} e^{(\frac{2}{3}k\pi-\sigma\psi)i}} \right).
\end{aligned}$$

We obtain

$$\begin{aligned}
&\frac{1}{2\pi i} \int_{r=0}^{x_1} e^{-i\psi} I_4(x_1, \psi, r) r dr = \\
&= \frac{1}{2} \left(\frac{e^{-i\psi}}{1-x_1 e^{-i\psi}} - \frac{e^{i\psi}}{1-x_1 e^{i\psi}} + \frac{e^{\frac{2}{3}\pi i} e^{-i\psi}}{1-x_1 e^{\frac{2}{3}\pi i} e^{-i\psi}} + \right. \\
&\quad \left. - \frac{e^{-\frac{2}{3}\pi i} e^{i\psi}}{1-x_1 e^{-\frac{2}{3}\pi i} e^{i\psi}} + \frac{e^{-\frac{2}{3}\pi i} e^{-i\psi}}{1-x_1 e^{-\frac{2}{3}\pi i} e^{-i\psi}} - \frac{e^{\frac{2}{3}\pi i} e^{i\psi}}{1-x_1 e^{\frac{2}{3}\pi i} e^{i\psi}} \right) x_1^2 + \\
&+ \frac{1}{2} \left(\frac{e^{i\psi}}{1-x_1^{-1} e^{i\psi}} - \frac{e^{-i\psi}}{1-x_1^{-1} e^{-i\psi}} + \frac{e^{-\frac{2}{3}\pi i} e^{i\psi}}{1-x_1^{-1} e^{-\frac{2}{3}\pi i} e^{i\psi}} + \right. \\
&\quad \left. - \frac{e^{\frac{2}{3}\pi i} e^{-i\psi}}{1-x_1^{-1} e^{\frac{2}{3}\pi i} e^{-i\psi}} + \frac{e^{\frac{2}{3}\pi i} e^{i\psi}}{1-x_1^{-1} e^{\frac{2}{3}\pi i} e^{i\psi}} - \frac{e^{-\frac{2}{3}\pi i} e^{-i\psi}}{1-x_1^{-1} e^{-\frac{2}{3}\pi i} e^{-i\psi}} \right) + \\
&+ \frac{1}{2} [e^{i\psi} \text{Ln}(x_1 e^{i\psi} - r^2) - e^{-i\psi} \text{Ln}(x_1 e^{-i\psi} - r^2)]_0^{x_1} + \\
&+ \frac{1}{2} \left[e^{-\frac{2}{3}\pi i} e^{i\psi} \text{Ln}\left(x_1 e^{-\frac{2}{3}\pi i} e^{i\psi} - r^2\right) - e^{\frac{2}{3}\pi i} e^{-i\psi} \text{Ln}\left(x_1 e^{\frac{2}{3}\pi i} e^{-i\psi} - r^2\right) \right]_0^{x_1} + \\
&+ \frac{1}{2} \left[e^{\frac{2}{3}\pi i} e^{i\psi} \text{Ln}\left(x_1 e^{\frac{2}{3}\pi i} e^{i\psi} - r^2\right) - e^{-\frac{2}{3}\pi i} e^{-i\psi} \text{Ln}\left(x_1 e^{-\frac{2}{3}\pi i} e^{-i\psi} - r^2\right) \right]_0^{x_1} + \\
&- \frac{1}{2x_1^2} [e^{i\psi} \text{Ln}(x_1^{-1} e^{i\psi} - r^2) - e^{-i\psi} \text{Ln}(x_1^{-1} e^{-i\psi} - r^2)]_0^{x_1} + \\
&- \frac{1}{2} \left[e^{-\frac{2}{3}\pi i} e^{i\psi} \text{Ln}\left(x_1^{-1} e^{-\frac{2}{3}\pi i} e^{i\psi} - r^2\right) - e^{\frac{2}{3}\pi i} e^{-i\psi} \text{Ln}\left(x_1^{-1} e^{\frac{2}{3}\pi i} e^{-i\psi} - r^2\right) \right]_0^{x_1} + \\
&- \frac{1}{2} \left[e^{\frac{2}{3}\pi i} e^{i\psi} \text{Ln}\left(x_1^{-1} e^{\frac{2}{3}\pi i} e^{i\psi} - r^2\right) - e^{-\frac{2}{3}\pi i} e^{-i\psi} \text{Ln}\left(x_1^{-1} e^{-\frac{2}{3}\pi i} e^{-i\psi} - r^2\right) \right]_0^{x_1}
\end{aligned}$$

$$\begin{aligned}
&= \frac{1}{2} \left(\frac{e^{-i\psi}}{1-x_1 e^{-i\psi}} - \frac{e^{i\psi}}{1-x_1 e^{i\psi}} + \frac{e^{\frac{2}{3}\pi i} e^{-i\psi}}{1-x_1 e^{\frac{2}{3}\pi i} e^{-i\psi}} + \right. \\
&\quad \left. - \frac{e^{\frac{2}{3}\pi i} e^{i\psi}}{1-x_1 e^{\frac{2}{3}\pi i} e^{i\psi}} + \frac{e^{-\frac{2}{3}\pi i} e^{-i\psi}}{1-x_1 e^{-\frac{2}{3}\pi i} e^{-i\psi}} - \frac{e^{-\frac{2}{3}\pi i} e^{i\psi}}{1-x_1 e^{-\frac{2}{3}\pi i} e^{i\psi}} \right) x_1^2 + \\
&+ \frac{1}{2} \left(\frac{e^{i\psi}}{1-x_1^{-1} e^{i\psi}} - \frac{e^{-i\psi}}{1-x_1^{-1} e^{-i\psi}} + \frac{e^{-\frac{2}{3}\pi i} e^{i\psi}}{1-x_1^{-1} e^{-\frac{2}{3}\pi i} e^{i\psi}} + \right. \\
&\quad \left. - \frac{e^{-\frac{2}{3}\pi i} e^{-i\psi}}{1-x_1^{-1} e^{-\frac{2}{3}\pi i} e^{-i\psi}} + \frac{e^{\frac{2}{3}\pi i} e^{i\psi}}{1-x_1^{-1} e^{\frac{2}{3}\pi i} e^{i\psi}} - \frac{e^{\frac{2}{3}\pi i} e^{-i\psi}}{1-x_1^{-1} e^{\frac{2}{3}\pi i} e^{-i\psi}} \right) + \\
&+ \frac{1}{2} \left(e^{i\psi} \text{Ln} \left(\frac{x_1 e^{i\psi} - x_1^2}{x_1 e^{i\psi}} \right) - e^{-i\psi} \text{Ln} \left(\frac{x_1 e^{-i\psi} - x_1^2}{x_1 e^{-i\psi}} \right) \right) + \\
&+ \frac{1}{2} \left(e^{-\frac{2}{3}\pi i} e^{i\psi} \text{Ln} \left(\frac{x_1 e^{-\frac{2}{3}\pi i} e^{i\psi} - x_1^2}{x_1 e^{-\frac{2}{3}\pi i} e^{i\psi}} \right) - e^{\frac{2}{3}\pi i} e^{-i\psi} \text{Ln} \left(\frac{x_1 e^{\frac{2}{3}\pi i} e^{-i\psi} - x_1^2}{x_1 e^{\frac{2}{3}\pi i} e^{-i\psi}} \right) \right) + \\
&+ \frac{1}{2} \left(e^{\frac{2}{3}\pi i} e^{i\psi} \text{Ln} \left(\frac{x_1 e^{\frac{2}{3}\pi i} e^{i\psi} - x_1^2}{x_1 e^{\frac{2}{3}\pi i} e^{i\psi}} \right) - e^{-\frac{2}{3}\pi i} e^{-i\psi} \text{Ln} \left(\frac{x_1 e^{-\frac{2}{3}\pi i} e^{-i\psi} - x_1^2}{x_1 e^{-\frac{2}{3}\pi i} e^{-i\psi}} \right) \right) + \\
&- \frac{1}{2x_1^2} \left(e^{i\psi} \text{Ln} \left(\frac{x_1^{-1} e^{i\psi} - x_1^2}{x_1^{-1} e^{i\psi}} \right) - e^{-i\psi} \text{Ln} \left(\frac{x_1^{-1} e^{-i\psi} - x_1^2}{x_1^{-1} e^{-i\psi}} \right) \right) + \\
&- \frac{1}{2} \left(e^{-\frac{2}{3}\pi i} e^{i\psi} \text{Ln} \left(\frac{x_1^{-1} e^{-\frac{2}{3}\pi i} e^{i\psi} - x_1^2}{x_1^{-1} e^{-\frac{2}{3}\pi i} e^{i\psi}} \right) - e^{\frac{2}{3}\pi i} e^{-i\psi} \text{Ln} \left(\frac{x_1^{-1} e^{\frac{2}{3}\pi i} e^{-i\psi} - x_1^2}{x_1^{-1} e^{\frac{2}{3}\pi i} e^{-i\psi}} \right) \right) + \\
&- \frac{1}{2} \left(e^{\frac{2}{3}\pi i} e^{i\psi} \text{Ln} \left(\frac{x_1^{-1} e^{\frac{2}{3}\pi i} e^{i\psi} - x_1^2}{x_1^{-1} e^{\frac{2}{3}\pi i} e^{i\psi}} \right) - e^{-\frac{2}{3}\pi i} e^{-i\psi} \text{Ln} \left(\frac{x_1^{-1} e^{-\frac{2}{3}\pi i} e^{-i\psi} - x_1^2}{x_1^{-1} e^{-\frac{2}{3}\pi i} e^{-i\psi}} \right) \right) \\
&= -ix_1^2 \left(\frac{\sin \psi}{1-2x_1 \cos \psi + x_1^2} + \frac{\sin(\psi - \frac{2}{3}\pi)}{1-2x_1 \cos(\psi - \frac{2}{3}\pi) + x_1^2} + \frac{\sin(\psi + \frac{2}{3}\pi)}{1-2x_1 \cos(\psi + \frac{2}{3}\pi) + x_1^2} \right) + \\
&+ i \left(\frac{x_1^2 \sin \psi}{1-2x_1 \cos \psi + x_1^2} + \frac{x_1^2 \sin(\psi - \frac{2}{3}\pi)}{1-2x_1 \cos(\psi - \frac{2}{3}\pi) + x_1^2} + \frac{x_1^2 \sin(\psi + \frac{2}{3}\pi)}{1-2x_1 \cos(\psi + \frac{2}{3}\pi) + x_1^2} \right) + \\
&+ \frac{1}{2} (e^{i\psi} \text{Ln}(1-x_1 e^{-i\psi}) - e^{-i\psi} \text{Ln}(1-x_1 e^{i\psi})) + \\
&+ \frac{1}{2} \left(e^{-\frac{2}{3}\pi i + i\psi} \text{Ln} \left(1-x_1 e^{\frac{2}{3}\pi i - i\psi} \right) - e^{\frac{2}{3}\pi i - i\psi} \text{Ln} \left(1-x_1 e^{-\frac{2}{3}\pi i + i\psi} \right) \right) + \\
&+ \frac{1}{2} \left(e^{\frac{2}{3}\pi i + i\psi} \text{Ln} \left(1-x_1 e^{-\frac{2}{3}\pi i - i\psi} \right) - e^{-\frac{2}{3}\pi i - i\psi} \text{Ln} \left(1-x_1 e^{\frac{2}{3}\pi i + i\psi} \right) \right) + \\
&- \frac{1}{2x_1^2} (e^{i\psi} \text{Ln}(1-x_1^3 e^{-i\psi}) - e^{-i\psi} \text{Ln}(1-x_1^3 e^{i\psi})) + \\
&- \frac{1}{2} \left(e^{-\frac{2}{3}\pi i + i\psi} \text{Ln} \left(1-x_1^3 e^{\frac{2}{3}\pi i - i\psi} \right) - e^{\frac{2}{3}\pi i - i\psi} \text{Ln} \left(1-x_1^3 e^{-\frac{2}{3}\pi i + i\psi} \right) \right) + \\
&- \frac{1}{2} \left(e^{\frac{2}{3}\pi i + i\psi} \text{Ln} \left(1-x_1^3 e^{-\frac{2}{3}\pi i - i\psi} \right) - e^{-\frac{2}{3}\pi i - i\psi} \text{Ln} \left(1-x_1^3 e^{\frac{2}{3}\pi i + i\psi} \right) \right)
\end{aligned}$$

$$\begin{aligned}
&= i \sin \psi \ln \sqrt{1 - 2x_1 \cos \psi + x_1^2} + i \cos \psi \arctan \left(\frac{x_1 \sin \psi}{1 - x_1 \cos \psi} \right) + \\
&+ i \sin \left(\psi - \frac{2}{3} \pi \right) \ln \sqrt{1 - 2x_1 \cos \left(\psi - \frac{2}{3} \pi \right) + x_1^2} + i \cos \left(\psi - \frac{2}{3} \pi \right) \arctan \left(\frac{x_1 \sin \left(\psi - \frac{2}{3} \pi \right)}{1 - x_1 \cos \left(\psi - \frac{2}{3} \pi \right)} \right) + \\
&+ i \sin \left(\psi + \frac{2}{3} \pi \right) \ln \sqrt{1 - 2x_1 \cos \left(\psi + \frac{2}{3} \pi \right) + x_1^2} + i \cos \left(\psi + \frac{2}{3} \pi \right) \arctan \left(\frac{x_1 \sin \left(\psi + \frac{2}{3} \pi \right)}{1 - x_1 \cos \left(\psi + \frac{2}{3} \pi \right)} \right) + \\
&- \frac{i \sin \psi}{x_1^2} \ln \sqrt{1 - 2x_1^3 \cos \psi + x_1^6} - i \frac{\cos \psi}{x_1^2} \arctan \left(\frac{x_1^3 \sin \psi}{1 - x_1^3 \cos \psi} \right) + \\
&- \frac{i \sin \left(\psi - \frac{2}{3} \pi \right)}{x_1^2} \ln \sqrt{1 - 2x_1^3 \cos \left(\psi - \frac{2}{3} \pi \right) + x_1^6} - i \frac{\cos \left(\psi - \frac{2}{3} \pi \right)}{x_1^2} \arctan \left(\frac{x_1^3 \sin \left(\psi - \frac{2}{3} \pi \right)}{1 - x_1^3 \cos \left(\psi - \frac{2}{3} \pi \right)} \right) + \\
&- i \frac{\sin \left(\psi + \frac{2}{3} \pi \right)}{x_1^2} \ln \sqrt{1 - 2x_1^3 \cos \left(\psi + \frac{2}{3} \pi \right) + x_1^6} - i \frac{\cos \left(\psi + \frac{2}{3} \pi \right)}{x_1^2} \arctan \left(\frac{x_1^3 \sin \left(\psi + \frac{2}{3} \pi \right)}{1 - x_1^3 \cos \left(\psi + \frac{2}{3} \pi \right)} \right) + \\
&= i \sum_{k=0}^2 \sin \left(\psi + \frac{2}{3} k \pi \right) \ln \sqrt{1 - 2x_1 \cos \left(\psi + \frac{2}{3} k \pi \right) + x_1^2} + \\
&+ i \sum_{k=0}^2 \cos \left(\psi + \frac{2}{3} k \pi \right) \arctan \left(\frac{x_1 \sin \left(\psi + \frac{2}{3} k \pi \right)}{1 - x_1 \cos \left(\psi + \frac{2}{3} k \pi \right)} \right) + \\
&- i \sum_{k=0}^2 \frac{\sin \left(\psi + \frac{2}{3} k \pi \right)}{x_1^2} \ln \sqrt{1 - 2x_1^3 \cos \left(\psi + \frac{2}{3} k \pi \right) + x_1^6} + \\
&- i \sum_{k=0}^2 \frac{\cos \left(\psi + \frac{2}{3} k \pi \right)}{x_1^2} \arctan \left(\frac{x_1^3 \sin \left(\psi + \frac{2}{3} k \pi \right)}{1 - x_1^3 \cos \left(\psi + \frac{2}{3} k \pi \right)} \right) \tag{59}
\end{aligned}$$

• For $r \in (x_1, 1)$ the integral becomes

$$\int_{r=x_1}^1 I_5(x_1, \psi, r) r dr,$$

where

$$\frac{I_5(x_1, \psi, r)}{2\pi i} = e^{-i\psi} (\#_{(49)} + \#_{(51)} + \#_{(52)} + \#_{(54)} + \#_{(55)} + \#_{(57)} + \#_{(58)})$$

$$\begin{aligned}
&= \frac{e^{-i\psi}}{x_1^2} \left(\frac{1}{r^2 - x_1^{-1}e^{-i\psi}} - \frac{1}{1 - x_1^{-1}e^{-i\psi}} \right) + \\
&+ e^{i\psi} \left(\frac{1}{1 - x_1e^{i\psi}} - \frac{1}{r^2 - x_1e^{i\psi}} \right) + \\
&+ \frac{e^{\frac{2}{3}\pi i - i\psi}}{x_1^2} \left(\frac{1}{r^2 - x_1^{-1}e^{\frac{2}{3}\pi i - i\psi}} - \frac{1}{1 - x_1^{-1}e^{\frac{2}{3}\pi i - i\psi}} \right) + \\
&+ e^{-\frac{2}{3}\pi i + i\psi} \left(\frac{1}{1 - x_1e^{-\frac{2}{3}\pi i + i\psi}} - \frac{1}{r^2 - x_1e^{-\frac{2}{3}\pi i + i\psi}} \right) + \\
&+ \frac{e^{-\frac{2}{3}\pi i - i\psi}}{x_1^2} \left(\frac{1}{r^2 - x_1^{-1}e^{-\frac{2}{3}\pi i - i\psi}} - \frac{1}{1 - x_1^{-1}e^{-\frac{2}{3}\pi i - i\psi}} \right) + \\
&+ e^{\frac{2}{3}\pi i + i\psi} \left(\frac{1}{1 - x_1e^{\frac{2}{3}\pi i + i\psi}} - \frac{1}{r^2 - x_1e^{\frac{2}{3}\pi i + i\psi}} \right) + \\
&- e^{i\psi} \left(\frac{1}{1 - x_1e^{i\psi}} + \frac{e^{-\frac{2}{3}\pi i}}{1 - x_1e^{-\frac{2}{3}\pi i + i\psi}} + \frac{e^{\frac{2}{3}\pi i}}{1 - x_1e^{\frac{2}{3}\pi i + i\psi}} \right) + \\
&+ \frac{e^{i\psi}}{x_1^2} \left(\frac{1}{1 - x_1^{-1}e^{i\psi}} + \frac{e^{-\frac{2}{3}\pi i}}{1 - x_1^{-1}e^{-\frac{2}{3}\pi i + i\psi}} + \frac{e^{\frac{2}{3}\pi i}}{1 - x_1^{-1}e^{\frac{2}{3}\pi i + i\psi}} \right) + \\
&+ e^{i\psi} \left(\frac{1}{r^2 - x_1e^{i\psi}} + \frac{e^{-\frac{2}{3}\pi i}}{r^2 - x_1e^{-\frac{2}{3}\pi i + i\psi}} + \frac{e^{\frac{2}{3}\pi i}}{r^2 - x_1e^{\frac{2}{3}\pi i + i\psi}} \right) + \\
&- \frac{e^{i\psi}}{x_1^2} \left(\frac{1}{r^2 - x_1^{-1}e^{i\psi}} + \frac{e^{-\frac{2}{3}\pi i}}{r^2 - x_1^{-1}e^{-\frac{2}{3}\pi i + i\psi}} + \frac{e^{\frac{2}{3}\pi i}}{r^2 - x_1^{-1}e^{\frac{2}{3}\pi i + i\psi}} \right) \\
&= + \frac{1}{x_1^2} \left(\frac{e^{i\psi}}{1 - x_1^{-1}e^{i\psi}} - \frac{e^{-i\psi}}{1 - x_1^{-1}e^{-i\psi}} + \frac{e^{-\frac{2}{3}\pi i + i\psi}}{1 - x_1^{-1}e^{-\frac{2}{3}\pi i + i\psi}} + \right. \\
&\quad \left. - \frac{e^{\frac{2}{3}\pi i - i\psi}}{1 - x_1^{-1}e^{\frac{2}{3}\pi i - i\psi}} + \frac{e^{\frac{2}{3}\pi i + i\psi}}{1 - x_1^{-1}e^{\frac{2}{3}\pi i + i\psi}} - \frac{e^{-\frac{2}{3}\pi i - i\psi}}{1 - x_1^{-1}e^{-\frac{2}{3}\pi i - i\psi}} \right) + \\
&+ \frac{1}{x_1^2} \left(\frac{e^{-i\psi}}{r^2 - x_1^{-1}e^{-i\psi}} - \frac{e^{i\psi}}{r^2 - x_1^{-1}e^{i\psi}} + \frac{e^{\frac{2}{3}\pi i - i\psi}}{r^2 - x_1^{-1}e^{\frac{2}{3}\pi i - i\psi}} + \right. \\
&\quad \left. - \frac{e^{-\frac{2}{3}\pi i + i\psi}}{r^2 - x_1^{-1}e^{-\frac{2}{3}\pi i + i\psi}} + \frac{e^{-\frac{2}{3}\pi i - i\psi}}{r^2 - x_1^{-1}e^{-\frac{2}{3}\pi i - i\psi}} - \frac{e^{\frac{2}{3}\pi i + i\psi}}{r^2 - x_1^{-1}e^{\frac{2}{3}\pi i + i\psi}} \right). \\
&= \sum_{k=0}^2 \sum_{\sigma=\pm 1} \frac{\sigma}{x_1^2} \left(\frac{e^{(\frac{2}{3}k\pi - \sigma\psi)i}}{r^2 - x_1^{-1}e^{(\frac{2}{3}k\pi - \sigma\psi)i}} - \frac{e^{(\frac{2}{3}k\pi - \sigma\psi)i}}{1 - x_1^{-1}e^{(\frac{2}{3}k\pi - \sigma\psi)i}} \right). \tag{60}
\end{aligned}$$

We obtain

$$\frac{1}{2\pi i} \int_{r=x_1}^1 e^{-i\psi} I_5(x_1, \psi, r) r dr =$$

$$\begin{aligned}
&= \frac{1}{2x_1^2} \left(\frac{e^{i\psi}}{1-x_1^{-1}e^{i\psi}} - \frac{e^{-i\psi}}{1-x_1^{-1}e^{-i\psi}} + \frac{e^{-\frac{2}{3}\pi i+i\psi}}{1-x_1^{-1}e^{-\frac{2}{3}\pi i+i\psi}} + \right. \\
&\quad \left. - \frac{e^{\frac{2}{3}\pi i-i\psi}}{1-x_1^{-1}e^{\frac{2}{3}\pi i-i\psi}} + \frac{e^{\frac{2}{3}\pi i+i\psi}}{1-x_1^{-1}e^{\frac{2}{3}\pi i+i\psi}} - \frac{e^{-\frac{2}{3}\pi i-i\psi}}{1-x_1^{-1}e^{-\frac{2}{3}\pi i-i\psi}} \right) (1-x_1^2) + \\
&\quad + \frac{1}{2x_1^2} \left[e^{-i\psi} \text{Ln} (x_1^{-1}e^{-i\psi} - r^2) - e^{i\psi} \text{Ln} (x_1^{-1}e^{i\psi} - r^2) + + \right. \\
&\quad + e^{\frac{2}{3}\pi i-i\psi} \text{Ln} \left(x_1^{-1}e^{\frac{2}{3}\pi i-i\psi} - r^2 \right) - e^{-\frac{2}{3}\pi i+i\psi} \text{Ln} \left(x_1^{-1}e^{-\frac{2}{3}\pi i+i\psi} - r^2 \right) + \\
&\quad \left. + e^{-\frac{2}{3}\pi i-i\psi} \text{Ln} \left(x_1^{-1}e^{-\frac{2}{3}\pi i-i\psi} - r^2 \right) - e^{\frac{2}{3}\pi i+i\psi} \text{Ln} \left(x_1^{-1}e^{\frac{2}{3}\pi i+i\psi} - r^2 \right) \right]_{r=x_1}^1. \\
&= \frac{1}{2x_1^2} \left(\frac{e^{i\psi}}{1-x_1^{-1}e^{i\psi}} - \frac{e^{-i\psi}}{1-x_1^{-1}e^{-i\psi}} + \frac{e^{-\frac{2}{3}\pi i+i\psi}}{1-x_1^{-1}e^{-\frac{2}{3}\pi i+i\psi}} + \right. \\
&\quad \left. - \frac{e^{\frac{2}{3}\pi i-i\psi}}{1-x_1^{-1}e^{\frac{2}{3}\pi i-i\psi}} + \frac{e^{\frac{2}{3}\pi i+i\psi}}{1-x_1^{-1}e^{\frac{2}{3}\pi i+i\psi}} - \frac{e^{-\frac{2}{3}\pi i-i\psi}}{1-x_1^{-1}e^{-\frac{2}{3}\pi i-i\psi}} \right) (1-x_1^2) + \\
&\quad + \frac{1}{2x_1^2} \left(e^{-i\psi} \text{Ln} \left(\frac{x_1^{-1}e^{-i\psi} - 1}{x_1^{-1}e^{-i\psi} - x_1^2} \right) - e^{i\psi} \text{Ln} \left(\frac{x_1^{-1}e^{i\psi} - 1}{x_1^{-1}e^{i\psi} - x_1^2} \right) + \right. \\
&\quad + e^{\frac{2}{3}\pi i-i\psi} \text{Ln} \left(\frac{x_1^{-1}e^{\frac{2}{3}\pi i-i\psi} - 1}{x_1^{-1}e^{\frac{2}{3}\pi i-i\psi} - x_1^2} \right) - e^{-\frac{2}{3}\pi i+i\psi} \text{Ln} \left(\frac{x_1^{-1}e^{-\frac{2}{3}\pi i+i\psi} - 1}{x_1^{-1}e^{-\frac{2}{3}\pi i+i\psi} - x_1^2} \right) + \\
&\quad \left. + e^{-\frac{2}{3}\pi i-i\psi} \text{Ln} \left(\frac{x_1^{-1}e^{-\frac{2}{3}\pi i-i\psi} - 1}{x_1^{-1}e^{-\frac{2}{3}\pi i-i\psi} - x_1^2} \right) - e^{\frac{2}{3}\pi i+i\psi} \text{Ln} \left(\frac{x_1^{-1}e^{\frac{2}{3}\pi i+i\psi} - 1}{x_1^{-1}e^{\frac{2}{3}\pi i+i\psi} - x_1^2} \right) \right). \\
&= \frac{1}{2x_1^2} \left(\frac{2i \sin \psi}{1-2x_1^{-1} \cos \psi + x_1^{-2}} + \frac{2i \sin \left(\psi - \frac{2}{3}\pi \right)}{1-2x_1^{-1} \cos \left(\psi - \frac{2}{3}\pi \right) + x_1^{-2}} + \frac{2i \sin \left(\psi + \frac{2}{3}\pi \right)}{1-2x_1^{-1} \cos \left(\psi + \frac{2}{3}\pi \right) + x_1^{-2}} \right) (1-x_1^2) + \\
&\quad + \frac{1}{2x_1^2} \left(e^{-i\psi} \text{Ln} \left(\frac{1-x_1e^{i\psi}}{1-x_1^3e^{i\psi}} \right) - e^{i\psi} \text{Ln} \left(\frac{1-x_1e^{-i\psi}}{1-x_1^3e^{-i\psi}} \right) + e^{\frac{2}{3}\pi i-i\psi} \text{Ln} \left(\frac{1-x_1e^{-\frac{2}{3}\pi i+i\psi}}{1-x_1^3e^{-\frac{2}{3}\pi i+i\psi}} \right) + \right. \\
&\quad \left. - e^{-\frac{2}{3}\pi i+i\psi} \text{Ln} \left(\frac{1-x_1e^{\frac{2}{3}\pi i-i\psi}}{1-x_1^3e^{\frac{2}{3}\pi i-i\psi}} \right) + e^{-\frac{2}{3}\pi i-i\psi} \text{Ln} \left(\frac{1-x_1e^{\frac{2}{3}\pi i+i\psi}}{1-x_1^3e^{\frac{2}{3}\pi i+i\psi}} \right) - e^{\frac{2}{3}\pi i+i\psi} \text{Ln} \left(\frac{1-x_1e^{-\frac{2}{3}\pi i-i\psi}}{1-x_1^3e^{-\frac{2}{3}\pi i-i\psi}} \right) \right).
\end{aligned}$$

$$\begin{aligned}
&= \frac{1}{2x_1^2} \left(\frac{2i \sin \psi}{1 - 2x_1^{-1} \cos \psi + x_1^{-2}} + \frac{2i \sin \left(\psi - \frac{2}{3}\pi\right)}{1 - 2x_1^{-1} \cos \left(\psi - \frac{2}{3}\pi\right) + x_1^{-2}} + \frac{2i \sin \left(\psi + \frac{2}{3}\pi\right)}{1 - 2x_1^{-1} \cos \left(\psi + \frac{2}{3}\pi\right) + x_1^{-2}} \right) (1 - x_1^2) + \\
&+ \frac{1}{2x_1^2} \left(-2i \sin \psi \ln \sqrt{1 - 2x_1 \cos \psi + x_1^2} - 2i \cos \psi \arctan \left(\frac{x_1 \sin \psi}{1 - x_1 \cos \psi} \right) + \right. \\
&+ 2i \sin \psi \ln \sqrt{1 - 2x_1^3 \cos \psi + x_1^6} + 2i \cos \psi \arctan \left(\frac{x_1^3 \sin \psi}{1 - x_1^3 \cos \psi} \right) + \\
&- 2i \sin \left(\psi - \frac{2}{3}\pi\right) \ln \sqrt{1 - 2x_1 \cos \left(\psi - \frac{2}{3}\pi\right) + x_1^2} - 2i \cos \left(\psi - \frac{2}{3}\pi\right) \arctan \left(\frac{x_1 \sin \left(\psi - \frac{2}{3}\pi\right)}{1 - x_1 \cos \left(\psi - \frac{2}{3}\pi\right)} \right) + \\
&+ 2i \sin \left(\psi - \frac{2}{3}\pi\right) \ln \sqrt{1 - 2x_1^3 \cos \left(\psi - \frac{2}{3}\pi\right) + x_1^6} + 2i \cos \left(\psi - \frac{2}{3}\pi\right) \arctan \left(\frac{x_1^3 \sin \left(\psi - \frac{2}{3}\pi\right)}{1 - x_1^3 \cos \left(\psi - \frac{2}{3}\pi\right)} \right) + \\
&- 2i \sin \left(\psi + \frac{2}{3}\pi\right) \ln \sqrt{1 - 2x_1 \cos \left(\psi + \frac{2}{3}\pi\right) + x_1^2} - 2i \cos \left(\psi + \frac{2}{3}\pi\right) \arctan \left(\frac{x_1 \sin \left(\psi + \frac{2}{3}\pi\right)}{1 - x_1 \cos \left(\psi + \frac{2}{3}\pi\right)} \right) + \\
&+ 2i \sin \left(\psi + \frac{2}{3}\pi\right) \ln \sqrt{1 - 2x_1^3 \cos \left(\psi + \frac{2}{3}\pi\right) + x_1^6} + 2i \cos \left(\psi + \frac{2}{3}\pi\right) \arctan \left(\frac{x_1^3 \sin \left(\psi + \frac{2}{3}\pi\right)}{1 - x_1^3 \cos \left(\psi + \frac{2}{3}\pi\right)} \right) \Big) \\
&= i \left(\frac{\sin \psi}{x_1^2 - 2x_1 \cos \psi + 1} + \frac{\sin \left(\psi - \frac{2}{3}\pi\right)}{x_1^2 - 2x_1 \cos \left(\psi - \frac{2}{3}\pi\right) + 1} + \frac{\sin \left(\psi + \frac{2}{3}\pi\right)}{x_1^2 - 2x_1 \cos \left(\psi + \frac{2}{3}\pi\right) + 1} \right) (1 - x_1^2) + \\
&+ \frac{i}{x_1^2} \left(-\sin \psi \ln \sqrt{1 - 2x_1 \cos \psi + x_1^2} - \cos \psi \arctan \left(\frac{x_1 \sin \psi}{1 - x_1 \cos \psi} \right) + \right. \\
&+ \sin \psi \ln \sqrt{1 - 2x_1^3 \cos \psi + x_1^6} + \cos \psi \arctan \left(\frac{x_1^3 \sin \psi}{1 - x_1^3 \cos \psi} \right) + \\
&- \sin \left(\psi - \frac{2}{3}\pi\right) \ln \sqrt{1 - 2x_1 \cos \left(\psi - \frac{2}{3}\pi\right) + x_1^2} - \cos \left(\psi - \frac{2}{3}\pi\right) \arctan \left(\frac{x_1 \sin \left(\psi - \frac{2}{3}\pi\right)}{1 - x_1 \cos \left(\psi - \frac{2}{3}\pi\right)} \right) + \\
&+ \sin \left(\psi - \frac{2}{3}\pi\right) \ln \sqrt{1 - 2x_1^3 \cos \left(\psi - \frac{2}{3}\pi\right) + x_1^6} + \cos \left(\psi - \frac{2}{3}\pi\right) \arctan \left(\frac{x_1^3 \sin \left(\psi - \frac{2}{3}\pi\right)}{1 - x_1^3 \cos \left(\psi - \frac{2}{3}\pi\right)} \right) + \\
&- \sin \left(\psi + \frac{2}{3}\pi\right) \ln \sqrt{1 - 2x_1 \cos \left(\psi + \frac{2}{3}\pi\right) + x_1^2} - \cos \left(\psi + \frac{2}{3}\pi\right) \arctan \left(\frac{x_1 \sin \left(\psi + \frac{2}{3}\pi\right)}{1 - x_1 \cos \left(\psi + \frac{2}{3}\pi\right)} \right) + \\
&+ \sin \left(\psi + \frac{2}{3}\pi\right) \ln \sqrt{1 - 2x_1^3 \cos \left(\psi + \frac{2}{3}\pi\right) + x_1^6} + \cos \left(\psi + \frac{2}{3}\pi\right) \arctan \left(\frac{x_1^3 \sin \left(\psi + \frac{2}{3}\pi\right)}{1 - x_1^3 \cos \left(\psi + \frac{2}{3}\pi\right)} \right) \Big)
\end{aligned}$$

$$\begin{aligned}
&= i(1-x_1^2) \sum_{k=0}^2 \frac{\sin(\psi + \frac{2}{3}k\pi)}{x_1^2 - 2x_1 \cos(\psi + \frac{2}{3}k\pi) + 1} + \\
&\quad + \frac{i}{x_1^2} \sum_{k=0}^2 \left(-\sin(\psi + \frac{2}{3}k\pi) \ln \sqrt{1 - 2x_1 \cos(\psi + \frac{2}{3}k\pi) + x_1^2} + \right. \\
&\quad \quad \quad - \cos(\psi + \frac{2}{3}k\pi) \arctan \left(\frac{x_1 \sin(\psi + \frac{2}{3}k\pi)}{1 - x_1 \cos(\psi + \frac{2}{3}k\pi)} \right) + \\
&\quad \quad \quad + \sin(\psi + \frac{2}{3}k\pi) \ln \sqrt{1 - 2x_1^3 \cos(\psi + \frac{2}{3}k\pi) + x_1^6} + \\
&\quad \quad \quad \left. + \cos(\psi + \frac{2}{3}k\pi) \arctan \left(\frac{x_1^3 \sin(\psi + \frac{2}{3}k\pi)}{1 - x_1^3 \cos(\psi + \frac{2}{3}k\pi)} \right) \right) \tag{61}
\end{aligned}$$

5.5 Conclusion of case 2)

In the present case the enumerator of (5) becomes

$$\begin{aligned}
&\int_{z \in S} \lim_{x_2 \downarrow 0} \frac{G_S(x, z)}{x_2} \lim_{|y| \uparrow 1} \frac{G_S(z, y)}{1 - |y|^2} dz = \frac{2\pi i}{8\pi^2} (\#_{(59)} + \#_{(61)}) = \\
&= \frac{1}{4\pi} \left(\sin \psi \ln \sqrt{1 - 2x_1 \cos \psi + x_1^2} + \cos \psi \arctan \left(\frac{x_1 \sin \psi}{1 - x_1 \cos \psi} \right) + \right. \\
&\quad + \sin(\psi - \frac{2}{3}\pi) \ln \sqrt{1 - 2x_1 \cos(\psi - \frac{2}{3}\pi) + x_1^2} + \cos(\psi - \frac{2}{3}\pi) \arctan \left(\frac{x_1 \sin(\psi - \frac{2}{3}\pi)}{1 - x_1 \cos(\psi - \frac{2}{3}\pi)} \right) + \\
&\quad + \sin(\psi + \frac{2}{3}\pi) \ln \sqrt{1 - 2x_1 \cos(\psi + \frac{2}{3}\pi) + x_1^2} + \cos(\psi + \frac{2}{3}\pi) \arctan \left(\frac{x_1 \sin(\psi + \frac{2}{3}\pi)}{1 - x_1 \cos(\psi + \frac{2}{3}\pi)} \right) + \\
&\quad - \frac{\sin \psi}{x_1^2} \ln \sqrt{1 - 2x_1^3 \cos \psi + x_1^6} - \frac{\cos \psi}{x_1^2} \arctan \left(\frac{x_1^3 \sin \psi}{1 - x_1^3 \cos \psi} \right) + \\
&\quad - \frac{\sin(\psi - \frac{2}{3}\pi)}{x_1^2} \ln \sqrt{1 - 2x_1^3 \cos(\psi - \frac{2}{3}\pi) + x_1^6} + \frac{\cos(\psi - \frac{2}{3}\pi)}{x_1^2} \arctan \left(\frac{x_1^3 \sin(\psi - \frac{2}{3}\pi)}{1 - x_1^3 \cos(\psi - \frac{2}{3}\pi)} \right) + \\
&\quad \left. - \frac{\sin(\psi + \frac{2}{3}\pi)}{x_1^2} \ln \sqrt{1 - 2x_1^3 \cos(\psi + \frac{2}{3}\pi) + x_1^6} + \frac{\cos(\psi + \frac{2}{3}\pi)}{x_1^2} \arctan \left(\frac{x_1^3 \sin(\psi + \frac{2}{3}\pi)}{1 - x_1^3 \cos(\psi + \frac{2}{3}\pi)} \right) \right) + \\
&\quad + \frac{1}{4\pi} \left(\frac{\sin \psi}{x_1^2 - 2x_1 \cos \psi + 1} + \frac{\sin(\psi - \frac{2}{3}\pi)}{x_1^2 - 2x_1 \cos(\psi - \frac{2}{3}\pi) + 1} + \frac{\sin(\psi + \frac{2}{3}\pi)}{x_1^2 - 2x_1 \cos(\psi + \frac{2}{3}\pi) + 1} \right) (1 - x_1^2) + \\
&\quad + \frac{1}{4\pi x_1^2} \left(-\sin \psi \ln \sqrt{1 - 2x_1 \cos \psi + x_1^2} - \cos \psi \arctan \left(\frac{x_1 \sin \psi}{1 - x_1 \cos \psi} \right) + \right. \\
&\quad \left. + \sin \psi \ln \sqrt{1 - 2x_1^3 \cos \psi + x_1^6} + \cos \psi \arctan \left(\frac{x_1^3 \sin \psi}{1 - x_1^3 \cos \psi} \right) + \right.
\end{aligned}$$

$$\begin{aligned}
& - \sin\left(\psi - \frac{2}{3}\pi\right) \ln \sqrt{1 - 2x_1 \cos\left(\psi - \frac{2}{3}\pi\right) + x_1^2} - \cos\left(\psi - \frac{2}{3}\pi\right) \arctan\left(\frac{x_1 \sin\left(\psi - \frac{2}{3}\pi\right)}{1 - x_1 \cos\left(\psi - \frac{2}{3}\pi\right)}\right) + \\
& + \sin\left(\psi - \frac{2}{3}\pi\right) \ln \sqrt{1 - 2x_1^3 \cos\left(\psi - \frac{2}{3}\pi\right) + x_1^6} + \cos\left(\psi - \frac{2}{3}\pi\right) \arctan\left(\frac{x_1^3 \sin\left(\psi - \frac{2}{3}\pi\right)}{1 - x_1^3 \cos\left(\psi - \frac{2}{3}\pi\right)}\right) + \\
& - \sin\left(\psi + \frac{2}{3}\pi\right) \ln \sqrt{1 - 2x_1 \cos\left(\psi + \frac{2}{3}\pi\right) + x_1^2} - \cos\left(\psi + \frac{2}{3}\pi\right) \arctan\left(\frac{x_1 \sin\left(\psi + \frac{2}{3}\pi\right)}{1 - x_1 \cos\left(\psi + \frac{2}{3}\pi\right)}\right) + \\
& + \sin\left(\psi + \frac{2}{3}\pi\right) \ln \sqrt{1 - 2x_1^3 \cos\left(\psi + \frac{2}{3}\pi\right) + x_1^6} + \cos\left(\psi + \frac{2}{3}\pi\right) \arctan\left(\frac{x_1^3 \sin\left(\psi + \frac{2}{3}\pi\right)}{1 - x_1^3 \cos\left(\psi + \frac{2}{3}\pi\right)}\right) \\
= & \frac{1 - x_1^{-2}}{\pi} \sin \psi \ln \sqrt{1 - 2x_1 \cos \psi + x_1^2} + \frac{1 - x_1^{-2}}{\pi} \cos \psi \arctan\left(\frac{x_1 \sin \psi}{1 - x_1 \cos \psi}\right) + \\
& + \frac{1 - x_1^{-2}}{\pi} \sin\left(\psi - \frac{2}{3}\pi\right) \ln \sqrt{1 - 2x_1 \cos\left(\psi - \frac{2}{3}\pi\right) + x_1^2} + \\
& + \frac{1 - x_1^{-2}}{\pi} \cos\left(\psi - \frac{2}{3}\pi\right) \arctan\left(\frac{x_1 \sin\left(\psi - \frac{2}{3}\pi\right)}{1 - x_1 \cos\left(\psi - \frac{2}{3}\pi\right)}\right) + \\
& + \frac{1 - x_1^{-2}}{\pi} \sin\left(\psi + \frac{2}{3}\pi\right) \ln \sqrt{1 - 2x_1 \cos\left(\psi + \frac{2}{3}\pi\right) + x_1^2} + \\
& + \frac{1 - x_1^{-2}}{\pi} \cos\left(\psi + \frac{2}{3}\pi\right) \arctan\left(\frac{x_1 \sin\left(\psi + \frac{2}{3}\pi\right)}{1 - x_1 \cos\left(\psi + \frac{2}{3}\pi\right)}\right) + \\
& + \frac{1 - x_1^2}{\pi} \frac{\sin \psi}{x_1^2 - 2x_1 \cos \psi + 1} + \frac{1 - x_1^2}{\pi} \frac{\sin\left(\psi - \frac{2}{3}\pi\right)}{x_1^2 - 2x_1 \cos\left(\psi - \frac{2}{3}\pi\right) + 1} + \frac{1 - x_1^2}{\pi} \frac{\sin\left(\psi + \frac{2}{3}\pi\right)}{x_1^2 - 2x_1 \cos\left(\psi + \frac{2}{3}\pi\right) + 1} \\
= & - \frac{1 - x_1^2}{\pi} \frac{\cos \psi \arctan\left(\frac{x_1 \sin \psi}{1 - x_1 \cos \psi}\right) + \sin \psi \ln \sqrt{1 - 2x_1 \cos \psi + x_1^2}}{x_1^2} + \\
& - \frac{1 - x_1^2}{\pi} \frac{\cos\left(\psi - \frac{2}{3}\pi\right) \arctan\left(\frac{x_1 \sin\left(\psi - \frac{2}{3}\pi\right)}{1 - x_1 \cos\left(\psi - \frac{2}{3}\pi\right)}\right) + \sin\left(\psi - \frac{2}{3}\pi\right) \ln \sqrt{1 - 2x_1 \cos\left(\psi - \frac{2}{3}\pi\right) + x_1^2}}{x_1^2} + \\
& - \frac{1 - x_1^2}{\pi} \frac{\cos\left(\psi + \frac{2}{3}\pi\right) \arctan\left(\frac{x_1 \sin\left(\psi + \frac{2}{3}\pi\right)}{1 - x_1 \cos\left(\psi + \frac{2}{3}\pi\right)}\right) + \sin\left(\psi + \frac{2}{3}\pi\right) \ln \sqrt{1 - 2x_1 \cos\left(\psi + \frac{2}{3}\pi\right) + x_1^2}}{x_1^2} + \\
& + \frac{1 - x_1^2}{\pi} \left(\frac{\sin \psi}{x_1^2 - 2x_1 \cos \psi + 1} + \frac{\sin\left(\psi - \frac{2}{3}\pi\right)}{x_1^2 - 2x_1 \cos\left(\psi - \frac{2}{3}\pi\right) + 1} + \frac{\sin\left(\psi + \frac{2}{3}\pi\right)}{x_1^2 - 2x_1 \cos\left(\psi + \frac{2}{3}\pi\right) + 1} \right) \\
= & - \frac{1 - x_1^2}{\pi x_1^2} \sum_{k=0}^2 \cos\left(\psi + \frac{2}{3}k\pi\right) \arctan\left(\frac{x_1 \sin\left(\psi + \frac{2}{3}k\pi\right)}{1 - x_1 \cos\left(\psi + \frac{2}{3}k\pi\right)}\right) + \\
& + \frac{1 - x_1^2}{\pi x_1^2} \sum_{k=0}^2 \sin\left(\psi + \frac{2}{3}k\pi\right) \ln \sqrt{1 - 2x_1 \cos\left(\psi + \frac{2}{3}k\pi\right) + x_1^2} + \\
& + \frac{1 - x_1^2}{\pi} \sum_{k=0}^2 \frac{\sin\left(\psi + \frac{2}{3}k\pi\right)}{x_1^2 - 2x_1 \cos\left(\psi + \frac{2}{3}k\pi\right) + 1}. \tag{62}
\end{aligned}$$

Together with the denominator (41) we find

$$\begin{aligned}
T_{12}(x_1, \psi) &= \frac{\#(62)}{\#(41)} = \\
&= \frac{\frac{\sin \psi}{x_1^2 - 2x_1 \cos \psi + 1} + \frac{\sin(\psi - \frac{2}{3}\pi)}{x_1^2 - 2x_1 \cos(\psi - \frac{2}{3}\pi) + 1} + \frac{\sin(\psi + \frac{2}{3}\pi)}{x_1^2 - 2x_1 \cos(\psi + \frac{2}{3}\pi) + 1}}{4 \left(\frac{\sin \psi}{(x_1^2 - 2x_1 \cos \psi + 1)^2} + \frac{\sin(\psi - \frac{2}{3}\pi)}{(x_1^2 - 2x_1 \cos(\psi - \frac{2}{3}\pi) + 1)^2} + \frac{\sin(\psi + \frac{2}{3}\pi)}{(x_1^2 - 2x_1 \cos(\psi + \frac{2}{3}\pi) + 1)^2} \right)} + \\
&\quad - \frac{\cos \psi \arctan\left(\frac{x_1 \sin \psi}{1 - x_1 \cos \psi}\right) + \sin \psi \ln \sqrt{1 - 2x_1 \cos \psi + x_1^2}}{4x_1^2 \left(\frac{\sin \psi}{(x_1^2 - 2x_1 \cos \psi + 1)^2} + \frac{\sin(\psi - \frac{2}{3}\pi)}{(x_1^2 - 2x_1 \cos(\psi - \frac{2}{3}\pi) + 1)^2} + \frac{\sin(\psi + \frac{2}{3}\pi)}{(x_1^2 - 2x_1 \cos(\psi + \frac{2}{3}\pi) + 1)^2} \right)} + \\
&\quad - \frac{\cos(\psi - \frac{2}{3}\pi) \arctan\left(\frac{x_1 \sin(\psi - \frac{2}{3}\pi)}{1 - x_1 \cos(\psi - \frac{2}{3}\pi)}\right) + \sin(\psi - \frac{2}{3}\pi) \ln \sqrt{1 - 2x_1 \cos(\psi - \frac{2}{3}\pi) + x_1^2}}{4x_1^2 \left(\frac{\sin \psi}{(x_1^2 - 2x_1 \cos \psi + 1)^2} + \frac{\sin(\psi - \frac{2}{3}\pi)}{(x_1^2 - 2x_1 \cos(\psi - \frac{2}{3}\pi) + 1)^2} + \frac{\sin(\psi + \frac{2}{3}\pi)}{(x_1^2 - 2x_1 \cos(\psi + \frac{2}{3}\pi) + 1)^2} \right)} + \\
&\quad - \frac{\cos(\psi + \frac{2}{3}\pi) \arctan\left(\frac{x_1 \sin(\psi + \frac{2}{3}\pi)}{1 - x_1 \cos(\psi + \frac{2}{3}\pi)}\right) + \sin(\psi + \frac{2}{3}\pi) \ln \sqrt{1 - 2x_1 \cos(\psi + \frac{2}{3}\pi) + x_1^2}}{4x_1^2 \left(\frac{\sin \psi}{(x_1^2 - 2x_1 \cos \psi + 1)^2} + \frac{\sin(\psi - \frac{2}{3}\pi)}{(x_1^2 - 2x_1 \cos(\psi - \frac{2}{3}\pi) + 1)^2} + \frac{\sin(\psi + \frac{2}{3}\pi)}{(x_1^2 - 2x_1 \cos(\psi + \frac{2}{3}\pi) + 1)^2} \right)}.
\end{aligned}$$

and a careful analysis shows that

$$\sup_{\substack{0 < x_1 < 1 \\ 0 < \psi < \frac{1}{3}\pi}} T_{12}(x_1, \psi) = \lim_{x_1 \downarrow 0} T_{12}(x_1, \alpha) = \frac{1}{16}$$

holds for all $\alpha \in (0, \frac{1}{3}\pi)$. Aside from this global maximum T_{12} has a local maximum at $x_1 = 1$ and $\psi = \frac{1}{3}\pi$, but

$$T_{12}\left(1, \frac{1}{3}\pi\right) = \frac{5}{27} - \frac{4}{243}\sqrt{3}\pi - \frac{4}{81}\ln 2 < \frac{1}{16}.$$

See Figure 2 on page 51.

6 The case that $x, y \in \Gamma_2$

The third section is concerned with $x \rightarrow (\cos \phi, \sin \phi)$ and $y \rightarrow (\cos \psi, \sin \psi)$. We study

$$T_{22}(\phi, \psi) := \frac{\int_S \lim_{|x| \uparrow 1} \frac{G_S(x, z)}{1 - |x|^2} \lim_{|y| \uparrow 1} \frac{G_S(z, y)}{1 - |y|^2} dz}{\lim_{\substack{|x| \uparrow 1 \\ |y| \uparrow 1}} \frac{G_S(x, y)}{(1 - |x|^2)(1 - |y|^2)}}. \quad (63)$$

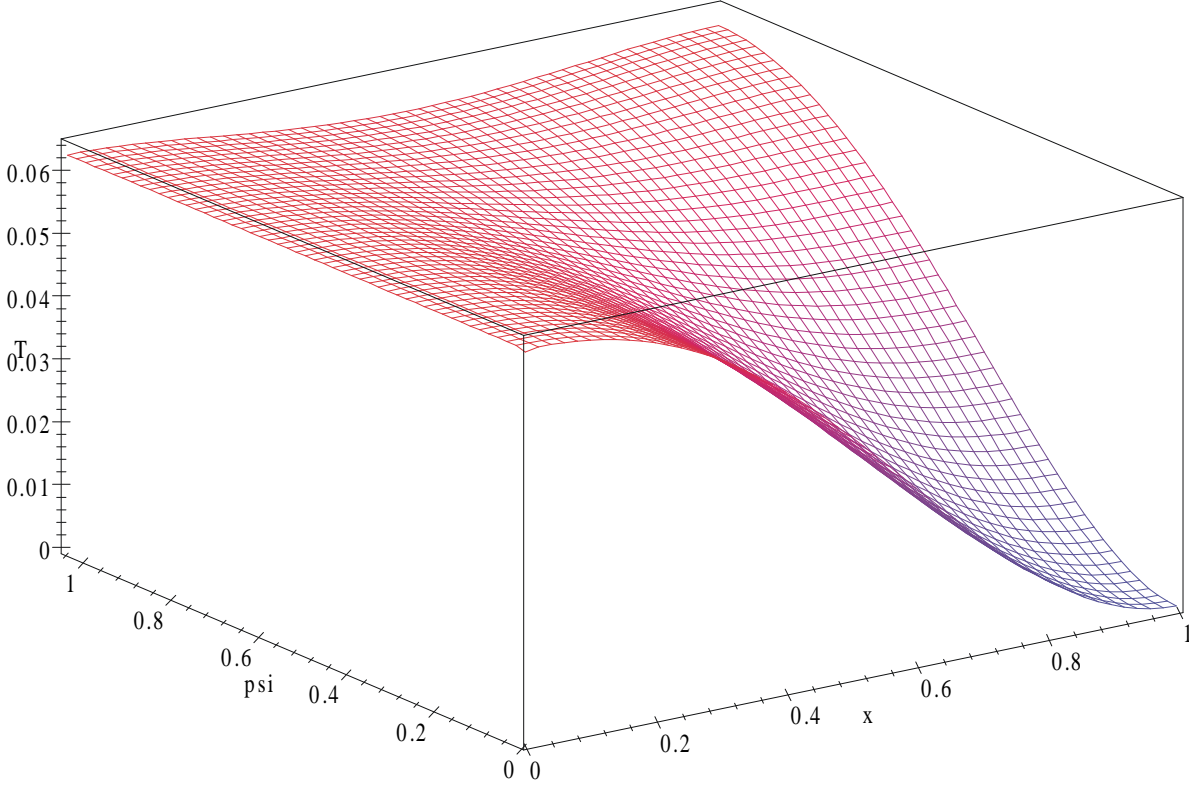


Figure 2: $x \in \Gamma_1$ and $y \in \Gamma_2$: $T_{12}(x_1, \psi)$.

6.1 Limit of the Green function

By (7) the denominator becomes

$$\begin{aligned}
 & \lim_{\substack{y \rightarrow (\cos \psi, \sin \psi) \\ x \rightarrow (\cos \phi, \sin \phi)}} \frac{G_S(x, y)}{(1 - |y|^2)(1 - |x|^2)} = \\
 & = \frac{1}{8\pi} \sum_{k=0}^2 \left(\frac{1}{1 - \cos(\psi + \phi + \frac{2}{3}k\pi)} - \frac{1}{1 - \cos(\psi - \phi + \frac{2}{3}k\pi)} \right), \tag{64}
 \end{aligned}$$

while for the numerator we have as in case 2, denoting $z = (r \cos \theta, r \sin \theta)$,

$$\begin{aligned}
 & \lim_{y \rightarrow (\cos \psi, \sin \psi)} \frac{G_S(z, y)}{1 - |y|^2} = \\
 & = \frac{1 - r^2}{4\pi} \left(\frac{1}{r^2 - 2r \cos(\theta + \psi) + 1} - \frac{1}{r^2 - 2r \cos(\theta - \psi) + 1} \right) \\
 & + \frac{1 - r^2}{4\pi} \left(\frac{1}{r^2 - 2r \cos(\theta + \psi + \frac{2}{3}\pi) + 1} - \frac{1}{r^2 - 2r \cos(\theta - \psi + \frac{2}{3}\pi) + 1} \right) + \\
 & + \frac{1 - r^2}{4\pi} \left(\frac{1}{r^2 - 2r \cos(\theta + \psi - \frac{2}{3}\pi) + 1} - \frac{1}{r^2 - 2r \cos(\theta - \psi - \frac{2}{3}\pi) + 1} \right),
 \end{aligned}$$

and by using symmetry

$$\begin{aligned}
& \lim_{x \rightarrow (\cos \phi, \sin \phi)} \frac{G_S(x, z)}{1 - |x|^2} = \\
&= \frac{1 - r^2}{4\pi} \left(\frac{1}{r^2 - 2r \cos(\theta + \phi) + 1} - \frac{1}{r^2 - 2r \cos(\theta - \phi) + 1} \right) \\
&+ \frac{1 - r^2}{4\pi} \left(\frac{1}{r^2 - 2r \cos(\theta + \phi + \frac{2}{3}\pi) + 1} - \frac{1}{r^2 - 2r \cos(\theta - \phi + \frac{2}{3}\pi) + 1} \right) + \\
&+ \frac{1 - r^2}{4\pi} \left(\frac{1}{r^2 - 2r \cos(\theta + \phi - \frac{2}{3}\pi) + 1} - \frac{1}{r^2 - 2r \cos(\theta - \phi - \frac{2}{3}\pi) + 1} \right).
\end{aligned}$$

6.2 Derivation of a contour integral

We want to compute

$$\begin{aligned}
& \pi^2 \int_{z \in S} \lim_{x \rightarrow (\cos \phi, \sin \phi)} \frac{G_S(x, z)}{1 - |x|^2} \lim_{\substack{y \rightarrow (\cos \psi, \sin \psi) \\ y \in S}} \frac{G_S(z, y)}{1 - |y|^2} dz = \\
&= \int_{r=0}^1 \int_{\theta=0}^{\frac{1}{3}\pi} \sum_{k=0}^2 g\left(\theta + k\frac{2}{3}\pi\right) \sum_{m=0}^2 f\left(\theta + m\frac{2}{3}\pi\right) d\theta r dr
\end{aligned} \tag{65}$$

with f as in (42) and

$$g(\theta) = \frac{1 - r^2}{4} \left(\frac{1}{r^2 - 2r \cos(\theta + \phi) + 1} - \frac{1}{r^2 - 2r \cos(\theta - \phi) + 1} \right). \tag{66}$$

Again we find

$$\begin{aligned}
& \int_{\theta=0}^{\frac{1}{3}\pi} \sum_{k=0}^2 g\left(\theta + k\frac{2}{3}\pi\right) \sum_{m=0}^2 f\left(\theta + m\frac{2}{3}\pi\right) d\theta = \\
&= \frac{1}{6} \int_{\theta=0}^{2\pi} \sum_{k=0}^2 g\left(\theta + k\frac{2}{3}\pi\right) \sum_{m=0}^2 f\left(\theta + m\frac{2}{3}\pi\right) d\theta = \\
&= \frac{1}{2} \int_{\theta=0}^{2\pi} \sum_{k=0}^2 g\left(\theta + k\frac{2}{3}\pi\right) f(\theta) d\theta = \\
&= \frac{1}{2} \int_{\theta=0}^{2\pi} \sum_{k=0}^2 g\left(\theta + k\frac{2}{3}\pi\right) \left(f_\psi(\theta) - f_{-\psi}(\theta) \right) d\theta,
\end{aligned}$$

which means that with $w = re^{i\theta}$ as before

$$\begin{aligned}
f_\psi(\theta) &= \frac{1}{4} \left(\frac{r^2 e^{-i\psi}}{w - r^2 e^{-i\psi}} - \frac{e^{-i\psi}}{w - e^{-i\psi}} \right), \\
g(\theta) &= \frac{1}{4} \left(\frac{r^2 e^{-i\phi}}{w - r^2 e^{-i\phi}} - \frac{e^{-i\phi}}{w - e^{-i\phi}} - \frac{r^2 e^{i\phi}}{w - r^2 e^{i\phi}} + \frac{e^{i\phi}}{w - e^{i\phi}} \right), \\
g\left(\theta - \frac{2}{3}\pi\right) &= \frac{1}{4} \left(\frac{r^2 e^{-i\phi - \frac{2}{3}\pi i}}{w - r^2 e^{-i\phi - \frac{2}{3}\pi i}} - \frac{e^{-i\phi - \frac{2}{3}\pi i}}{w - e^{-i\phi - \frac{2}{3}\pi i}} - \frac{r^2 e^{i\phi - \frac{2}{3}\pi i}}{w - r^2 e^{i\phi - \frac{2}{3}\pi i}} + \frac{e^{i\phi - \frac{2}{3}\pi i}}{w - e^{i\phi - \frac{2}{3}\pi i}} \right), \\
g\left(\theta + \frac{2}{3}\pi\right) &= \frac{1}{4} \left(\frac{r^2 e^{-i\phi + \frac{2}{3}\pi i}}{w - r^2 e^{-i\phi + \frac{2}{3}\pi i}} - \frac{e^{-i\phi + \frac{2}{3}\pi i}}{w - e^{-i\phi + \frac{2}{3}\pi i}} - \frac{r^2 e^{i\phi + \frac{2}{3}\pi i}}{w - r^2 e^{i\phi + \frac{2}{3}\pi i}} + \frac{e^{i\phi + \frac{2}{3}\pi i}}{w - e^{i\phi + \frac{2}{3}\pi i}} \right).
\end{aligned}$$

Then

$$\begin{aligned}
& \pi^2 \int_{z \in S} \lim_{x \rightarrow (\cos \phi, \sin \phi)} \frac{G_S(x, z)}{1 - |x|^2} \lim_{y \rightarrow (\cos \psi, \sin \psi)} \frac{G_S(z, y)}{1 - |y|^2} dz = \\
&= \frac{1}{2} \int_{r=0}^1 \oint_{|w|=r} \left(\frac{1}{4} \left(\frac{r^2 e^{-i\phi}}{w - r^2 e^{-i\phi}} - \frac{e^{-i\phi}}{w - e^{-i\phi}} - \frac{r^2 e^{i\phi}}{w - r^2 e^{i\phi}} + \frac{e^{i\phi}}{w - e^{i\phi}} \right) \right. \\
& \quad \frac{1}{4} \left(\frac{r^2 e^{-i\phi - \frac{2}{3}\pi i}}{w - r^2 e^{-i\phi - \frac{2}{3}\pi i}} - \frac{e^{-i\phi - \frac{2}{3}\pi i}}{w - e^{-i\phi - \frac{2}{3}\pi i}} - \frac{r^2 e^{i\phi - \frac{2}{3}\pi i}}{w - r^2 e^{i\phi - \frac{2}{3}\pi i}} + \frac{e^{i\phi - \frac{2}{3}\pi i}}{w - e^{i\phi - \frac{2}{3}\pi i}} \right) \\
& \quad \left. \frac{1}{4} \left(\frac{r^2 e^{-i\phi + \frac{2}{3}\pi i}}{w - r^2 e^{-i\phi + \frac{2}{3}\pi i}} - \frac{e^{-i\phi + \frac{2}{3}\pi i}}{w - e^{-i\phi + \frac{2}{3}\pi i}} - \frac{r^2 e^{i\phi + \frac{2}{3}\pi i}}{w - r^2 e^{i\phi + \frac{2}{3}\pi i}} + \frac{e^{i\phi + \frac{2}{3}\pi i}}{w - e^{i\phi + \frac{2}{3}\pi i}} \right) \right) \times \\
& \quad \times \frac{1}{4} \left(\frac{r^2 e^{-i\psi}}{w - r^2 e^{-i\psi}} - \frac{e^{-i\psi}}{w - e^{-i\psi}} - \frac{r^2 e^{i\psi}}{w - r^2 e^{i\psi}} + \frac{e^{i\psi}}{w - e^{i\psi}} \right) \frac{dw}{iw} r dr.
\end{aligned}$$

6.3 Computation of the contour integral

Let us first consider

$$\frac{1}{2} \int_{\theta=0}^{2\pi} \sum_{k=0}^2 g(\theta + k\frac{2}{3}\pi) f_\psi(\theta) d\theta.$$

The integrand has the following poles within the unit circle:

$$P = \left\{ r^2 e^{-i\phi}, r^2 e^{i\phi}, r^2 e^{-i\phi - \frac{2}{3}\pi i}, r^2 e^{i\phi - \frac{2}{3}\pi i}, r^2 e^{-i\phi + \frac{2}{3}\pi i}, r^2 e^{i\phi + \frac{2}{3}\pi i} \right\} \text{ and } \{0, r^2 e^{-i\psi}\}.$$

With the exception of 0 all these poles satisfy $|w| = r^2 < r$. Hence, independently of r , we find

$$\begin{aligned}
& \frac{1}{32i} \oint_{|w|=r} \left(\frac{r^2 e^{-i\phi}}{w - r^2 e^{-i\phi}} - \frac{e^{-i\phi}}{w - e^{-i\phi}} - \frac{r^2 e^{i\phi}}{w - r^2 e^{i\phi}} + \frac{e^{i\phi}}{w - e^{i\phi}} + \right. \\
& \quad + \frac{r^2 e^{-i\phi - \frac{2}{3}\pi i}}{w - r^2 e^{-i\phi - \frac{2}{3}\pi i}} - \frac{e^{-i\phi - \frac{2}{3}\pi i}}{w - e^{-i\phi - \frac{2}{3}\pi i}} - \frac{r^2 e^{i\phi - \frac{2}{3}\pi i}}{w - r^2 e^{i\phi - \frac{2}{3}\pi i}} + \frac{e^{i\phi - \frac{2}{3}\pi i}}{w - e^{i\phi - \frac{2}{3}\pi i}} + \\
& \quad \left. + \frac{r^2 e^{-i\phi + \frac{2}{3}\pi i}}{w - r^2 e^{-i\phi + \frac{2}{3}\pi i}} - \frac{e^{-i\phi + \frac{2}{3}\pi i}}{w - e^{-i\phi + \frac{2}{3}\pi i}} - \frac{r^2 e^{i\phi + \frac{2}{3}\pi i}}{w - r^2 e^{i\phi + \frac{2}{3}\pi i}} + \frac{e^{i\phi + \frac{2}{3}\pi i}}{w - e^{i\phi + \frac{2}{3}\pi i}} \right) \times \\
& \quad \times \left(\frac{r^2 e^{-i\psi}}{w - r^2 e^{-i\psi}} - \frac{e^{-i\psi}}{w - e^{-i\psi}} \right) \frac{1}{w} dw. \\
&= \frac{\pi}{16} \sum_{w_i \in P \cup \{0, r^2 e^{-i\psi}\}} \text{Res} \{F_\psi(w)\}_{w=w_i}
\end{aligned}$$

with F_ψ the integrand. Again the contribution by $w_1 = 0$ cancels and we find

$$\begin{aligned}
& \sum_{w_i \in P \cup \{r^2 e^{-i\psi}\}} \text{Res} \{F_\psi(w)\}_{w=w_i} = \\
& = \left(\frac{r^2 e^{-i\psi}}{w - r^2 e^{-i\psi}} - \frac{e^{-i\psi}}{w - e^{-i\psi}} \right)_{w=r^2 e^{-i\phi}} - \left(\frac{r^2 e^{-i\psi}}{w - r^2 e^{-i\psi}} - \frac{e^{-i\psi}}{w - e^{-i\psi}} \right)_{w=r^2 e^{i\phi}} + \\
& + \left(\frac{r^2 e^{-i\psi}}{w - r^2 e^{-i\psi}} - \frac{e^{-i\psi}}{w - e^{-i\psi}} \right)_{w=r^2 e^{-i\phi - \frac{2}{3}\pi i}} - \left(\frac{r^2 e^{-i\psi}}{w - r^2 e^{-i\psi}} - \frac{e^{-i\psi}}{w - e^{-i\psi}} \right)_{w=r^2 e^{i\phi - \frac{2}{3}\pi i}} + \\
& + \left(\frac{r^2 e^{-i\psi}}{w - r^2 e^{-i\psi}} - \frac{e^{-i\psi}}{w - e^{-i\psi}} \right)_{w=r^2 e^{-i\phi + \frac{2}{3}\pi i}} - \left(\frac{r^2 e^{-i\psi}}{w - r^2 e^{-i\psi}} - \frac{e^{-i\psi}}{w - e^{-i\psi}} \right)_{w=r^2 e^{i\phi + \frac{2}{3}\pi i}} + \\
& + \left(\frac{r^2 e^{-i\phi}}{w - r^2 e^{-i\phi}} - \frac{e^{-i\phi}}{w - e^{-i\phi}} - \frac{r^2 e^{i\phi}}{w - r^2 e^{i\phi}} + \frac{e^{i\phi}}{w - e^{i\phi}} + \right. \\
& + \frac{r^2 e^{-i\phi - \frac{2}{3}\pi i}}{w - r^2 e^{-i\phi - \frac{2}{3}\pi i}} - \frac{e^{-i\phi - \frac{2}{3}\pi i}}{w - e^{-i\phi - \frac{2}{3}\pi i}} - \frac{r^2 e^{i\phi - \frac{2}{3}\pi i}}{w - r^2 e^{i\phi - \frac{2}{3}\pi i}} + \frac{e^{i\phi - \frac{2}{3}\pi i}}{w - e^{i\phi - \frac{2}{3}\pi i}} + \\
& \left. + \frac{r^2 e^{-i\phi + \frac{2}{3}\pi i}}{w - r^2 e^{-i\phi + \frac{2}{3}\pi i}} - \frac{e^{-i\phi + \frac{2}{3}\pi i}}{w - e^{-i\phi + \frac{2}{3}\pi i}} - \frac{r^2 e^{i\phi + \frac{2}{3}\pi i}}{w - r^2 e^{i\phi + \frac{2}{3}\pi i}} + \frac{e^{i\phi + \frac{2}{3}\pi i}}{w - e^{i\phi + \frac{2}{3}\pi i}} \right)_{w=r^2 e^{-i\psi}} \\
& = \frac{r^2 e^{-i\psi}}{r^2 e^{-i\phi} - r^2 e^{-i\psi}} - \frac{e^{-i\psi}}{r^2 e^{-i\phi} - e^{-i\psi}} - \frac{r^2 e^{-i\psi}}{r^2 e^{i\phi} - r^2 e^{-i\psi}} + \frac{e^{-i\psi}}{r^2 e^{i\phi} - e^{-i\psi}} + \\
& + \frac{r^2 e^{-i\psi}}{r^2 e^{-i\phi - \frac{2}{3}\pi i} - r^2 e^{-i\psi}} - \frac{e^{-i\psi}}{r^2 e^{-i\phi - \frac{2}{3}\pi i} - e^{-i\psi}} - \frac{r^2 e^{-i\psi}}{r^2 e^{i\phi - \frac{2}{3}\pi i} - r^2 e^{-i\psi}} + \frac{e^{-i\psi}}{r^2 e^{i\phi - \frac{2}{3}\pi i} - e^{-i\psi}} + \\
& + \frac{r^2 e^{-i\psi}}{r^2 e^{-i\phi + \frac{2}{3}\pi i} - r^2 e^{-i\psi}} - \frac{e^{-i\psi}}{r^2 e^{-i\phi + \frac{2}{3}\pi i} - e^{-i\psi}} - \frac{r^2 e^{-i\psi}}{r^2 e^{i\phi + \frac{2}{3}\pi i} - r^2 e^{-i\psi}} + \frac{e^{-i\psi}}{r^2 e^{i\phi + \frac{2}{3}\pi i} - e^{-i\psi}} + \\
& + \frac{r^2 e^{-i\phi}}{r^2 e^{-i\psi} - r^2 e^{-i\phi}} - \frac{e^{-i\phi}}{r^2 e^{-i\psi} - e^{-i\phi}} - \frac{r^2 e^{i\phi}}{r^2 e^{-i\psi} - r^2 e^{i\phi}} + \frac{e^{i\phi}}{r^2 e^{-i\psi} - e^{i\phi}} + \\
& + \frac{r^2 e^{-i\phi - \frac{2}{3}\pi i}}{r^2 e^{-i\psi} - r^2 e^{-i\phi - \frac{2}{3}\pi i}} - \frac{e^{-i\phi - \frac{2}{3}\pi i}}{r^2 e^{-i\psi} - e^{-i\phi - \frac{2}{3}\pi i}} - \frac{r^2 e^{i\phi - \frac{2}{3}\pi i}}{r^2 e^{-i\psi} - r^2 e^{i\phi - \frac{2}{3}\pi i}} + \frac{e^{i\phi - \frac{2}{3}\pi i}}{r^2 e^{-i\psi} - e^{i\phi - \frac{2}{3}\pi i}} + \\
& + \frac{r^2 e^{-i\phi + \frac{2}{3}\pi i}}{r^2 e^{-i\psi} - r^2 e^{-i\phi + \frac{2}{3}\pi i}} - \frac{e^{-i\phi + \frac{2}{3}\pi i}}{r^2 e^{-i\psi} - e^{-i\phi + \frac{2}{3}\pi i}} - \frac{r^2 e^{i\phi + \frac{2}{3}\pi i}}{r^2 e^{-i\psi} - r^2 e^{i\phi + \frac{2}{3}\pi i}} + \frac{e^{i\phi + \frac{2}{3}\pi i}}{r^2 e^{-i\psi} - e^{i\phi + \frac{2}{3}\pi i}} \\
& = \frac{e^{-i\psi}}{e^{-i\phi} - e^{-i\psi}} - \frac{e^{-i\psi}}{r^2 e^{-i\phi} - e^{-i\psi}} - \frac{e^{-i\psi}}{e^{i\phi} - e^{-i\psi}} + \frac{e^{-i\psi}}{r^2 e^{i\phi} - e^{-i\psi}} + \\
& + \frac{e^{-i\psi}}{e^{-i\phi - \frac{2}{3}\pi i} - e^{-i\psi}} - \frac{e^{-i\psi}}{r^2 e^{-i\phi - \frac{2}{3}\pi i} - e^{-i\psi}} - \frac{e^{-i\psi}}{e^{i\phi - \frac{2}{3}\pi i} - e^{-i\psi}} + \frac{e^{-i\psi}}{r^2 e^{i\phi - \frac{2}{3}\pi i} - e^{-i\psi}} + \\
& + \frac{e^{-i\psi}}{e^{-i\phi + \frac{2}{3}\pi i} - e^{-i\psi}} - \frac{e^{-i\psi}}{r^2 e^{-i\phi + \frac{2}{3}\pi i} - e^{-i\psi}} - \frac{e^{-i\psi}}{e^{i\phi + \frac{2}{3}\pi i} - e^{-i\psi}} + \frac{e^{-i\psi}}{r^2 e^{i\phi + \frac{2}{3}\pi i} - e^{-i\psi}} + \\
& + \frac{e^{-i\phi}}{e^{-i\psi} - e^{-i\phi}} - \frac{e^{-i\phi}}{r^2 e^{-i\psi} - e^{-i\phi}} - \frac{e^{i\phi}}{e^{-i\psi} - e^{i\phi}} + \frac{e^{i\phi}}{r^2 e^{-i\psi} - e^{i\phi}} + \\
& + \frac{e^{-i\phi - \frac{2}{3}\pi i}}{e^{-i\psi} - e^{-i\phi - \frac{2}{3}\pi i}} - \frac{e^{-i\phi - \frac{2}{3}\pi i}}{r^2 e^{-i\psi} - e^{-i\phi - \frac{2}{3}\pi i}} - \frac{e^{i\phi - \frac{2}{3}\pi i}}{e^{-i\psi} - e^{i\phi - \frac{2}{3}\pi i}} + \frac{e^{i\phi - \frac{2}{3}\pi i}}{r^2 e^{-i\psi} - e^{i\phi - \frac{2}{3}\pi i}} + \\
& + \frac{e^{-i\phi + \frac{2}{3}\pi i}}{e^{-i\psi} - e^{-i\phi + \frac{2}{3}\pi i}} - \frac{e^{-i\phi + \frac{2}{3}\pi i}}{r^2 e^{-i\psi} - e^{-i\phi + \frac{2}{3}\pi i}} - \frac{e^{i\phi + \frac{2}{3}\pi i}}{e^{-i\psi} - e^{i\phi + \frac{2}{3}\pi i}} + \frac{e^{i\phi + \frac{2}{3}\pi i}}{r^2 e^{-i\psi} - e^{i\phi + \frac{2}{3}\pi i}}
\end{aligned}$$

$$\begin{aligned}
&= \frac{e^{-i\psi}}{e^{-i\phi} - e^{-i\psi}} - \frac{e^{-i\psi}}{e^{i\phi} - e^{-i\psi}} + \frac{e^{-i\phi}}{e^{-i\psi} - e^{-i\phi}} - \frac{e^{i\phi}}{e^{-i\psi} - e^{i\phi}} + \\
&+ \frac{e^{-i\psi}}{e^{-i\phi - \frac{2}{3}\pi i} - e^{-i\psi}} - \frac{e^{-i\psi}}{e^{i\phi - \frac{2}{3}\pi i} - e^{-i\psi}} + \frac{e^{-i\psi}}{e^{-i\phi + \frac{2}{3}\pi i} - e^{-i\psi}} - \frac{e^{-i\psi}}{e^{i\phi + \frac{2}{3}\pi i} - e^{-i\psi}} + \\
&+ \frac{e^{-i\phi - \frac{2}{3}\pi i}}{e^{-i\psi} - e^{-i\phi - \frac{2}{3}\pi i}} - \frac{e^{i\phi - \frac{2}{3}\pi i}}{e^{-i\psi} - e^{i\phi - \frac{2}{3}\pi i}} + \frac{e^{-i\phi + \frac{2}{3}\pi i}}{e^{-i\psi} - e^{-i\phi + \frac{2}{3}\pi i}} - \frac{e^{i\phi + \frac{2}{3}\pi i}}{e^{-i\psi} - e^{i\phi + \frac{2}{3}\pi i}} + \\
&- \frac{e^{i\phi - i\psi}}{r^2 - e^{i\phi - i\psi}} + \frac{e^{-i\phi - i\psi}}{r^2 - e^{-i\phi - i\psi}} - \frac{e^{i\phi - i\psi + \frac{2}{3}\pi i}}{r^2 - e^{i\phi - i\psi + \frac{2}{3}\pi i}} + \frac{e^{-i\phi - i\psi + \frac{2}{3}\pi i}}{r^2 - e^{-i\phi - i\psi + \frac{2}{3}\pi i}} + \\
&- \frac{e^{i\phi - i\psi - \frac{2}{3}\pi i}}{r^2 - e^{i\phi - i\psi - \frac{2}{3}\pi i}} + \frac{e^{-i\phi - i\psi - \frac{2}{3}\pi i}}{r^2 - e^{-i\phi - i\psi - \frac{2}{3}\pi i}} - \frac{e^{-i\phi + i\psi}}{r^2 - e^{-i\phi + i\psi}} + \frac{e^{i\phi + i\psi}}{r^2 - e^{i\phi + i\psi}} + \\
&- \frac{e^{-i\phi + i\psi - \frac{2}{3}\pi i}}{r^2 - e^{-i\phi + i\psi - \frac{2}{3}\pi i}} + \frac{e^{i\phi + i\psi - \frac{2}{3}\pi i}}{r^2 - e^{i\phi + i\psi - \frac{2}{3}\pi i}} - \frac{e^{-i\phi + i\psi + \frac{2}{3}\pi i}}{r^2 - e^{-i\phi + i\psi + \frac{2}{3}\pi i}} + \frac{e^{i\phi + i\psi + \frac{2}{3}\pi i}}{r^2 - e^{i\phi + i\psi + \frac{2}{3}\pi i}} \\
&= - \frac{e^{i\phi - i\psi}}{r^2 - e^{i\phi - i\psi}} + \frac{e^{-i\phi - i\psi}}{r^2 - e^{-i\phi - i\psi}} - \frac{e^{i\phi - i\psi + \frac{2}{3}\pi i}}{r^2 - e^{i\phi - i\psi + \frac{2}{3}\pi i}} + \frac{e^{-i\phi - i\psi + \frac{2}{3}\pi i}}{r^2 - e^{-i\phi - i\psi + \frac{2}{3}\pi i}} + \\
&- \frac{e^{i\phi - i\psi - \frac{2}{3}\pi i}}{r^2 - e^{i\phi - i\psi - \frac{2}{3}\pi i}} + \frac{e^{-i\phi - i\psi - \frac{2}{3}\pi i}}{r^2 - e^{-i\phi - i\psi - \frac{2}{3}\pi i}} - \frac{e^{-i\phi + i\psi}}{r^2 - e^{-i\phi + i\psi}} + \frac{e^{i\phi + i\psi}}{r^2 - e^{i\phi + i\psi}} + \\
&- \frac{e^{-i\phi + i\psi - \frac{2}{3}\pi i}}{r^2 - e^{-i\phi + i\psi - \frac{2}{3}\pi i}} + \frac{e^{i\phi + i\psi - \frac{2}{3}\pi i}}{r^2 - e^{i\phi + i\psi - \frac{2}{3}\pi i}} - \frac{e^{-i\phi + i\psi + \frac{2}{3}\pi i}}{r^2 - e^{-i\phi + i\psi + \frac{2}{3}\pi i}} + \frac{e^{i\phi + i\psi + \frac{2}{3}\pi i}}{r^2 - e^{i\phi + i\psi + \frac{2}{3}\pi i}}. \tag{67}
\end{aligned}$$

6.4 Integration in the radial direction

Remember that

$$\begin{aligned}
&\int_{z \in S} \lim_{x \rightarrow (\cos \phi, \sin \phi)} \frac{G_S(x, z)}{1 - |x|^2} \lim_{y \rightarrow (\cos \psi, \sin \psi)} \frac{G_S(z, y)}{1 - |y|^2} dz = \\
&= \frac{1}{\pi^2} \frac{\pi}{16} \int_{r=0}^1 \left(\sum_{w_i \in P \cup \{r^2 e^{-i\psi}\}} \text{Res} \{F_\psi(w)\}_{w=w_i} - \sum_{w_i \in P \cup \{r^2 e^{i\psi}\}} \text{Res} \{F_{-\psi}(w)\}_{w=w_i} \right) r dr \tag{68}
\end{aligned}$$

Let us first compute the left half:

$$\begin{aligned}
&\frac{1}{\pi^2} \frac{\pi}{16} \int_{r=0}^1 \left(\sum_{w_i \in P \cup \{r^2 e^{-i\psi}\}} \text{Res} \{F_\psi(w)\}_{w=w_i} \right) r dr = \\
&= \frac{1}{16\pi} \int_{r=0}^1 \left(- \frac{e^{i\phi - i\psi}}{r^2 - e^{i\phi - i\psi}} + \frac{e^{-i\phi - i\psi}}{r^2 - e^{-i\phi - i\psi}} - \frac{e^{i\phi - i\psi + \frac{2}{3}\pi i}}{r^2 - e^{i\phi - i\psi + \frac{2}{3}\pi i}} + \frac{e^{-i\phi - i\psi + \frac{2}{3}\pi i}}{r^2 - e^{-i\phi - i\psi + \frac{2}{3}\pi i}} + \right. \\
&- \frac{e^{i\phi - i\psi - \frac{2}{3}\pi i}}{r^2 - e^{i\phi - i\psi - \frac{2}{3}\pi i}} + \frac{e^{-i\phi - i\psi - \frac{2}{3}\pi i}}{r^2 - e^{-i\phi - i\psi - \frac{2}{3}\pi i}} - \frac{e^{-i\phi + i\psi}}{r^2 - e^{-i\phi + i\psi}} + \frac{e^{i\phi + i\psi}}{r^2 - e^{i\phi + i\psi}} + \\
&\left. - \frac{e^{-i\phi + i\psi - \frac{2}{3}\pi i}}{r^2 - e^{-i\phi + i\psi - \frac{2}{3}\pi i}} + \frac{e^{i\phi + i\psi - \frac{2}{3}\pi i}}{r^2 - e^{i\phi + i\psi - \frac{2}{3}\pi i}} - \frac{e^{-i\phi + i\psi + \frac{2}{3}\pi i}}{r^2 - e^{-i\phi + i\psi + \frac{2}{3}\pi i}} + \frac{e^{i\phi + i\psi + \frac{2}{3}\pi i}}{r^2 - e^{i\phi + i\psi + \frac{2}{3}\pi i}} \right) r dr
\end{aligned}$$

$$\begin{aligned}
&= \frac{1}{32\pi} \left[-e^{i\phi-i\psi} \text{Ln} \left(1 - r^2 e^{-i\phi+i\psi} \right) + e^{-i\phi-i\psi} \text{Ln} \left(1 - r^2 e^{i\phi+i\psi} \right) + \right. \\
&\quad - e^{i\phi-i\psi+\frac{2}{3}\pi i} \text{Ln} \left(1 - r^2 e^{-i\phi+i\psi-\frac{2}{3}\pi i} \right) + e^{-i\phi-i\psi+\frac{2}{3}\pi i} \text{Ln} \left(1 - r^2 e^{i\phi+i\psi-\frac{2}{3}\pi i} \right) + \\
&\quad - e^{i\phi-i\psi-\frac{2}{3}\pi i} \text{Ln} \left(1 - r^2 e^{-i\phi+i\psi+\frac{2}{3}\pi i} \right) + e^{-i\phi-i\psi-\frac{2}{3}\pi i} \text{Ln} \left(1 - r^2 e^{i\phi+i\psi+\frac{2}{3}\pi i} \right) + \\
&\quad - e^{-i\phi+i\psi} \text{Ln} \left(1 - r^2 e^{i\phi-i\psi} \right) + e^{i\phi+i\psi} \text{Ln} \left(1 - r^2 e^{-i\phi-i\psi} \right) + \\
&\quad - e^{-i\phi+i\psi-\frac{2}{3}\pi i} \text{Ln} \left(1 - r^2 e^{i\phi-i\psi+\frac{2}{3}\pi i} \right) + e^{i\phi+i\psi-\frac{2}{3}\pi i} \text{Ln} \left(1 - r^2 e^{-i\phi-i\psi+\frac{2}{3}\pi i} \right) + \\
&\quad \left. - e^{-i\phi+i\psi+\frac{2}{3}\pi i} \text{Ln} \left(1 - r^2 e^{i\phi-i\psi-\frac{2}{3}\pi i} \right) + e^{i\phi+i\psi+\frac{2}{3}\pi i} \text{Ln} \left(1 - r^2 e^{-i\phi-i\psi-\frac{2}{3}\pi i} \right) \right]_{r=0}^1 \\
&= \frac{1}{32\pi} \left(-e^{i\phi-i\psi} \text{Ln} \left(1 - e^{-i\phi+i\psi} \right) + e^{-i\phi-i\psi} \text{Ln} \left(1 - e^{i\phi+i\psi} \right) + \right. \\
&\quad - e^{i\phi-i\psi+\frac{2}{3}\pi i} \text{Ln} \left(1 - e^{-i\phi+i\psi-\frac{2}{3}\pi i} \right) + e^{-i\phi-i\psi+\frac{2}{3}\pi i} \text{Ln} \left(1 - e^{i\phi+i\psi-\frac{2}{3}\pi i} \right) + \\
&\quad - e^{i\phi-i\psi-\frac{2}{3}\pi i} \text{Ln} \left(1 - e^{-i\phi+i\psi+\frac{2}{3}\pi i} \right) + e^{-i\phi-i\psi-\frac{2}{3}\pi i} \text{Ln} \left(1 - e^{i\phi+i\psi+\frac{2}{3}\pi i} \right) + \\
&\quad - e^{-i\phi+i\psi} \text{Ln} \left(1 - e^{i\phi-i\psi} \right) + e^{i\phi+i\psi} \text{Ln} \left(1 - e^{-i\phi-i\psi} \right) + \\
&\quad - e^{-i\phi+i\psi-\frac{2}{3}\pi i} \text{Ln} \left(1 - e^{i\phi-i\psi+\frac{2}{3}\pi i} \right) + e^{i\phi+i\psi-\frac{2}{3}\pi i} \text{Ln} \left(1 - e^{-i\phi-i\psi+\frac{2}{3}\pi i} \right) + \\
&\quad \left. - e^{-i\phi+i\psi+\frac{2}{3}\pi i} \text{Ln} \left(1 - e^{i\phi-i\psi-\frac{2}{3}\pi i} \right) + e^{i\phi+i\psi+\frac{2}{3}\pi i} \text{Ln} \left(1 - e^{-i\phi-i\psi-\frac{2}{3}\pi i} \right) \right) \\
&= \frac{1}{32\pi} \left(-e^{i\phi-i\psi} \text{Ln} \left(1 - e^{-i\phi+i\psi} \right) - e^{-i\phi+i\psi} \text{Ln} \left(1 - e^{i\phi-i\psi} \right) + \right. \\
&\quad - e^{i\phi-i\psi+\frac{2}{3}\pi i} \text{Ln} \left(1 - e^{-i\phi+i\psi-\frac{2}{3}\pi i} \right) - e^{-i\phi+i\psi-\frac{2}{3}\pi i} \text{Ln} \left(1 - e^{i\phi-i\psi+\frac{2}{3}\pi i} \right) + \\
&\quad - e^{-i\phi+i\psi+\frac{2}{3}\pi i} \text{Ln} \left(1 - e^{i\phi-i\psi-\frac{2}{3}\pi i} \right) - e^{i\phi-i\psi-\frac{2}{3}\pi i} \text{Ln} \left(1 - e^{-i\phi+i\psi+\frac{2}{3}\pi i} \right) + \\
&\quad + e^{i\phi+i\psi} \text{Ln} \left(1 - e^{-i\phi-i\psi} \right) + e^{-i\phi-i\psi} \text{Ln} \left(1 - e^{i\phi+i\psi} \right) + \\
&\quad + e^{-i\phi-i\psi+\frac{2}{3}\pi i} \text{Ln} \left(1 - e^{i\phi+i\psi-\frac{2}{3}\pi i} \right) + e^{i\phi+i\psi-\frac{2}{3}\pi i} \text{Ln} \left(1 - e^{-i\phi-i\psi+\frac{2}{3}\pi i} \right) + \\
&\quad \left. + e^{i\phi+i\psi+\frac{2}{3}\pi i} \text{Ln} \left(1 - e^{-i\phi-i\psi-\frac{2}{3}\pi i} \right) + e^{-i\phi-i\psi-\frac{2}{3}\pi i} \text{Ln} \left(1 - e^{i\phi+i\psi+\frac{2}{3}\pi i} \right) \right). \tag{69}
\end{aligned}$$

Using that for $\alpha \in (0, 2\pi)$

$$\begin{aligned}
& e^{i\alpha} \text{Ln}(1 - e^{-i\alpha}) + e^{-i\alpha} \text{Ln}(1 - e^{i\alpha}) = \\
&= (e^{i\alpha} + e^{-i\alpha}) \ln |1 - e^{i\alpha}| + i(e^{-i\alpha} - e^{i\alpha}) \text{Arg}(1 - e^{i\alpha}) \\
&= 2 \frac{e^{i\alpha} + e^{-i\alpha}}{2} \ln \sqrt{1 - e^{i\alpha} - e^{-i\alpha} + e^{i\alpha} e^{-i\alpha}} + 2 \frac{e^{i\alpha} - e^{-i\alpha}}{2i} \text{Arg}(1 - e^{i\alpha}) \\
&= 2(\cos \alpha) \ln \sqrt{2 - 2 \cos \alpha} + 2(\sin \alpha) \frac{\alpha - \pi}{2} \\
&= \cos \alpha \ln(2 - 2 \cos \alpha) + (\alpha - \pi) \sin \alpha
\end{aligned}$$

and assuming $|\psi| < \phi < \frac{1}{3}\pi$ we continue (69) by

$$\begin{aligned}
&= \frac{1}{32\pi} \left(- \left(\cos(\phi - \psi) \ln(2 - 2 \cos(\phi - \psi)) + (\phi - \psi - \pi) \sin(\phi - \psi) \right) + \right. \\
&\quad - \left(\cos\left(\phi - \psi + \frac{2}{3}\pi\right) \ln\left(2 - 2 \cos\left(\phi - \psi + \frac{2}{3}\pi\right)\right) + \left(\phi - \psi + \frac{2}{3}\pi - \pi\right) \sin\left(\phi - \psi + \frac{2}{3}\pi\right) \right) + \\
&\quad - \left(\cos\left(-\phi + \psi + \frac{2}{3}\pi\right) \ln\left(2 - 2 \cos\left(-\phi + \psi + \frac{2}{3}\pi\right)\right) + \left(-\phi + \psi + \frac{2}{3}\pi - \pi\right) \sin\left(-\phi + \psi + \frac{2}{3}\pi\right) \right) + \\
&\quad + \left(\cos(\phi + \psi) \ln(2 - 2 \cos(\phi + \psi)) + (\phi + \psi - \pi) \sin(\phi + \psi) \right) + \\
&\quad + \left(\cos\left(-\phi - \psi + \frac{2}{3}\pi\right) \ln\left(2 - 2 \cos\left(-\phi - \psi + \frac{2}{3}\pi\right)\right) + \left(-\phi - \psi + \frac{2}{3}\pi - \pi\right) \sin\left(-\phi - \psi + \frac{2}{3}\pi\right) \right) + \\
&\quad \left. + \left(\cos\left(\phi + \psi + \frac{2}{3}\pi\right) \ln\left(2 - 2 \cos\left(\phi + \psi + \frac{2}{3}\pi\right)\right) + \left(\phi + \psi + \frac{2}{3}\pi - \pi\right) \sin\left(\phi + \psi + \frac{2}{3}\pi\right) \right) \right). \quad (70)
\end{aligned}$$

Since $\cos(\alpha) + \cos\left(\alpha + \frac{2}{3}\pi\right) + \cos\left(\alpha - \frac{2}{3}\pi\right) = 0$ and a similar identity with sin one proceeds by

$$\begin{aligned}
&= \frac{1}{32\pi} \left(- \left(\cos(\phi - \psi) \ln(1 - \cos(\phi - \psi)) - \pi \sin(\phi - \psi) \right) + \right. \\
&\quad - \left(\cos\left(\phi - \psi + \frac{2}{3}\pi\right) \ln\left(1 - \cos\left(\phi - \psi + \frac{2}{3}\pi\right)\right) - \frac{1}{3}\pi \sin\left(\phi - \psi + \frac{2}{3}\pi\right) \right) + \\
&\quad - \left(\cos\left(\phi - \psi - \frac{2}{3}\pi\right) \ln\left(1 - \cos\left(\phi - \psi - \frac{2}{3}\pi\right)\right) + \frac{1}{3}\pi \sin\left(\phi - \psi - \frac{2}{3}\pi\right) \right) + \\
&\quad + \left(\cos(\phi + \psi) \ln(1 - \cos(\phi + \psi)) - \pi \sin(\phi + \psi) \right) + \\
&\quad + \left(\cos\left(\phi + \psi - \frac{2}{3}\pi\right) \ln\left(1 - \cos\left(\phi + \psi - \frac{2}{3}\pi\right)\right) + \frac{1}{3}\pi \sin\left(\phi + \psi - \frac{2}{3}\pi\right) \right) + \\
&\quad \left. + \left(\cos\left(\phi + \psi + \frac{2}{3}\pi\right) \ln\left(1 - \cos\left(\phi + \psi + \frac{2}{3}\pi\right)\right) - \frac{1}{3}\pi \sin\left(\phi + \psi + \frac{2}{3}\pi\right) \right) \right). \quad (71)
\end{aligned}$$

$$\begin{aligned}
&= \frac{1}{32\pi} \left(-\cos(\phi - \psi) \ln(1 - \cos(\phi - \psi)) + \pi \sin(\phi - \psi) + \right. \\
&\quad - \cos\left(\phi - \psi + \frac{2}{3}\pi\right) \ln\left(1 - \cos\left(\phi - \psi + \frac{2}{3}\pi\right)\right) + \frac{1}{3}\pi \sin\left(\phi - \psi + \frac{2}{3}\pi\right) + \\
&\quad - \cos\left(\phi - \psi - \frac{2}{3}\pi\right) \ln\left(1 - \cos\left(\phi - \psi - \frac{2}{3}\pi\right)\right) - \frac{1}{3}\pi \sin\left(\phi - \psi - \frac{2}{3}\pi\right) + \\
&\quad + \cos(\phi + \psi) \ln(1 - \cos(\phi + \psi)) - \pi \sin(\phi + \psi) + \\
&\quad + \cos\left(\phi + \psi - \frac{2}{3}\pi\right) \ln\left(1 - \cos\left(\phi + \psi - \frac{2}{3}\pi\right)\right) + \frac{1}{3}\pi \sin\left(\phi + \psi - \frac{2}{3}\pi\right) + \\
&\quad \left. + \cos\left(\phi + \psi + \frac{2}{3}\pi\right) \ln\left(1 - \cos\left(\phi + \psi + \frac{2}{3}\pi\right)\right) - \frac{1}{3}\pi \sin\left(\phi + \psi + \frac{2}{3}\pi\right) \right) \quad (72)
\end{aligned}$$

$$\begin{aligned}
&= \frac{1}{32} \left(\sin(\phi - \psi) - \sin(\phi + \psi) \right. \\
&\quad + \frac{1}{3} \sin\left(\phi - \psi + \frac{2}{3}\pi\right) - \frac{1}{3} \sin\left(\phi + \psi + \frac{2}{3}\pi\right) + \\
&\quad \left. - \frac{1}{3} \sin\left(\phi - \psi - \frac{2}{3}\pi\right) + \frac{1}{3} \sin\left(\phi + \psi - \frac{2}{3}\pi\right) \right) + \\
&\quad + \frac{1}{32\pi} \left(-\cos(\phi - \psi) \ln(1 - \cos(\phi - \psi)) + \right. \\
&\quad - \cos\left(\phi - \psi + \frac{2}{3}\pi\right) \ln\left(1 - \cos\left(\phi - \psi + \frac{2}{3}\pi\right)\right) + \\
&\quad - \cos\left(\phi - \psi - \frac{2}{3}\pi\right) \ln\left(1 - \cos\left(\phi - \psi - \frac{2}{3}\pi\right)\right) + \\
&\quad + \cos(\phi + \psi) \ln(1 - \cos(\phi + \psi)) + \\
&\quad + \cos\left(\phi + \psi - \frac{2}{3}\pi\right) \ln\left(1 - \cos\left(\phi + \psi - \frac{2}{3}\pi\right)\right) + \\
&\quad \left. + \cos\left(\phi + \psi + \frac{2}{3}\pi\right) \ln\left(1 - \cos\left(\phi + \psi + \frac{2}{3}\pi\right)\right) \right) \quad (73)
\end{aligned}$$

$$\begin{aligned}
&= \frac{1}{32} \sin \psi \left(-2 \cos \phi - \frac{2}{3} \cos\left(\phi + \frac{2}{3}\pi\right) + \frac{2}{3} \cos\left(\phi - \frac{2}{3}\pi\right) \right) + \\
&\quad + \frac{1}{32\pi} \left(-\cos(\phi - \psi) \ln(1 - \cos(\phi - \psi)) + \right. \\
&\quad - \cos\left(\phi - \psi + \frac{2}{3}\pi\right) \ln\left(1 - \cos\left(\phi - \psi + \frac{2}{3}\pi\right)\right) + \\
&\quad - \cos\left(\phi - \psi - \frac{2}{3}\pi\right) \ln\left(1 - \cos\left(\phi - \psi - \frac{2}{3}\pi\right)\right) + \\
&\quad + \cos(\phi + \psi) \ln(1 - \cos(\phi + \psi)) + \\
&\quad + \cos\left(\phi + \psi - \frac{2}{3}\pi\right) \ln\left(1 - \cos\left(\phi + \psi - \frac{2}{3}\pi\right)\right) + \\
&\quad \left. + \cos\left(\phi + \psi + \frac{2}{3}\pi\right) \ln\left(1 - \cos\left(\phi + \psi + \frac{2}{3}\pi\right)\right) \right) \quad (74)
\end{aligned}$$

$$\begin{aligned}
&= \frac{1}{16} \sin \psi \left(-\cos \phi + \frac{1}{3} \sqrt{3} \sin \phi \right) + \\
&\quad + \frac{1}{32\pi} \left(-\cos(\phi - \psi) \ln(1 - \cos(\phi - \psi)) + \right. \\
&\quad - \cos\left(\phi - \psi + \frac{2}{3}\pi\right) \ln\left(1 - \cos\left(\phi - \psi + \frac{2}{3}\pi\right)\right) + \\
&\quad - \cos\left(\phi - \psi - \frac{2}{3}\pi\right) \ln\left(1 - \cos\left(\phi - \psi - \frac{2}{3}\pi\right)\right) + \\
&\quad + \cos(\phi + \psi) \ln(1 - \cos(\phi + \psi)) + \\
&\quad + \cos\left(\phi + \psi - \frac{2}{3}\pi\right) \ln\left(1 - \cos\left(\phi + \psi - \frac{2}{3}\pi\right)\right) + \\
&\quad \left. + \cos\left(\phi + \psi + \frac{2}{3}\pi\right) \ln\left(1 - \cos\left(\phi + \psi + \frac{2}{3}\pi\right)\right) \right). \tag{75}
\end{aligned}$$

Since we still have to subtract in (68) the expression for $-\psi$ and since (75) is odd in ψ , (75) doubles.

6.5 Conclusion of Case 3)

The enumerator turns out to be (for $\psi < \phi$)

$$\begin{aligned}
&\int_{z \in S} \lim_{x \rightarrow (\cos \phi, \sin \phi)} \frac{G_S(x, z)}{1 - |x|^2} \lim_{y \rightarrow (\cos \psi, \sin \psi)} \frac{G_S(z, y)}{1 - |y|^2} dz = \\
&= \frac{1}{8} \sin \psi \left(-\cos \phi + \frac{1}{3} \sqrt{3} \sin \phi \right) + \\
&\quad - \frac{1}{16\pi} \sum_{k=0}^2 \cos\left(\phi - \psi + \frac{2}{3}k\pi\right) \ln\left(1 - \cos\left(\phi - \psi + \frac{2}{3}k\pi\right)\right) + \\
&\quad + \frac{1}{16\pi} \sum_{k=0}^2 \cos\left(\phi + \psi + \frac{2}{3}k\pi\right) \ln\left(1 - \cos\left(\phi + \psi + \frac{2}{3}k\pi\right)\right). \tag{76}
\end{aligned}$$

For $\psi > \phi$, by symmetry $T_{22}(\phi, \psi) := T_{22}(\psi, \phi)$. Again we find

$$\sup_{\phi, \psi \in (0, \frac{1}{3}\pi)} T_{22}(\phi, \psi) = \lim_{\phi \uparrow \frac{1}{3}\pi, \psi \downarrow 0} T_{22}(\phi, \psi) = \frac{4}{243} \pi \sqrt{3} - \frac{5}{27} + \frac{4}{81} \ln 2.$$

See Figure 3 on p. 60.

7 The case that $x \in \Gamma_1$, $y \in \Gamma_3$

We set $y = (\rho \cos \psi, \rho \sin \psi)$ and we are interested in

$$T_{13}(x_1, \rho) = \frac{\int_S \lim_{x_2 \downarrow 0} \frac{G_S(x, z)}{x_2} \lim_{\psi \uparrow \frac{1}{3}\pi} \frac{G_S(z, y)}{\rho \sin\left(\frac{1}{3}\pi - \psi\right)} dz}{\lim_{x_2 \downarrow 0, \psi \uparrow \frac{1}{3}\pi} \frac{G_S(x, y)}{x_2 \rho \sin\left(\frac{1}{3}\pi - \psi\right)}}. \tag{77}$$

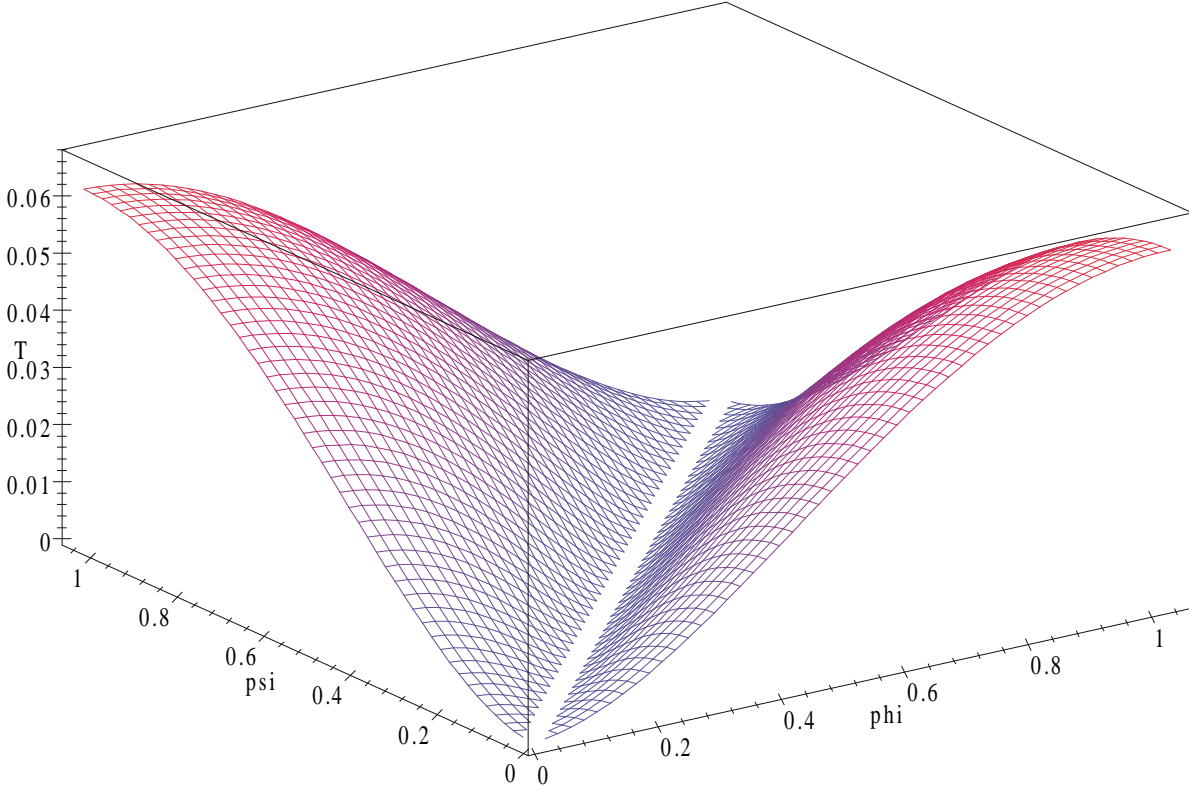


Figure 3: x and y both on Γ_2 : $T_{22}(\phi, \psi)$

7.1 Limit of the Green function

First we look at the factors in the enumerator: As before we have with $z = (r \cos \theta, r \sin \theta)$ that

$$\begin{aligned}
 & \lim_{x_2 \downarrow 0} \frac{G_S(x, z)}{x_2} = \\
 &= \frac{1}{\pi} \left(\frac{r \sin \theta}{x_1^2 - 2x_1 r \cos \theta + r^2} - \frac{r \sin \theta}{1 - 2x_1 r \cos \theta + x_1^2 r^2} \right) + \\
 &+ \frac{1}{\pi} \left(\frac{r \sin \left(\theta - \frac{2}{3} \pi \right)}{x_1^2 - 2x_1 r \cos \left(\theta - \frac{2}{3} \pi \right) + r^2} - \frac{r \sin \left(\theta - \frac{2}{3} \pi \right)}{1 - 2x_1 r \cos \left(\theta - \frac{2}{3} \pi \right) + x_1^2 r^2} \right) + \\
 &+ \frac{1}{\pi} \left(\frac{r \sin \left(\theta + \frac{2}{3} \pi \right)}{x_1^2 - 2x_1 r \cos \left(\theta + \frac{2}{3} \pi \right) + r^2} - \frac{r \sin \left(\theta + \frac{2}{3} \pi \right)}{1 - 2x_1 r \cos \left(\theta + \frac{2}{3} \pi \right) + x_1^2 r^2} \right).
 \end{aligned}$$

By symmetry we find

$$\begin{aligned}
& \lim_{\substack{\psi \uparrow \frac{1}{3}\pi \\ y=(\rho \cos \psi, \rho \sin \psi)}} \frac{G_S(z, y)}{\rho \sin\left(\frac{1}{3}\pi - \psi\right)} = \\
&= \frac{1}{\pi} \left(\frac{r \sin\left(\frac{1}{3}\pi - \theta\right)}{\rho^2 - 2\rho r \cos\left(\frac{1}{3}\pi - \theta\right) + r^2} - \frac{r \sin\left(\frac{1}{3}\pi - \theta\right)}{1 - 2\rho r \cos\left(\frac{1}{3}\pi - \theta\right) + \rho^2 r^2} \right) + \\
&+ \frac{1}{\pi} \left(\frac{r \sin\left(-\frac{1}{3}\pi - \theta\right)}{\rho^2 - 2\rho r \cos\left(-\frac{1}{3}\pi - \theta\right) + r^2} - \frac{r \sin\left(-\frac{1}{3}\pi - \theta\right)}{1 - 2\rho r \cos\left(-\frac{1}{3}\pi - \theta\right) + \rho^2 r^2} \right) + \\
&+ \frac{1}{\pi} \left(\frac{r \sin(\pi - \theta)}{\rho^2 - 2\rho r \cos(\pi - \theta) + r^2} - \frac{r \sin(\pi - \theta)}{1 - 2\rho r \cos(\pi - \theta) + \rho^2 r^2} \right). \\
&= \frac{1}{\pi} \left(\frac{r \sin\left(\theta - \frac{1}{3}\pi\right)}{1 - 2\rho r \cos\left(\theta - \frac{1}{3}\pi\right) + \rho^2 r^2} - \frac{r \sin\left(\theta - \frac{1}{3}\pi\right)}{\rho^2 - 2\rho r \cos\left(\theta - \frac{1}{3}\pi\right) + r^2} \right) + \\
&+ \frac{1}{\pi} \left(\frac{r \sin\left(\theta + \frac{1}{3}\pi\right)}{1 - 2\rho r \cos\left(\theta + \frac{1}{3}\pi\right) + \rho^2 r^2} - \frac{r \sin\left(\theta + \frac{1}{3}\pi\right)}{\rho^2 - 2\rho r \cos\left(\theta + \frac{1}{3}\pi\right) + r^2} \right) + \\
&+ \frac{1}{\pi} \left(\frac{r \sin(\theta - \pi)}{1 - 2\rho r \cos(\theta - \pi) + \rho^2 r^2} - \frac{r \sin(\theta - \pi)}{\rho^2 - 2\rho r \cos(\theta - \pi) + r^2} \right).
\end{aligned}$$

The denominator becomes

$$\begin{aligned}
& \lim_{\substack{x_2 \downarrow 0 \text{ and } \psi \uparrow \frac{1}{3}\pi \\ y=(\rho \cos \psi, \rho \sin \psi)}} \frac{G_S(x, y)}{x_2 \rho \sin\left(\frac{1}{3}\pi - \psi\right)} = \\
&= \lim_{\theta \downarrow 0} \frac{1}{\pi x_1 \sin \theta} \left(\frac{x_1 \sin\left(\frac{1}{3}\pi - \theta\right)}{\rho^2 - 2\rho x_1 \cos\left(\frac{1}{3}\pi - \theta\right) + x_1^2} - \frac{x_1 \sin\left(\frac{1}{3}\pi - \theta\right)}{1 - 2\rho x_1 \cos\left(\frac{1}{3}\pi - \theta\right) + \rho^2 x_1^2} + \right. \\
&+ \frac{x_1 \sin\left(-\frac{1}{3}\pi - \theta\right)}{\rho^2 - 2\rho x_1 \cos\left(-\frac{1}{3}\pi - \theta\right) + x_1^2} - \frac{x_1 \sin\left(-\frac{1}{3}\pi - \theta\right)}{1 - 2\rho x_1 \cos\left(-\frac{1}{3}\pi - \theta\right) + \rho^2 x_1^2} + \\
&\left. + \frac{x_1 \sin(\pi - \theta)}{\rho^2 - 2\rho x_1 \cos(\pi - \theta) + x_1^2} - \frac{x_1 \sin(\pi - \theta)}{1 - 2\rho x_1 \cos(\pi - \theta) + \rho^2 x_1^2} \right) \\
&= \lim_{\theta \downarrow 0} \frac{1}{\pi \sin \theta} \left(\frac{\sin\left(\frac{1}{3}\pi - \theta\right)}{\rho^2 - 2\rho x_1 \cos\left(\frac{1}{3}\pi - \theta\right) + x_1^2} + \frac{\sin\left(-\frac{1}{3}\pi - \theta\right)}{\rho^2 - 2\rho x_1 \cos\left(-\frac{1}{3}\pi - \theta\right) + x_1^2} + \right. \\
&- \frac{\sin\left(\frac{1}{3}\pi - \theta\right)}{1 - 2\rho x_1 \cos\left(\frac{1}{3}\pi - \theta\right) + \rho^2 x_1^2} - \frac{\sin\left(-\frac{1}{3}\pi - \theta\right)}{1 - 2\rho x_1 \cos\left(-\frac{1}{3}\pi - \theta\right) + \rho^2 x_1^2} + \\
&\left. + \frac{\sin(\pi - \theta)}{\rho^2 - 2\rho x_1 \cos(\pi - \theta) + x_1^2} - \frac{\sin(\pi - \theta)}{1 - 2\rho x_1 \cos(\pi - \theta) + \rho^2 x_1^2} \right)
\end{aligned}$$

$$\begin{aligned}
&= \lim_{\theta \downarrow 0} \frac{1}{\pi \sin \theta} \left(\frac{\sin(\frac{1}{3}\pi) \cos \theta - \cos(\frac{1}{3}\pi) \sin \theta}{\rho^2 - 2\rho x_1 \cos(\frac{1}{3}\pi - \theta) + x_1^2} - \frac{\sin(\frac{1}{3}\pi) \cos \theta + \cos(\frac{1}{3}\pi) \sin \theta}{\rho^2 - 2\rho x_1 \cos(\frac{1}{3}\pi + \theta) + x_1^2} + \right. \\
&\quad - \frac{\sin(\frac{1}{3}\pi) \cos \theta - \cos(\frac{1}{3}\pi) \sin \theta}{1 - 2\rho x_1 \cos(\frac{1}{3}\pi - \theta) + \rho^2 x_1^2} + \frac{\sin(\frac{1}{3}\pi) \cos \theta + \cos(\frac{1}{3}\pi) \sin \theta}{1 - 2\rho x_1 \cos(\frac{1}{3}\pi + \theta) + \rho^2 x_1^2} + \\
&\quad \left. + \frac{\sin \theta}{\rho^2 + 2\rho x_1 \cos \theta + x_1^2} - \frac{\sin \theta}{1 + 2\rho x_1 \cos \theta + \rho^2 x_1^2} \right) \\
&= \lim_{\theta \downarrow 0} \frac{1}{\pi \sin \theta} \left(\frac{\sin(\frac{1}{3}\pi) \cos \theta}{\rho^2 - 2\rho x_1 \cos(\frac{1}{3}\pi - \theta) + x_1^2} - \frac{\cos(\frac{1}{3}\pi) \sin \theta}{\rho^2 - 2\rho x_1 \cos(\frac{1}{3}\pi - \theta) + x_1^2} + \right. \\
&\quad - \frac{\sin(\frac{1}{3}\pi) \cos \theta}{\rho^2 - 2\rho x_1 \cos(\frac{1}{3}\pi + \theta) + x_1^2} - \frac{\cos(\frac{1}{3}\pi) \sin \theta}{\rho^2 - 2\rho x_1 \cos(\frac{1}{3}\pi + \theta) + x_1^2} + \\
&\quad - \frac{\sin(\frac{1}{3}\pi) \cos \theta}{1 - 2\rho x_1 \cos(\frac{1}{3}\pi - \theta) + \rho^2 x_1^2} + \frac{\cos(\frac{1}{3}\pi) \sin \theta}{1 - 2\rho x_1 \cos(\frac{1}{3}\pi - \theta) + \rho^2 x_1^2} + \\
&\quad + \frac{\sin(\frac{1}{3}\pi) \cos \theta}{1 - 2\rho x_1 \cos(\frac{1}{3}\pi + \theta) + \rho^2 x_1^2} + \frac{\cos(\frac{1}{3}\pi) \sin \theta}{1 - 2\rho x_1 \cos(\frac{1}{3}\pi + \theta) + \rho^2 x_1^2} + \\
&\quad \left. + \frac{\sin \theta}{\rho^2 + 2\rho x_1 \cos \theta + x_1^2} - \frac{\sin \theta}{1 + 2\rho x_1 \cos \theta + \rho^2 x_1^2} \right) \\
&= \lim_{\theta \downarrow 0} \frac{1}{\pi \sin \theta} \left(\frac{\sin(\frac{1}{3}\pi) \cos \theta}{\rho^2 - 2\rho x_1 \cos(\frac{1}{3}\pi - \theta) + x_1^2} - \frac{\sin(\frac{1}{3}\pi) \cos \theta}{\rho^2 - 2\rho x_1 \cos(\frac{1}{3}\pi + \theta) + x_1^2} + \right. \\
&\quad \left. + \frac{\sin(\frac{1}{3}\pi) \cos \theta}{1 - 2\rho x_1 \cos(\frac{1}{3}\pi + \theta) + \rho^2 x_1^2} - \frac{\sin(\frac{1}{3}\pi) \cos \theta}{1 - 2\rho x_1 \cos(\frac{1}{3}\pi - \theta) + \rho^2 x_1^2} \right) + \\
&\quad + \lim_{\theta \downarrow 0} \frac{1}{\pi \sin \theta} \left(\frac{\sin \theta}{\rho^2 + 2\rho x_1 \cos \theta + x_1^2} - \frac{\sin \theta}{1 + 2\rho x_1 \cos \theta + \rho^2 x_1^2} + \right. \\
&\quad - \frac{\cos(\frac{1}{3}\pi) \sin \theta}{\rho^2 - 2\rho x_1 \cos(\frac{1}{3}\pi + \theta) + x_1^2} - \frac{\cos(\frac{1}{3}\pi) \sin \theta}{\rho^2 - 2\rho x_1 \cos(\frac{1}{3}\pi - \theta) + x_1^2} + \\
&\quad \left. + \frac{\cos(\frac{1}{3}\pi) \sin \theta}{1 - 2\rho x_1 \cos(\frac{1}{3}\pi - \theta) + \rho^2 x_1^2} + \frac{\cos(\frac{1}{3}\pi) \sin \theta}{1 - 2\rho x_1 \cos(\frac{1}{3}\pi + \theta) + \rho^2 x_1^2} \right)
\end{aligned}$$

$$\begin{aligned}
&= \lim_{\theta \downarrow 0} \frac{\frac{1}{2}\sqrt{3} \cos \theta}{\pi \sin \theta} \left(\frac{(\rho^2 - 2\rho x_1 \cos(\frac{1}{3}\pi + \theta) + x_1^2) - (\rho^2 - 2\rho x_1 \cos(\frac{1}{3}\pi - \theta) + x_1^2) 1}{(\rho^2 - 2\rho x_1 \cos(\frac{1}{3}\pi - \theta) + x_1^2) (\rho^2 - 2\rho x_1 \cos(\frac{1}{3}\pi + \theta) + x_1^2)} + \right. \\
&\quad \left. + \frac{(1 - 2\rho x_1 \cos(\frac{1}{3}\pi - \theta) + \rho^2 x_1^2) - (1 - 2\rho x_1 \cos(\frac{1}{3}\pi + \theta) + \rho^2 x_1^2)}{(1 - 2\rho x_1 \cos(\frac{1}{3}\pi + \theta) + \rho^2 x_1^2) (1 - 2\rho x_1 \cos(\frac{1}{3}\pi - \theta) + \rho^2 x_1^2)} \right) + \\
&\quad + \lim_{\theta \downarrow 0} \frac{1}{\pi} \left(\frac{1}{\rho^2 + 2\rho x_1 \cos \theta + x_1^2} - \frac{1}{1 + 2\rho x_1 \cos \theta + \rho^2 x_1^2} + \right. \\
&\quad - \frac{\frac{1}{2}}{\rho^2 - 2\rho x_1 \cos(\frac{1}{3}\pi + \theta) + x_1^2} - \frac{\frac{1}{2}}{\rho^2 - 2\rho x_1 \cos(\frac{1}{3}\pi - \theta) + x_1^2} + \\
&\quad \left. + \frac{\frac{1}{2}}{1 - 2\rho x_1 \cos(\frac{1}{3}\pi - \theta) + \rho^2 x_1^2} + \frac{\frac{1}{2}}{1 - 2\rho x_1 \cos(\frac{1}{3}\pi + \theta) + \rho^2 x_1^2} \right) \\
&= \lim_{\theta \downarrow 0} \frac{\frac{1}{2}\sqrt{3}2\rho x_1 \cos \theta}{\pi \sin \theta} \left(\frac{\cos(\frac{1}{3}\pi - \theta) - \cos(\frac{1}{3}\pi + \theta)}{(\rho^2 - 2\rho x_1 \cos(\frac{1}{3}\pi - \theta) + x_1^2) (\rho^2 - 2\rho x_1 \cos(\frac{1}{3}\pi + \theta) + x_1^2)} + \right. \\
&\quad \left. + \frac{\cos(\frac{1}{3}\pi + \theta) - \cos(\frac{1}{3}\pi - \theta)}{(1 - 2\rho x_1 \cos(\frac{1}{3}\pi + \theta) + \rho^2 x_1^2) (1 - 2\rho x_1 \cos(\frac{1}{3}\pi - \theta) + \rho^2 x_1^2)} \right) + \\
&\quad + \frac{1}{\pi} \left(\frac{1}{\rho^2 + 2\rho x_1 + x_1^2} - \frac{1}{1 + 2\rho x_1 + \rho^2 x_1^2} + \right. \\
&\quad \left. - \frac{\frac{1}{2}}{\rho^2 - \rho x_1 + x_1^2} - \frac{\frac{1}{2}}{\rho^2 - \rho x_1 + x_1^2} + \frac{\frac{1}{2}}{1 - \rho x_1 + \rho^2 x_1^2} + \frac{\frac{1}{2}}{1 - \rho x_1 + \rho^2 x_1^2} \right) \\
&= \lim_{\theta \downarrow 0} \frac{\sqrt{3}\rho x_1}{\pi \sin \theta} \left(\frac{2 \sin(\frac{1}{3}\pi) \sin \theta}{(\rho^2 - 2\rho x_1 \cos(\frac{1}{3}\pi - \theta) + x_1^2) (\rho^2 - 2\rho x_1 \cos(\frac{1}{3}\pi + \theta) + x_1^2)} + \right. \\
&\quad \left. - \frac{2 \sin(\frac{1}{3}\pi) \sin \theta}{(1 - 2\rho x_1 \cos(\frac{1}{3}\pi + \theta) + \rho^2 x_1^2) (1 - 2\rho x_1 \cos(\frac{1}{3}\pi - \theta) + \rho^2 x_1^2)} \right) + \\
&\quad + \frac{1}{\pi} \left(\frac{1}{\rho^2 + 2\rho x_1 + x_1^2} - \frac{1}{1 + 2\rho x_1 + \rho^2 x_1^2} - \frac{1}{\rho^2 - \rho x_1 + x_1^2} + \frac{1}{1 - \rho x_1 + \rho^2 x_1^2} \right)
\end{aligned}$$

$$\begin{aligned}
&= \lim_{\theta \downarrow 0} \frac{\rho x_1}{\pi} \left(\frac{3}{(\rho^2 - 2\rho x_1 \cos(\frac{1}{3}\pi - \theta) + x_1^2)(\rho^2 - 2\rho x_1 \cos(\frac{1}{3}\pi + \theta) + x_1^2)} + \right. \\
&\quad \left. - \frac{3}{(1 - 2\rho x_1 \cos(\frac{1}{3}\pi + \theta) + \rho^2 x_1^2)(1 - 2\rho x_1 \cos(\frac{1}{3}\pi - \theta) + \rho^2 x_1^2)} \right) + \\
&\quad + \frac{1}{\pi} \left(\frac{1}{\rho^2 + 2\rho x_1 + x_1^2} - \frac{1}{1 + 2\rho x_1 + \rho^2 x_1^2} - \frac{1}{\rho^2 - \rho x_1 + x_1^2} + \frac{1}{1 - \rho x_1 + \rho^2 x_1^2} \right) \\
&= \frac{\rho x_1}{\pi} \left(\frac{3}{(\rho^2 - \rho x_1 + x_1^2)(\rho^2 - \rho x_1 + x_1^2)} - \frac{3}{(1 - \rho x_1 + \rho^2 x_1^2)(1 - \rho x_1 + \rho^2 x_1^2)} \right) + \\
&\quad + \frac{1}{\pi} \left(\frac{1}{\rho^2 + 2\rho x_1 + x_1^2} - \frac{1}{1 + 2\rho x_1 + \rho^2 x_1^2} - \frac{1}{\rho^2 - \rho x_1 + x_1^2} + \frac{1}{1 - \rho x_1 + \rho^2 x_1^2} \right) \\
&= \frac{1}{\pi} \left(\frac{3\rho x_1}{(\rho^2 - \rho x_1 + x_1^2)^2} - \frac{3\rho x_1}{(1 - \rho x_1 + \rho^2 x_1^2)^2} + \frac{1}{\rho^2 + 2\rho x_1 + x_1^2} + \right. \\
&\quad \left. - \frac{1}{1 + 2\rho x_1 + \rho^2 x_1^2} - \frac{1}{\rho^2 - \rho x_1 + x_1^2} + \frac{1}{1 - \rho x_1 + \rho^2 x_1^2} \right).
\end{aligned}$$

7.2 Derivation of a contour integral

As before

$$\begin{aligned}
&\lim_{x_2 \downarrow 0} \frac{G_S(x, z)}{x_2} = \\
&= \frac{1}{\pi} \left(\frac{r \sin \theta}{x_1^2 - 2x_1 r \cos \theta + r^2} - \frac{r \sin \theta}{1 - 2x_1 r \cos \theta + x_1^2 r^2} \right) + \\
&\quad + \frac{1}{\pi} \left(\frac{r \sin(\theta - \frac{2}{3}\pi)}{x_1^2 - 2x_1 r \cos(\theta - \frac{2}{3}\pi) + r^2} - \frac{r \sin(\theta - \frac{2}{3}\pi)}{1 - 2x_1 r \cos(\theta - \frac{2}{3}\pi) + x_1^2 r^2} \right) + \\
&\quad + \frac{1}{\pi} \left(\frac{r \sin(\theta + \frac{2}{3}\pi)}{x_1^2 - 2x_1 r \cos(\theta + \frac{2}{3}\pi) + r^2} - \frac{r \sin(\theta + \frac{2}{3}\pi)}{1 - 2x_1 r \cos(\theta + \frac{2}{3}\pi) + x_1^2 r^2} \right)
\end{aligned}$$

By symmetry we find

$$\begin{aligned}
&\lim_{\substack{\psi \uparrow \frac{1}{3}\pi \\ y = (\rho \cos \psi, \rho \sin \psi)}} \frac{G_S(z, y)}{\rho \sin(\frac{1}{3}\pi - \psi)} = \\
&= \frac{1}{\pi} \left(\frac{r \sin(\frac{1}{3}\pi - \theta)}{\rho^2 - 2\rho r \cos(\frac{1}{3}\pi - \theta) + r^2} - \frac{r \sin(\frac{1}{3}\pi - \theta)}{1 - 2\rho r \cos(\frac{1}{3}\pi - \theta) + \rho^2 r^2} \right) + \\
&\quad + \frac{1}{\pi} \left(\frac{r \sin(-\frac{1}{3}\pi - \theta)}{\rho^2 - 2\rho r \cos(-\frac{1}{3}\pi - \theta) + r^2} - \frac{r \sin(-\frac{1}{3}\pi - \theta)}{1 - 2\rho r \cos(-\frac{1}{3}\pi - \theta) + \rho^2 r^2} \right) + \\
&\quad + \frac{1}{\pi} \left(\frac{r \sin(\pi - \theta)}{\rho^2 - 2\rho r \cos(\pi - \theta) + r^2} - \frac{r \sin(\pi - \theta)}{1 - 2\rho r \cos(\pi - \theta) + \rho^2 r^2} \right).
\end{aligned}$$

The expression we want to compute explicitly is

$$\begin{aligned}
& \pi^2 \int_{z \in S} \left(\lim_{x_2 \downarrow 0} \frac{G_S(x, z)}{x_2} \right) \left(\lim_{\substack{\psi \uparrow \frac{1}{3}\pi \\ y = (\rho \cos \psi, \rho \sin \psi)}} \frac{G_S(z, y)}{\rho \sin(\frac{1}{3}\pi - \psi)} \right) dz = \\
& = \int_{r=0}^1 \int_{\theta=0}^{\frac{1}{3}\pi} \sum_{k=-1}^1 \sum_{m=-1}^1 h(\theta + k\frac{2}{3}\pi) \ell(\pi - \theta + m\frac{2}{3}\pi) d\theta r dr. \tag{78}
\end{aligned}$$

As before with $w = re^{i\theta}$ we have $m = 1$

$$\begin{aligned}
h(\theta + k\frac{2}{3}\pi) &= \frac{1}{2i} \left(\frac{r^2 e^{-k\frac{2}{3}\pi i}}{w - x_1 r^2 e^{-k\frac{2}{3}\pi i}} + \frac{x_1^{-2} e^{-k\frac{2}{3}\pi i}}{w - x_1^{-1} e^{-k\frac{2}{3}\pi i}} - \frac{x_1^{-2} r^2 e^{-k\frac{2}{3}\pi i}}{w - x_1^{-1} r^2 e^{-k\frac{2}{3}\pi i}} - \frac{e^{-k\frac{2}{3}\pi i}}{w - x_1 e^{-k\frac{2}{3}\pi i}} \right), \\
\ell(\pi - \theta + m\frac{2}{3}\pi) &= \\
&= \frac{1}{2i} \left(\frac{r^2 e^{-m\frac{2}{3}\pi i}}{-\bar{w} - \rho r^2 e^{-m\frac{2}{3}\pi i}} + \frac{\rho^{-2} e^{-m\frac{2}{3}\pi i}}{-\bar{w} - \rho^{-1} e^{-m\frac{2}{3}\pi i}} - \frac{\rho^{-2} r^2 e^{-m\frac{2}{3}\pi i}}{-\bar{w} - \rho^{-1} r^2 e^{-m\frac{2}{3}\pi i}} - \frac{e^{-m\frac{2}{3}\pi i}}{-\bar{w} - \rho e^{-m\frac{2}{3}\pi i}} \right) \\
&= \frac{1}{2i} \left(\frac{wr^2 e^{-m\frac{2}{3}\pi i}}{-r^2 - w\rho r^2 e^{-m\frac{2}{3}\pi i}} + \frac{w\rho^{-2} e^{-m\frac{2}{3}\pi i}}{-r^2 - w\rho^{-1} e^{-m\frac{2}{3}\pi i}} - \frac{wr^2 \rho^{-2} e^{-m\frac{2}{3}\pi i}}{-r^2 - w\rho^{-1} r^2 e^{-m\frac{2}{3}\pi i}} - \frac{we^{-m\frac{2}{3}\pi i}}{-r^2 - w\rho e^{-m\frac{2}{3}\pi i}} \right) \\
&= \frac{1}{2i\rho} \left(-\frac{w}{w + \rho^{-1} e^{m\frac{2}{3}\pi i}} - \frac{w}{w + r^2 \rho e^{m\frac{2}{3}\pi i}} + \frac{w}{w + \rho e^{m\frac{2}{3}\pi i}} + \frac{w}{w + r^2 \rho^{-1} e^{m\frac{2}{3}\pi i}} \right) \\
&= \frac{1}{2i\rho} \left(\frac{\rho^{-1} e^{m\frac{2}{3}\pi i}}{w + \rho^{-1} e^{m\frac{2}{3}\pi i}} + \frac{r^2 \rho e^{m\frac{2}{3}\pi i}}{w + r^2 \rho e^{m\frac{2}{3}\pi i}} - \frac{\rho e^{m\frac{2}{3}\pi i}}{w + \rho e^{m\frac{2}{3}\pi i}} - \frac{r^2 \rho^{-1} e^{m\frac{2}{3}\pi i}}{w + r^2 \rho^{-1} e^{m\frac{2}{3}\pi i}} \right).
\end{aligned}$$

Again the inner integral in (78) equals

$$\begin{aligned}
& \int_{\theta=0}^{\frac{1}{3}\pi} \sum_{k=-1}^1 \sum_{m=-1}^1 h(\theta + k\frac{2}{3}\pi) \ell(\pi - \theta + m\frac{2}{3}\pi) d\theta = \\
&= \frac{1}{2} \int_{\theta=0}^{2\pi} \sum_{k=-1}^1 h\left(\theta + k\frac{2}{3}\pi\right) \ell(\pi - \theta) d\theta \\
&= \frac{i}{8\rho x_1} \int_{\theta=0}^{2\pi} \left(\frac{x_1 r^2}{w - x_1 r^2} + \frac{x_1^{-1}}{w - x_1^{-1}} - \frac{x_1^{-1} r^2}{w - x_1^{-1} r^2} - \frac{x_1}{w - x_1} + \right. \\
&+ \frac{x_1 r^2 e^{-\frac{2}{3}\pi i}}{w - x_1 r^2 e^{-\frac{2}{3}\pi i}} + \frac{x_1^{-1} e^{-\frac{2}{3}\pi i}}{w - x_1^{-1} e^{-\frac{2}{3}\pi i}} - \frac{x_1^{-1} r^2 e^{-\frac{2}{3}\pi i}}{w - x_1^{-1} r^2 e^{-\frac{2}{3}\pi i}} - \frac{x_1 e^{-\frac{2}{3}\pi i}}{w - x_1 e^{-\frac{2}{3}\pi i}} + \\
&+ \left. \frac{x_1 r^2 e^{\frac{2}{3}\pi i}}{w - x_1 r^2 e^{\frac{2}{3}\pi i}} + \frac{x_1^{-1} e^{\frac{2}{3}\pi i}}{w - x_1^{-1} e^{\frac{2}{3}\pi i}} - \frac{x_1^{-1} r^2 e^{\frac{2}{3}\pi i}}{w - x_1^{-1} r^2 e^{\frac{2}{3}\pi i}} - \frac{x_1 e^{\frac{2}{3}\pi i}}{w - x_1 e^{\frac{2}{3}\pi i}} \right) \times \\
&\times \left(\frac{\rho^{-1}}{w + \rho^{-1}} + \frac{r^2 \rho}{w + r^2 \rho} - \frac{\rho}{w + \rho} - \frac{r^2 \rho^{-1}}{w + r^2 \rho^{-1}} \right) \frac{dw}{w}
\end{aligned}$$

leaving us with three different cases with each having 6 poles. Note that a pole in 0 does not contribute.

7.3 Computation of the contour integral

Let us assume that $x_1 < \rho$. Then according to the size of r the integrand has different sets of poles. In the following table we give a scheme in which we denote how we will split the integral (and which range of r corresponds with which poles):

poles due to: range:		h for $k = 0$		h for $k = -1$		h for $k = 1$		ℓ	
		$a_1.$	$a_2.$	$b_1.$	$b_2.$	$c_1.$	$c_2.$	$d_1.$	$d_2.$
$r \in (0, x_1)$	I.	$x_1 r^2$	$x_1^{-1} r^2$	$x_1 r^2 e^{-\frac{2}{3}\pi i}$	$x_1^{-1} r^2 e^{-\frac{2}{3}\pi i}$	$x_1 r^2 e^{\frac{2}{3}\pi i}$	$x_1^{-1} r^2 e^{\frac{2}{3}\pi i}$	$-r^2 \rho$	$-r^2 \rho^{-1}$
$r \in (x_1, \rho)$	II.	$x_1 r^2$	x_1	$x_1 r^2 e^{-\frac{2}{3}\pi i}$	$x_1 e^{-\frac{2}{3}\pi i}$	$x_1 r^2 e^{\frac{2}{3}\pi i}$	$x_1 e^{\frac{2}{3}\pi i}$	$-r^2 \rho$	$-r^2 \rho^{-1}$
$r \in (\rho, 1)$	III.	$x_1 r^2$	x_1	$x_1 r^2 e^{-\frac{2}{3}\pi i}$	$x_1 e^{-\frac{2}{3}\pi i}$	$x_1 r^2 e^{\frac{2}{3}\pi i}$	$x_1 e^{\frac{2}{3}\pi i}$	$-r^2 \rho$	$-\rho$

We consider the contribution in the following integral by each of the poles separately:

$$\begin{aligned}
& \frac{i}{8\rho x_1} \int_{\theta=0}^{2\pi} \left(\frac{x_1 r^2}{w - x_1 r^2} + \frac{x_1^{-1}}{w - x_1^{-1}} - \frac{x_1^{-1} r^2}{w - x_1^{-1} r^2} - \frac{x_1}{w - x_1} + \right. \\
& + \frac{x_1 r^2 e^{-\frac{2}{3}\pi i}}{w - x_1 r^2 e^{-\frac{2}{3}\pi i}} + \frac{x_1^{-1} e^{-\frac{2}{3}\pi i}}{w - x_1^{-1} e^{-\frac{2}{3}\pi i}} - \frac{x_1^{-1} r^2 e^{-\frac{2}{3}\pi i}}{w - x_1^{-1} r^2 e^{-\frac{2}{3}\pi i}} - \frac{x_1 e^{-\frac{2}{3}\pi i}}{w - x_1 e^{-\frac{2}{3}\pi i}} + \\
& \left. + \frac{x_1 r^2 e^{\frac{2}{3}\pi i}}{w - x_1 r^2 e^{\frac{2}{3}\pi i}} + \frac{x_1^{-1} e^{\frac{2}{3}\pi i}}{w - x_1^{-1} e^{\frac{2}{3}\pi i}} - \frac{x_1^{-1} r^2 e^{\frac{2}{3}\pi i}}{w - x_1^{-1} r^2 e^{\frac{2}{3}\pi i}} - \frac{x_1 e^{\frac{2}{3}\pi i}}{w - x_1 e^{\frac{2}{3}\pi i}} \right) \times \\
& \times \left(\frac{\rho^{-1}}{w + \rho^{-1}} + \frac{r^2 \rho}{w + r^2 \rho} - \frac{\rho}{w + \rho} - \frac{r^2 \rho^{-1}}{w + r^2 \rho^{-1}} \right) \frac{dw}{w}
\end{aligned}$$

I, II and III, a_1 : pole at $w = x_1 r^2$.

$$\begin{aligned}
& \frac{-\pi}{4x_1 \rho} \left[\frac{\rho^{-1}}{w + \rho^{-1}} + \frac{r^2 \rho}{w + r^2 \rho} - \frac{\rho}{w + \rho} - \frac{r^2 \rho^{-1}}{w + r^2 \rho^{-1}} \right]_{w=x_1 r^2} = \\
& = \frac{-\pi}{4x_1 \rho} \left(\frac{\rho^{-1}}{x_1 r^2 + \rho^{-1}} + \frac{r^2 \rho}{x_1 r^2 + r^2 \rho} - \frac{\rho}{x_1 r^2 + \rho} - \frac{r^2 \rho^{-1}}{x_1 r^2 + r^2 \rho^{-1}} \right) \\
& = \frac{-\pi}{4x_1 \rho} \left(\frac{\rho^{-1}}{x_1 r^2 + \rho^{-1}} + \frac{\rho}{x_1 + \rho} - \frac{\rho}{x_1 r^2 + \rho} - \frac{\rho^{-1}}{x_1 + \rho^{-1}} \right) \\
& = \frac{\pi}{4x_1 \rho} \left(\frac{1}{x_1 \rho + 1} - \frac{\rho}{x_1 + \rho} + \frac{x_1^{-1} \rho}{r^2 + x_1^{-1} \rho} - \frac{x_1^{-1} \rho^{-1}}{r^2 + x_1^{-1} \rho^{-1}} \right) \tag{79}
\end{aligned}$$

I, a_2 : pole at $w = x_1^{-1} r^2$.

$$\begin{aligned}
& -\frac{\pi}{4x_1 \rho} \left[\frac{\rho^{-1}}{w + \rho^{-1}} + \frac{r^2 \rho}{w + r^2 \rho} - \frac{\rho}{w + \rho} - \frac{r^2 \rho^{-1}}{w + r^2 \rho^{-1}} \right]_{w=x_1^{-1} r^2} = \\
& = \frac{\pi}{4x_1 \rho} \left(\frac{\rho^{-1}}{x_1^{-1} r^2 + \rho^{-1}} + \frac{r^2 \rho}{x_1^{-1} r^2 + r^2 \rho} - \frac{\rho}{x_1^{-1} r^2 + \rho} - \frac{r^2 \rho^{-1}}{x_1^{-1} r^2 + r^2 \rho^{-1}} \right) \\
& = \frac{\pi}{4x_1 \rho} \left(\frac{x_1 \rho^{-1}}{r^2 + x_1 \rho^{-1}} + \frac{x_1 \rho}{1 + x_1 \rho} - \frac{x_1 \rho}{r^2 + x_1 \rho} - \frac{x_1}{\rho + x_1} \right) \\
& = \frac{\pi}{4x_1 \rho} \left(\frac{x_1 \rho}{1 + x_1 \rho} - \frac{x_1}{\rho + x_1} + \frac{x_1 \rho^{-1}}{r^2 + x_1 \rho^{-1}} - \frac{x_1 \rho}{r^2 + x_1 \rho} \right) \tag{80}
\end{aligned}$$

II and III, a_2 : pole at $w = x_1$.

$$\begin{aligned}
& -\frac{-\pi}{4x_1\rho} \left[\frac{\rho^{-1}}{w + \rho^{-1}} + \frac{r^2\rho}{w + r^2\rho} - \frac{\rho}{w + \rho} - \frac{r^2\rho^{-1}}{w + r^2\rho^{-1}} \right]_{w=x_1} = \\
& = \frac{\pi}{4x_1\rho} \left(\frac{\rho^{-1}}{x_1 + \rho^{-1}} + \frac{r^2\rho}{x_1 + r^2\rho} - \frac{\rho}{x_1 + \rho} - \frac{r^2\rho^{-1}}{x_1 + r^2\rho^{-1}} \right) \\
& = \frac{\pi}{4x_1\rho} \left(\frac{1}{x_1\rho + 1} + 1 - \frac{x_1}{x_1 + r^2\rho} - \frac{\rho}{x_1 + \rho} - 1 + \frac{x_1}{x_1 + r^2\rho^{-1}} \right) \\
& = \frac{\pi}{4x_1\rho} \left(\frac{1}{x_1\rho + 1} - \frac{\rho}{x_1 + \rho} + \frac{x_1\rho}{r^2 + x_1\rho} - \frac{x_1\rho^{-1}}{r^2 + x_1\rho^{-1}} \right)
\end{aligned} \tag{81}$$

I, II and III, b_1 : pole at $w = x_1r^2e^{-\frac{2}{3}\pi i}$.

$$\begin{aligned}
& \frac{-\pi}{4x_1\rho} \left[\frac{\rho^{-1}}{w + \rho^{-1}} + \frac{r^2\rho}{w + r^2\rho} - \frac{\rho}{w + \rho} - \frac{r^2\rho^{-1}}{w + r^2\rho^{-1}} \right]_{w=x_1r^2e^{-\frac{2}{3}\pi i}} = \\
& = \frac{-\pi}{4x_1\rho} \left(\frac{\rho^{-1}}{x_1r^2e^{-\frac{2}{3}\pi i} + \rho^{-1}} + \frac{r^2\rho}{x_1r^2e^{-\frac{2}{3}\pi i} + r^2\rho} - \frac{\rho}{x_1r^2e^{-\frac{2}{3}\pi i} + \rho} - \frac{r^2\rho^{-1}}{x_1r^2e^{-\frac{2}{3}\pi i} + r^2\rho^{-1}} \right) \\
& = \frac{-\pi}{4x_1\rho} \left(\frac{x_1^{-1}\rho^{-1}e^{\frac{2}{3}\pi i}}{r^2 + x_1^{-1}\rho^{-1}e^{\frac{2}{3}\pi i}} + \frac{\rho}{x_1e^{-\frac{2}{3}\pi i} + \rho} - \frac{x_1^{-1}\rho e^{\frac{2}{3}\pi i}}{r^2 + x_1^{-1}\rho e^{\frac{2}{3}\pi i}} - \frac{1}{x_1\rho e^{-\frac{2}{3}\pi i} + 1} \right) \\
& = \frac{\pi}{4x_1\rho} \left(\frac{1}{x_1\rho e^{-\frac{2}{3}\pi i} + 1} - \frac{\rho}{x_1e^{-\frac{2}{3}\pi i} + \rho} + \frac{x_1^{-1}\rho e^{\frac{2}{3}\pi i}}{r^2 + x_1^{-1}\rho e^{\frac{2}{3}\pi i}} - \frac{x_1^{-1}\rho^{-1}e^{\frac{2}{3}\pi i}}{r^2 + x_1^{-1}\rho^{-1}e^{\frac{2}{3}\pi i}} \right)
\end{aligned} \tag{82}$$

I, b_2 : pole at $w = x_1^{-1}r^2e^{-\frac{2}{3}\pi i}$.

$$\begin{aligned}
& -\frac{-\pi}{4x_1\rho} \left[\frac{\rho^{-1}}{w + \rho^{-1}} + \frac{r^2\rho}{w + r^2\rho} - \frac{\rho}{w + \rho} - \frac{r^2\rho^{-1}}{w + r^2\rho^{-1}} \right]_{w=x_1^{-1}r^2e^{-\frac{2}{3}\pi i}} = \\
& = \frac{\pi}{4x_1\rho} \left(\frac{\rho^{-1}}{x_1^{-1}r^2e^{-\frac{2}{3}\pi i} + \rho^{-1}} + \frac{r^2\rho}{x_1^{-1}r^2e^{-\frac{2}{3}\pi i} + r^2\rho} - \frac{\rho}{x_1^{-1}r^2e^{-\frac{2}{3}\pi i} + \rho} - \frac{r^2\rho^{-1}}{x_1^{-1}r^2e^{-\frac{2}{3}\pi i} + r^2\rho^{-1}} \right) \\
& = \frac{\pi}{4x_1\rho} \left(\frac{x_1\rho^{-1}e^{\frac{2}{3}\pi i}}{r^2 + x_1\rho^{-1}e^{\frac{2}{3}\pi i}} + \frac{x_1\rho e^{\frac{2}{3}\pi i}}{1 + x_1\rho e^{\frac{2}{3}\pi i}} - \frac{x_1\rho e^{\frac{2}{3}\pi i}}{r^2 + x_1\rho e^{\frac{2}{3}\pi i}} - \frac{x_1}{\rho e^{-\frac{2}{3}\pi i} + x_1} \right) \\
& = \frac{\pi}{4x_1\rho} \left(\frac{x_1\rho e^{\frac{2}{3}\pi i}}{1 + x_1\rho e^{\frac{2}{3}\pi i}} - \frac{x_1e^{\frac{2}{3}\pi i}}{\rho + x_1e^{\frac{2}{3}\pi i}} + \frac{x_1\rho^{-1}e^{\frac{2}{3}\pi i}}{r^2 + x_1\rho^{-1}e^{\frac{2}{3}\pi i}} - \frac{x_1\rho e^{\frac{2}{3}\pi i}}{r^2 + x_1\rho e^{\frac{2}{3}\pi i}} \right)
\end{aligned} \tag{83}$$

II and III, b_2 : pole at $w = x_1 e^{-\frac{2}{3}\pi i}$.

$$\begin{aligned}
& -\frac{-\pi}{4x_1\rho} \left[\frac{\rho^{-1}}{w+\rho^{-1}} + \frac{r^2\rho}{w+r^2\rho} - \frac{\rho}{w+\rho} - \frac{r^2\rho^{-1}}{w+r^2\rho^{-1}} \right]_{w=x_1 e^{-\frac{2}{3}\pi i}} = \\
& = \frac{\pi}{4x_1\rho} \left(\frac{\rho^{-1}}{x_1 e^{-\frac{2}{3}\pi i} + \rho^{-1}} + \frac{r^2\rho}{x_1 e^{-\frac{2}{3}\pi i} + r^2\rho} - \frac{\rho}{x_1 e^{-\frac{2}{3}\pi i} + \rho} - \frac{r^2\rho^{-1}}{x_1 e^{-\frac{2}{3}\pi i} + r^2\rho^{-1}} \right) \\
& = \frac{\pi}{4x_1\rho} \left(\frac{\rho^{-1}}{x_1 e^{-\frac{2}{3}\pi i} + \rho^{-1}} + 1 - \frac{x_1 e^{-\frac{2}{3}\pi i}}{x_1 e^{-\frac{2}{3}\pi i} + r^2\rho} - \frac{\rho}{x_1 e^{-\frac{2}{3}\pi i} + \rho} - 1 + \frac{x_1 e^{-\frac{2}{3}\pi i}}{x_1 e^{-\frac{2}{3}\pi i} + r^2\rho^{-1}} \right) \\
& = \frac{\pi}{4x_1\rho} \left(\frac{1}{x_1 \rho e^{-\frac{2}{3}\pi i} + 1} - \frac{\rho}{x_1 e^{-\frac{2}{3}\pi i} + \rho} + \frac{x_1 \rho e^{-\frac{2}{3}\pi i}}{r^2 + x_1 \rho e^{-\frac{2}{3}\pi i}} - \frac{x_1 \rho^{-1} e^{-\frac{2}{3}\pi i}}{r^2 + x_1 \rho^{-1} e^{-\frac{2}{3}\pi i}} \right) \tag{84}
\end{aligned}$$

I, II and III, c_1 : pole at $w = x_1 r^2 e^{\frac{2}{3}\pi i}$.

$$\begin{aligned}
& -\frac{-\pi}{4x_1\rho} \left[\frac{\rho^{-1}}{w+\rho^{-1}} + \frac{r^2\rho}{w+r^2\rho} - \frac{\rho}{w+\rho} - \frac{r^2\rho^{-1}}{w+r^2\rho^{-1}} \right]_{w=x_1 r^2 e^{\frac{2}{3}\pi i}} = \\
& = \frac{-\pi}{4x_1\rho} \left(\frac{\rho^{-1}}{x_1 r^2 e^{\frac{2}{3}\pi i} + \rho^{-1}} + \frac{r^2\rho}{x_1 r^2 e^{\frac{2}{3}\pi i} + r^2\rho} - \frac{\rho}{x_1 r^2 e^{\frac{2}{3}\pi i} + \rho} - \frac{r^2\rho^{-1}}{x_1 r^2 e^{\frac{2}{3}\pi i} + r^2\rho^{-1}} \right) \\
& = \frac{\pi}{4x_1\rho} \left(\frac{1}{x_1 \rho e^{\frac{2}{3}\pi i} + 1} - \frac{\rho}{x_1 e^{\frac{2}{3}\pi i} + \rho} + \frac{x_1^{-1} \rho e^{-\frac{2}{3}\pi i}}{r^2 + x_1^{-1} \rho e^{-\frac{2}{3}\pi i}} - \frac{x_1^{-1} \rho^{-1} e^{-\frac{2}{3}\pi i}}{r^2 + x_1^{-1} \rho^{-1} e^{-\frac{2}{3}\pi i}} \right) \tag{85}
\end{aligned}$$

I, c_2 : pole at $w = x_1^{-1} r^2 e^{\frac{2}{3}\pi i}$.

$$\begin{aligned}
& -\frac{-\pi}{4x_1\rho} \left[\frac{\rho^{-1}}{w+\rho^{-1}} + \frac{r^2\rho}{w+r^2\rho} - \frac{\rho}{w+\rho} - \frac{r^2\rho^{-1}}{w+r^2\rho^{-1}} \right]_{w=x_1^{-1} r^2 e^{\frac{2}{3}\pi i}} = \\
& = \frac{\pi}{4x_1\rho} \left(\frac{\rho^{-1}}{x_1^{-1} r^2 e^{\frac{2}{3}\pi i} + \rho^{-1}} + \frac{r^2\rho}{x_1^{-1} r^2 e^{\frac{2}{3}\pi i} + r^2\rho} - \frac{\rho}{x_1^{-1} r^2 e^{\frac{2}{3}\pi i} + \rho} - \frac{r^2\rho^{-1}}{x_1^{-1} r^2 e^{\frac{2}{3}\pi i} + r^2\rho^{-1}} \right) \\
& = \frac{\pi}{4x_1\rho} \left(\frac{x_1 \rho e^{-\frac{2}{3}\pi i}}{1 + x_1 \rho e^{-\frac{2}{3}\pi i}} - \frac{x_1 e^{-\frac{2}{3}\pi i}}{\rho + x_1 e^{-\frac{2}{3}\pi i}} + \frac{x_1 \rho^{-1} e^{-\frac{2}{3}\pi i}}{r^2 + x_1 \rho^{-1} e^{-\frac{2}{3}\pi i}} - \frac{x_1 \rho e^{-\frac{2}{3}\pi i}}{r^2 + x_1 \rho e^{-\frac{2}{3}\pi i}} \right) \tag{86}
\end{aligned}$$

II and III, c_2 : pole at $w = x_1 e^{\frac{2}{3}\pi i}$.

$$\begin{aligned}
& -\frac{-\pi}{4x_1\rho} \left[\frac{\rho^{-1}}{w+\rho^{-1}} + \frac{r^2\rho}{w+r^2\rho} - \frac{\rho}{w+\rho} - \frac{r^2\rho^{-1}}{w+r^2\rho^{-1}} \right]_{w=x_1 e^{\frac{2}{3}\pi i}} = \\
& = \frac{\pi}{4x_1\rho} \left(\frac{\rho^{-1}}{x_1 e^{\frac{2}{3}\pi i} + \rho^{-1}} + \frac{r^2\rho}{x_1 e^{\frac{2}{3}\pi i} + r^2\rho} - \frac{\rho}{x_1 e^{\frac{2}{3}\pi i} + \rho} - \frac{r^2\rho^{-1}}{x_1 e^{\frac{2}{3}\pi i} + r^2\rho^{-1}} \right) \\
& = \frac{\pi}{4x_1\rho} \left(\frac{1}{x_1 \rho e^{\frac{2}{3}\pi i} + 1} + 1 - \frac{x_1 e^{\frac{2}{3}\pi i}}{x_1 e^{\frac{2}{3}\pi i} + r^2\rho} - \frac{\rho}{x_1 e^{\frac{2}{3}\pi i} + \rho} - 1 + \frac{x_1 e^{\frac{2}{3}\pi i}}{x_1 e^{\frac{2}{3}\pi i} + r^2\rho^{-1}} \right) \\
& = \frac{\pi}{4x_1\rho} \left(\frac{1}{x_1 \rho e^{\frac{2}{3}\pi i} + 1} - \frac{\rho}{x_1 e^{\frac{2}{3}\pi i} + \rho} + \frac{x_1 \rho e^{\frac{2}{3}\pi i}}{r^2 + x_1 \rho e^{\frac{2}{3}\pi i}} - \frac{x_1 \rho^{-1} e^{\frac{2}{3}\pi i}}{r^2 + x_1 \rho^{-1} e^{\frac{2}{3}\pi i}} \right). \tag{87}
\end{aligned}$$

I, II and III, d_1 : pole at $w = -r^2\rho$.

$$\begin{aligned}
& \frac{\pi}{4\rho x_1} \left[\frac{x_1 r^2}{w - x_1 r^2} + \frac{x_1^{-1}}{w - x_1^{-1}} - \frac{x_1^{-1} r^2}{w - x_1^{-1} r^2} - \frac{x_1}{w - x_1} \right]_{w=-r^2\rho} + \\
& + \frac{\pi}{4\rho x_1} \left[\frac{x_1 r^2 e^{-\frac{2}{3}\pi i}}{w - x_1 r^2 e^{-\frac{2}{3}\pi i}} + \frac{x_1^{-1} e^{-\frac{2}{3}\pi i}}{w - x_1^{-1} e^{-\frac{2}{3}\pi i}} - \frac{x_1^{-1} r^2 e^{-\frac{2}{3}\pi i}}{w - x_1^{-1} r^2 e^{-\frac{2}{3}\pi i}} - \frac{x_1 e^{-\frac{2}{3}\pi i}}{w - x_1 e^{-\frac{2}{3}\pi i}} \right]_{w=-r^2\rho} + \\
& + \frac{\pi}{4\rho x_1} \left[\frac{x_1 r^2 e^{\frac{2}{3}\pi i}}{w - x_1 r^2 e^{\frac{2}{3}\pi i}} + \frac{x_1^{-1} e^{\frac{2}{3}\pi i}}{w - x_1^{-1} e^{\frac{2}{3}\pi i}} - \frac{x_1^{-1} r^2 e^{\frac{2}{3}\pi i}}{w - x_1^{-1} r^2 e^{\frac{2}{3}\pi i}} - \frac{x_1 e^{\frac{2}{3}\pi i}}{w - x_1 e^{\frac{2}{3}\pi i}} \right]_{w=-r^2\rho} = \\
= & \frac{\pi}{4\rho x_1} \left(\frac{x_1 r^2}{-r^2\rho - x_1 r^2} + \frac{x_1^{-1}}{-r^2\rho - x_1^{-1}} - \frac{x_1^{-1} r^2}{-r^2\rho - x_1^{-1} r^2} - \frac{x_1}{-r^2\rho - x_1} \right) + \\
& + \frac{\pi}{4\rho x_1} \left(\frac{x_1 r^2 e^{-\frac{2}{3}\pi i}}{-r^2\rho - x_1 r^2 e^{-\frac{2}{3}\pi i}} + \frac{x_1^{-1} e^{-\frac{2}{3}\pi i}}{-r^2\rho - x_1^{-1} e^{-\frac{2}{3}\pi i}} - \frac{x_1^{-1} r^2 e^{-\frac{2}{3}\pi i}}{-r^2\rho - x_1^{-1} r^2 e^{-\frac{2}{3}\pi i}} - \frac{x_1 e^{-\frac{2}{3}\pi i}}{-r^2\rho - x_1 e^{-\frac{2}{3}\pi i}} \right) + \\
& + \frac{\pi}{4\rho x_1} \left(\frac{x_1 r^2 e^{\frac{2}{3}\pi i}}{-r^2\rho - x_1 r^2 e^{\frac{2}{3}\pi i}} + \frac{x_1^{-1} e^{\frac{2}{3}\pi i}}{-r^2\rho - x_1^{-1} e^{\frac{2}{3}\pi i}} - \frac{x_1^{-1} r^2 e^{\frac{2}{3}\pi i}}{-r^2\rho - x_1^{-1} r^2 e^{\frac{2}{3}\pi i}} - \frac{x_1 e^{\frac{2}{3}\pi i}}{-r^2\rho - x_1 e^{\frac{2}{3}\pi i}} \right) \\
= & \frac{\pi}{4\rho x_1} \left(-\frac{x_1}{\rho + x_1} - \frac{x_1^{-1} \rho^{-1}}{r^2 + x_1^{-1} \rho^{-1}} + \frac{1}{x_1 \rho + 1} + \frac{x_1 \rho^{-1}}{r^2 + x_1 \rho^{-1}} \right) + \\
& + \frac{\pi}{4\rho x_1} \left(-\frac{x_1 e^{-\frac{2}{3}\pi i}}{\rho + x_1 e^{-\frac{2}{3}\pi i}} - \frac{x_1^{-1} \rho^{-1} e^{-\frac{2}{3}\pi i}}{r^2 + x_1^{-1} \rho^{-1} e^{-\frac{2}{3}\pi i}} + \frac{1}{1 + x_1 \rho e^{\frac{2}{3}\pi i}} + \frac{x_1 \rho^{-1} e^{-\frac{2}{3}\pi i}}{r^2 + x_1 \rho^{-1} e^{-\frac{2}{3}\pi i}} \right) + \\
& + \frac{\pi}{4\rho x_1} \left(-\frac{x_1 e^{\frac{2}{3}\pi i}}{\rho + x_1 e^{\frac{2}{3}\pi i}} - \frac{x_1^{-1} \rho^{-1} e^{\frac{2}{3}\pi i}}{r^2 + x_1^{-1} \rho^{-1} e^{\frac{2}{3}\pi i}} + \frac{1}{x_1 \rho e^{-\frac{2}{3}\pi i} + 1} + \frac{x_1 \rho^{-1} e^{\frac{2}{3}\pi i}}{r^2 + x_1 \rho^{-1} e^{\frac{2}{3}\pi i}} \right) \\
= & \frac{\pi}{4\rho x_1} \left(\frac{1}{1 + x_1 \rho} - \frac{x_1}{\rho + x_1} \right) + \\
& + \frac{\pi}{4\rho x_1} \left(\frac{1}{1 + x_1 \rho e^{\frac{2}{3}\pi i}} + \frac{1}{1 + x_1 \rho e^{-\frac{2}{3}\pi i}} - \frac{x_1 e^{-\frac{2}{3}\pi i}}{\rho + x_1 e^{-\frac{2}{3}\pi i}} - \frac{x_1 e^{\frac{2}{3}\pi i}}{\rho + x_1 e^{\frac{2}{3}\pi i}} \right) \\
& + \frac{\pi}{4\rho x_1} \left(\frac{x_1 \rho^{-1}}{r^2 + x_1 \rho^{-1}} - \frac{x_1^{-1} \rho^{-1}}{r^2 + x_1^{-1} \rho^{-1}} \right) + \\
& + \frac{\pi}{4\rho x_1} \left(\frac{x_1 \rho^{-1} e^{-\frac{2}{3}\pi i}}{r^2 + x_1 \rho^{-1} e^{-\frac{2}{3}\pi i}} - \frac{x_1^{-1} \rho^{-1} e^{-\frac{2}{3}\pi i}}{r^2 + x_1^{-1} \rho^{-1} e^{-\frac{2}{3}\pi i}} \right) + \\
& + \frac{\pi}{4\rho x_1} \left(\frac{x_1 \rho^{-1} e^{\frac{2}{3}\pi i}}{r^2 + x_1 \rho^{-1} e^{\frac{2}{3}\pi i}} - \frac{x_1^{-1} \rho^{-1} e^{\frac{2}{3}\pi i}}{r^2 + x_1^{-1} \rho^{-1} e^{\frac{2}{3}\pi i}} \right) \tag{88}
\end{aligned}$$

I and II, d_2 : pole at $w = -r^2\rho^{-1}$.

$$\begin{aligned}
& \frac{-\pi}{4\rho x_1} \left[\frac{x_1 r^2}{w - x_1 r^2} + \frac{x_1^{-1}}{w - x_1^{-1}} - \frac{x_1^{-1} r^2}{w - x_1^{-1} r^2} - \frac{x_1}{w - x_1} \right]_{w=-r^2\rho^{-1}} + \\
& \frac{-\pi}{4\rho x_1} \left[\frac{x_1 r^2 e^{-\frac{2}{3}\pi i}}{w - x_1 r^2 e^{-\frac{2}{3}\pi i}} + \frac{x_1^{-1} e^{-\frac{2}{3}\pi i}}{w - x_1^{-1} e^{-\frac{2}{3}\pi i}} - \frac{x_1^{-1} r^2 e^{-\frac{2}{3}\pi i}}{w - x_1^{-1} r^2 e^{-\frac{2}{3}\pi i}} - \frac{x_1 e^{-\frac{2}{3}\pi i}}{w - x_1 e^{-\frac{2}{3}\pi i}} \right]_{w=-r^2\rho^{-1}} + \\
& \frac{-\pi}{4\rho x_1} \left[\frac{x_1 r^2 e^{\frac{2}{3}\pi i}}{w - x_1 r^2 e^{\frac{2}{3}\pi i}} + \frac{x_1^{-1} e^{\frac{2}{3}\pi i}}{w - x_1^{-1} e^{\frac{2}{3}\pi i}} - \frac{x_1^{-1} r^2 e^{\frac{2}{3}\pi i}}{w - x_1^{-1} r^2 e^{\frac{2}{3}\pi i}} - \frac{x_1 e^{\frac{2}{3}\pi i}}{w - x_1 e^{\frac{2}{3}\pi i}} \right]_{w=-r^2\rho^{-1}} = \\
= & \frac{-\pi}{4\rho x_1} \left(\frac{x_1 r^2}{-r^2\rho^{-1} - x_1 r^2} + \frac{x_1^{-1}}{-r^2\rho^{-1} - x_1^{-1}} - \frac{x_1^{-1} r^2}{-r^2\rho^{-1} - x_1^{-1} r^2} - \frac{x_1}{-r^2\rho^{-1} - x_1} \right) + \\
& + \frac{-\pi}{4\rho x_1} \left(\frac{x_1 r^2 e^{-\frac{2}{3}\pi i}}{-r^2\rho^{-1} - x_1 r^2 e^{-\frac{2}{3}\pi i}} + \frac{x_1^{-1} e^{-\frac{2}{3}\pi i}}{-r^2\rho^{-1} - x_1^{-1} e^{-\frac{2}{3}\pi i}} - \frac{x_1^{-1} r^2 e^{-\frac{2}{3}\pi i}}{-r^2\rho^{-1} - x_1^{-1} r^2 e^{-\frac{2}{3}\pi i}} - \frac{x_1 e^{-\frac{2}{3}\pi i}}{-r^2\rho^{-1} - x_1 e^{-\frac{2}{3}\pi i}} \right) + \\
& + \frac{-\pi}{4\rho x_1} \left(\frac{x_1 r^2 e^{\frac{2}{3}\pi i}}{-r^2\rho^{-1} - x_1 r^2 e^{\frac{2}{3}\pi i}} + \frac{x_1^{-1} e^{\frac{2}{3}\pi i}}{-r^2\rho^{-1} - x_1^{-1} e^{\frac{2}{3}\pi i}} - \frac{x_1^{-1} r^2 e^{\frac{2}{3}\pi i}}{-r^2\rho^{-1} - x_1^{-1} r^2 e^{\frac{2}{3}\pi i}} - \frac{x_1 e^{\frac{2}{3}\pi i}}{-r^2\rho^{-1} - x_1 e^{\frac{2}{3}\pi i}} \right) \\
= & \frac{\pi}{4\rho x_1} \left(\frac{\rho}{x_1 + \rho} - \frac{x_1 \rho}{1 + x_1 \rho} + \frac{x_1 \rho}{r^2 + x_1 \rho} - \frac{x_1^{-1} \rho}{r^2 + x_1^{-1} \rho} \right) + \\
& + \frac{\pi}{4\rho x_1} \left(\frac{\rho e^{-\frac{2}{3}\pi i}}{x_1 + \rho e^{-\frac{2}{3}\pi i}} - \frac{x_1 \rho e^{-\frac{2}{3}\pi i}}{1 + x_1 \rho e^{-\frac{2}{3}\pi i}} + \frac{x_1 \rho e^{-\frac{2}{3}\pi i}}{r^2 + x_1 \rho e^{-\frac{2}{3}\pi i}} - \frac{x_1^{-1} \rho e^{-\frac{2}{3}\pi i}}{r^2 + x_1^{-1} \rho e^{-\frac{2}{3}\pi i}} \right) + \\
& + \frac{\pi}{4\rho x_1} \left(\frac{\rho e^{\frac{2}{3}\pi i}}{x_1 + \rho e^{\frac{2}{3}\pi i}} - \frac{x_1 \rho e^{\frac{2}{3}\pi i}}{1 + x_1 \rho e^{\frac{2}{3}\pi i}} + \frac{x_1 \rho e^{\frac{2}{3}\pi i}}{r^2 + x_1 \rho e^{\frac{2}{3}\pi i}} - \frac{x_1^{-1} \rho e^{\frac{2}{3}\pi i}}{r^2 + x_1^{-1} \rho e^{\frac{2}{3}\pi i}} \right) \\
= & \frac{\pi}{4\rho x_1} \left(\frac{x_1 \rho}{1 + x_1 \rho} - \frac{\rho}{x_1 + \rho} + \frac{x_1 \rho e^{-\frac{2}{3}\pi i}}{1 + x_1 \rho e^{-\frac{2}{3}\pi i}} + \frac{x_1 \rho e^{\frac{2}{3}\pi i}}{1 + x_1 \rho e^{\frac{2}{3}\pi i}} - \frac{\rho e^{-\frac{2}{3}\pi i}}{x_1 + \rho e^{-\frac{2}{3}\pi i}} - \frac{\rho e^{\frac{2}{3}\pi i}}{x_1 + \rho e^{\frac{2}{3}\pi i}} \right) + \\
& + \frac{\pi}{4\rho x_1} \left(\frac{x_1^{-1} \rho}{r^2 + x_1^{-1} \rho} - \frac{x_1 \rho}{r^2 + x_1 \rho} \right) + \\
& + \frac{\pi}{4\rho x_1} \left(\frac{x_1^{-1} \rho e^{-\frac{2}{3}\pi i}}{r^2 + x_1^{-1} \rho e^{-\frac{2}{3}\pi i}} - \frac{x_1 \rho e^{-\frac{2}{3}\pi i}}{r^2 + x_1 \rho e^{-\frac{2}{3}\pi i}} \right) + \\
& + \frac{\pi}{4\rho x_1} \left(\frac{x_1^{-1} \rho e^{\frac{2}{3}\pi i}}{r^2 + x_1^{-1} \rho e^{\frac{2}{3}\pi i}} - \frac{x_1 \rho e^{\frac{2}{3}\pi i}}{r^2 + x_1 \rho e^{\frac{2}{3}\pi i}} \right)
\end{aligned}$$

$$\begin{aligned}
&= \frac{\pi}{4\rho x_1} \left(\frac{x_1\rho}{1+x_1\rho} - \frac{\rho}{x_1+\rho} + \frac{x_1\rho(2x_1\rho-1)}{1-x_1\rho+x_1^2\rho^2} - \frac{\rho(2\rho-x_1)}{x_1^2-x_1\rho+\rho^2} \right) + \\
&+ \frac{\pi}{4\rho x_1} \left(\frac{x_1^{-1}\rho}{r^2+x_1^{-1}\rho} - \frac{x_1\rho}{r^2+x_1\rho} \right) + \\
&+ \frac{\pi}{4\rho x_1} \left(\frac{x_1^{-1}\rho e^{-\frac{2}{3}\pi i}}{r^2+x_1^{-1}\rho e^{-\frac{2}{3}\pi i}} - \frac{x_1\rho e^{-\frac{2}{3}\pi i}}{r^2+x_1\rho e^{-\frac{2}{3}\pi i}} \right) + \\
&+ \frac{\pi}{4\rho x_1} \left(\frac{x_1^{-1}\rho e^{\frac{2}{3}\pi i}}{r^2+x_1^{-1}\rho e^{\frac{2}{3}\pi i}} - \frac{x_1\rho e^{\frac{2}{3}\pi i}}{r^2+x_1\rho e^{\frac{2}{3}\pi i}} \right)
\end{aligned} \tag{89}$$

III, d_2 : pole at $w = -\rho$.

$$\begin{aligned}
&\frac{-\pi}{4\rho x_1} \left[\frac{x_1 r^2}{w-x_1 r^2} + \frac{x_1^{-1}}{w-x_1^{-1}} - \frac{x_1^{-1} r^2}{w-x_1^{-1} r^2} - \frac{x_1}{w-x_1} \right]_{w=-\rho} + \\
&\frac{-\pi}{4\rho x_1} \left[\frac{x_1 r^2 e^{-\frac{2}{3}\pi i}}{w-x_1 r^2 e^{-\frac{2}{3}\pi i}} + \frac{x_1^{-1} e^{-\frac{2}{3}\pi i}}{w-x_1^{-1} e^{-\frac{2}{3}\pi i}} - \frac{x_1^{-1} r^2 e^{-\frac{2}{3}\pi i}}{w-x_1^{-1} r^2 e^{-\frac{2}{3}\pi i}} - \frac{x_1 e^{-\frac{2}{3}\pi i}}{w-x_1 e^{-\frac{2}{3}\pi i}} \right]_{w=-\rho} + \\
&\frac{-\pi}{4\rho x_1} \left[\frac{x_1 r^2 e^{\frac{2}{3}\pi i}}{w-x_1 r^2 e^{\frac{2}{3}\pi i}} + \frac{x_1^{-1} e^{\frac{2}{3}\pi i}}{w-x_1^{-1} e^{\frac{2}{3}\pi i}} - \frac{x_1^{-1} r^2 e^{\frac{2}{3}\pi i}}{w-x_1^{-1} r^2 e^{\frac{2}{3}\pi i}} - \frac{x_1 e^{\frac{2}{3}\pi i}}{w-x_1 e^{\frac{2}{3}\pi i}} \right]_{w=-\rho} = \\
&= \frac{-\pi}{4\rho x_1} \left(\frac{x_1 r^2}{-\rho-x_1 r^2} + \frac{x_1^{-1}}{-\rho-x_1^{-1}} - \frac{x_1^{-1} r^2}{-\rho-x_1^{-1} r^2} - \frac{x_1}{-\rho-x_1} \right) + \\
&+ \frac{-\pi}{4\rho x_1} \left(\frac{x_1 r^2 e^{-\frac{2}{3}\pi i}}{-\rho-x_1 r^2 e^{-\frac{2}{3}\pi i}} + \frac{x_1^{-1} e^{-\frac{2}{3}\pi i}}{-\rho-x_1^{-1} e^{-\frac{2}{3}\pi i}} - \frac{x_1^{-1} r^2 e^{-\frac{2}{3}\pi i}}{-\rho-x_1^{-1} r^2 e^{-\frac{2}{3}\pi i}} - \frac{x_1 e^{-\frac{2}{3}\pi i}}{-\rho-x_1 e^{-\frac{2}{3}\pi i}} \right) + \\
&+ \frac{-\pi}{4\rho x_1} \left(\frac{x_1 r^2 e^{\frac{2}{3}\pi i}}{-\rho-x_1 r^2 e^{\frac{2}{3}\pi i}} + \frac{x_1^{-1} e^{\frac{2}{3}\pi i}}{-\rho-x_1^{-1} e^{\frac{2}{3}\pi i}} - \frac{x_1^{-1} r^2 e^{\frac{2}{3}\pi i}}{-\rho-x_1^{-1} r^2 e^{\frac{2}{3}\pi i}} - \frac{x_1 e^{\frac{2}{3}\pi i}}{-\rho-x_1 e^{\frac{2}{3}\pi i}} \right) \\
&= \frac{-\pi}{4\rho x_1} \left(-1 + \frac{\rho}{\rho+x_1 r^2} - \frac{x_1^{-1}}{\rho+x_1^{-1}} + 1 - \frac{\rho}{\rho+x_1^{-1} r^2} + \frac{x_1}{\rho+x_1} \right) + \\
&+ \frac{-\pi}{4\rho x_1} \left(-1 + \frac{\rho}{\rho+x_1 r^2 e^{-\frac{2}{3}\pi i}} - \frac{x_1^{-1} e^{-\frac{2}{3}\pi i}}{\rho+x_1^{-1} e^{-\frac{2}{3}\pi i}} + 1 - \frac{\rho}{\rho+x_1^{-1} r^2 e^{-\frac{2}{3}\pi i}} + \frac{x_1 e^{-\frac{2}{3}\pi i}}{\rho+x_1 e^{-\frac{2}{3}\pi i}} \right) + \\
&+ \frac{-\pi}{4\rho x_1} \left(-1 + \frac{\rho}{\rho+x_1 r^2 e^{\frac{2}{3}\pi i}} - \frac{x_1^{-1} e^{\frac{2}{3}\pi i}}{\rho+x_1^{-1} e^{\frac{2}{3}\pi i}} + 1 - \frac{\rho}{\rho+x_1^{-1} r^2 e^{\frac{2}{3}\pi i}} + \frac{x_1 e^{\frac{2}{3}\pi i}}{\rho+x_1 e^{\frac{2}{3}\pi i}} \right) \\
&= \frac{-\pi}{4\rho x_1} \left(\frac{x_1}{\rho+x_1} - \frac{1}{x_1\rho+1} + \frac{x_1^{-1}\rho}{r^2+x_1^{-1}\rho} - \frac{x_1\rho}{r^2+x_1\rho} \right) + \\
&+ \frac{-\pi}{4\rho x_1} \left(\frac{x_1 e^{-\frac{2}{3}\pi i}}{\rho+x_1 e^{-\frac{2}{3}\pi i}} - \frac{1}{1+x_1\rho e^{\frac{2}{3}\pi i}} + \frac{x_1^{-1}\rho e^{\frac{2}{3}\pi i}}{r^2+x_1^{-1}\rho e^{\frac{2}{3}\pi i}} - \frac{x_1\rho e^{\frac{2}{3}\pi i}}{r^2+x_1\rho e^{\frac{2}{3}\pi i}} \right) + \\
&+ \frac{-\pi}{4\rho x_1} \left(\frac{x_1 e^{\frac{2}{3}\pi i}}{\rho+x_1 e^{\frac{2}{3}\pi i}} - \frac{1}{1+x_1\rho e^{-\frac{2}{3}\pi i}} + \frac{x_1^{-1}\rho e^{-\frac{2}{3}\pi i}}{r^2+x_1^{-1}\rho e^{-\frac{2}{3}\pi i}} - \frac{x_1\rho e^{-\frac{2}{3}\pi i}}{r^2+x_1\rho e^{-\frac{2}{3}\pi i}} \right).
\end{aligned} \tag{90}$$

$$\begin{aligned}
&= \frac{-\pi}{4\rho x_1} \left(\frac{x_1}{\rho + x_1} - \frac{1}{x_1\rho + 1} - \frac{1}{1 + x_1\rho e^{\frac{2}{3}\pi i}} - \frac{1}{1 + x_1\rho e^{-\frac{2}{3}\pi i}} + \frac{x_1 e^{-\frac{2}{3}\pi i}}{\rho + x_1 e^{-\frac{2}{3}\pi i}} + \frac{x_1 e^{\frac{2}{3}\pi i}}{\rho + x_1 e^{\frac{2}{3}\pi i}} \right) + \\
&+ \frac{-\pi}{4\rho x_1} \left(\frac{x_1^{-1}\rho}{r^2 + x_1^{-1}\rho} - \frac{x_1\rho}{r^2 + x_1\rho} \right) + \\
&+ \frac{-\pi}{4\rho x_1} \left(\frac{x_1^{-1}\rho e^{\frac{2}{3}\pi i}}{r^2 + x_1^{-1}\rho e^{\frac{2}{3}\pi i}} - \frac{x_1\rho e^{\frac{2}{3}\pi i}}{r^2 + x_1\rho e^{\frac{2}{3}\pi i}} \right) + \\
&+ \frac{-\pi}{4\rho x_1} \left(\frac{x_1^{-1}\rho e^{-\frac{2}{3}\pi i}}{r^2 + x_1^{-1}\rho e^{-\frac{2}{3}\pi i}} - \frac{x_1\rho e^{-\frac{2}{3}\pi i}}{r^2 + x_1\rho e^{-\frac{2}{3}\pi i}} \right). \tag{91}
\end{aligned}$$

$$\begin{aligned}
&= \frac{\pi}{4\rho x_1} \left(\frac{1}{x_1\rho + 1} - \frac{x_1}{\rho + x_1} + \frac{2 - x_1\rho}{1 - x_1\rho + x_1^2\rho^2} + \frac{(\rho - 2x_1)x_1}{x_1^2 - x_1\rho + \rho^2} \right) + \\
&+ \frac{\pi}{4\rho x_1} \left(\frac{x_1\rho}{r^2 + x_1\rho} - \frac{x_1^{-1}\rho}{r^2 + x_1^{-1}\rho} \right) + \\
&+ \frac{\pi}{4\rho x_1} \left(\frac{x_1\rho e^{\frac{2}{3}\pi i}}{r^2 + x_1\rho e^{\frac{2}{3}\pi i}} - \frac{x_1^{-1}\rho e^{\frac{2}{3}\pi i}}{r^2 + x_1^{-1}\rho e^{\frac{2}{3}\pi i}} \right) + \\
&+ \frac{\pi}{4\rho x_1} \left(\frac{x_1\rho e^{-\frac{2}{3}\pi i}}{r^2 + x_1\rho e^{-\frac{2}{3}\pi i}} - \frac{x_1^{-1}\rho e^{-\frac{2}{3}\pi i}}{r^2 + x_1^{-1}\rho e^{-\frac{2}{3}\pi i}} \right). \tag{92}
\end{aligned}$$

7.4 Integration in the radial direction

It remains to combine the appropriate residues and integrate them with respect to r . We will still assume $x_1 < \rho$.

- From 0 to 1 :

The part that is integrated without distinction in formula from 0 to 1 we denote

$$\int_{r=0}^1 I_6(x_1, \rho; r) r dr, \tag{93}$$

where

$$I_6(x_1, \rho; r) = \#_{(79)} + \#_{(82)} + \#_{(85)} + \#_{(88)}.$$

We have

$$\begin{aligned}
& \int_{r=0}^1 I_6(x_1, \rho; r) r dr = \\
& = \frac{\pi}{4x_1\rho} \int_{r=0}^1 \left(\frac{1}{x_1\rho+1} - \frac{\rho}{x_1+\rho} + \frac{x_1^{-1}\rho}{r^2+x_1^{-1}\rho} - \frac{x_1^{-1}\rho^{-1}}{r^2+x_1^{-1}\rho^{-1}} + \right. \\
& \quad + \frac{1}{x_1\rho e^{-\frac{2}{3}\pi i}+1} - \frac{\rho}{x_1 e^{-\frac{2}{3}\pi i}+\rho} + \frac{x_1^{-1}\rho e^{\frac{2}{3}\pi i}}{r^2+x_1^{-1}\rho e^{\frac{2}{3}\pi i}} - \frac{x_1^{-1}\rho^{-1} e^{\frac{2}{3}\pi i}}{r^2+x_1^{-1}\rho^{-1} e^{\frac{2}{3}\pi i}} + \\
& \quad + \frac{1}{x_1\rho e^{\frac{2}{3}\pi i}+1} - \frac{\rho}{x_1 e^{\frac{2}{3}\pi i}+\rho} + \frac{x_1^{-1}\rho e^{-\frac{2}{3}\pi i}}{r^2+x_1^{-1}\rho e^{-\frac{2}{3}\pi i}} - \frac{x_1^{-1}\rho^{-1} e^{-\frac{2}{3}\pi i}}{r^2+x_1^{-1}\rho^{-1} e^{-\frac{2}{3}\pi i}} + \\
& \quad + \frac{1}{x_1\rho+1} - \frac{x_1}{\rho+x_1} + \\
& \quad + \frac{1}{1+x_1\rho e^{\frac{2}{3}\pi i}} - \frac{x_1 e^{-\frac{2}{3}\pi i}}{\rho+x_1 e^{-\frac{2}{3}\pi i}} + \frac{1}{x_1\rho e^{-\frac{2}{3}\pi i}+1} - \frac{x_1 e^{\frac{2}{3}\pi i}}{\rho+x_1 e^{\frac{2}{3}\pi i}} + \\
& \quad + \frac{x_1\rho^{-1}}{r^2+x_1\rho^{-1}} - \frac{x_1^{-1}\rho^{-1}}{r^2+x_1^{-1}\rho^{-1}} + \\
& \quad + \frac{x_1\rho^{-1} e^{-\frac{2}{3}\pi i}}{r^2+x_1\rho^{-1} e^{-\frac{2}{3}\pi i}} - \frac{x_1^{-1}\rho^{-1} e^{-\frac{2}{3}\pi i}}{r^2+x_1^{-1}\rho^{-1} e^{-\frac{2}{3}\pi i}} + \\
& \quad \left. + \frac{x_1\rho^{-1} e^{\frac{2}{3}\pi i}}{r^2+x_1\rho^{-1} e^{\frac{2}{3}\pi i}} - \frac{x_1^{-1}\rho^{-1} e^{\frac{2}{3}\pi i}}{r^2+x_1^{-1}\rho^{-1} e^{\frac{2}{3}\pi i}} \right) r dr \\
& = \frac{\pi}{4x_1\rho} \int_{r=0}^1 \left(\frac{1}{x_1\rho+1} + \frac{1}{x_1\rho+1} - \frac{x_1}{\rho+x_1} - \frac{\rho}{x_1+\rho} + \right. \\
& \quad + \frac{1}{x_1\rho e^{-\frac{2}{3}\pi i}+1} + \frac{1}{x_1\rho e^{\frac{2}{3}\pi i}+1} + \frac{1}{1+x_1\rho e^{\frac{2}{3}\pi i}} + \frac{1}{x_1\rho e^{-\frac{2}{3}\pi i}+1} + \\
& \quad - \frac{\rho}{x_1 e^{-\frac{2}{3}\pi i}+\rho} - \frac{\rho}{x_1 e^{\frac{2}{3}\pi i}+\rho} - \frac{x_1 e^{-\frac{2}{3}\pi i}}{\rho+x_1 e^{-\frac{2}{3}\pi i}} - \frac{x_1 e^{\frac{2}{3}\pi i}}{\rho+x_1 e^{\frac{2}{3}\pi i}} + \\
& \quad + \frac{x_1^{-1}\rho}{r^2+x_1^{-1}\rho} - \frac{x_1^{-1}\rho^{-1}}{r^2+x_1^{-1}\rho^{-1}} + \frac{x_1\rho^{-1}}{r^2+x_1\rho^{-1}} - \frac{x_1^{-1}\rho^{-1}}{r^2+x_1^{-1}\rho^{-1}} + \\
& \quad + \frac{x_1^{-1}\rho e^{-\frac{2}{3}\pi i}}{r^2+x_1^{-1}\rho e^{-\frac{2}{3}\pi i}} + \frac{x_1^{-1}\rho e^{\frac{2}{3}\pi i}}{r^2+x_1^{-1}\rho e^{\frac{2}{3}\pi i}} + \frac{x_1\rho^{-1} e^{\frac{2}{3}\pi i}}{r^2+x_1\rho^{-1} e^{\frac{2}{3}\pi i}} + \frac{x_1\rho^{-1} e^{-\frac{2}{3}\pi i}}{r^2+x_1\rho^{-1} e^{-\frac{2}{3}\pi i}} + \\
& \quad \left. - \frac{x_1^{-1}\rho^{-1} e^{-\frac{2}{3}\pi i}}{r^2+x_1^{-1}\rho^{-1} e^{-\frac{2}{3}\pi i}} - \frac{x_1^{-1}\rho^{-1} e^{\frac{2}{3}\pi i}}{r^2+x_1^{-1}\rho^{-1} e^{\frac{2}{3}\pi i}} - \frac{x_1^{-1}\rho^{-1} e^{-\frac{2}{3}\pi i}}{r^2+x_1^{-1}\rho^{-1} e^{-\frac{2}{3}\pi i}} - \frac{x_1^{-1}\rho^{-1} e^{\frac{2}{3}\pi i}}{r^2+x_1^{-1}\rho^{-1} e^{\frac{2}{3}\pi i}} \right) r dr
\end{aligned}$$

$$\begin{aligned}
&= \frac{\pi}{4x_1\rho} \int_{r=0}^1 \left(\frac{2}{x_1\rho+1} + \frac{2}{1+x_1\rho e^{-\frac{2}{3}\pi i}} + \frac{2}{1+x_1\rho e^{\frac{2}{3}\pi i}} - 3 + \right. \\
&\quad + \frac{x_1^{-1}\rho}{r^2+x_1^{-1}\rho} + \frac{x_1\rho^{-1}}{r^2+x_1\rho^{-1}} - 2\frac{x_1^{-1}\rho^{-1}}{r^2+x_1^{-1}\rho^{-1}} + \\
&\quad + \frac{x_1^{-1}\rho e^{-\frac{2}{3}\pi i}}{r^2+x_1^{-1}\rho e^{-\frac{2}{3}\pi i}} + \frac{x_1^{-1}\rho e^{\frac{2}{3}\pi i}}{r^2+x_1^{-1}\rho e^{\frac{2}{3}\pi i}} + \frac{x_1\rho^{-1}e^{\frac{2}{3}\pi i}}{r^2+x_1\rho^{-1}e^{\frac{2}{3}\pi i}} + \frac{x_1\rho^{-1}e^{-\frac{2}{3}\pi i}}{r^2+x_1\rho^{-1}e^{-\frac{2}{3}\pi i}} + \\
&\quad \left. - 2\frac{x_1^{-1}\rho^{-1}e^{-\frac{2}{3}\pi i}}{r^2+x_1^{-1}\rho^{-1}e^{-\frac{2}{3}\pi i}} - 2\frac{x_1^{-1}\rho^{-1}e^{\frac{2}{3}\pi i}}{r^2+x_1^{-1}\rho^{-1}e^{\frac{2}{3}\pi i}} \right) r dr \\
&= \frac{\pi}{8x_1\rho} \left(\frac{2}{x_1\rho+1} + \frac{2}{1+x_1\rho e^{-\frac{2}{3}\pi i}} + \frac{2}{1+x_1\rho e^{\frac{2}{3}\pi i}} - 3 + \right. \\
&\quad + x_1^{-1}\rho \text{Ln} \left(\frac{1+x_1^{-1}\rho}{x_1^{-1}\rho} \right) + x_1\rho^{-1} \text{Ln} \left(\frac{1+x_1\rho^{-1}}{x_1\rho^{-1}} \right) - 2x_1^{-1}\rho^{-1} \text{Ln} \left(\frac{1+x_1^{-1}\rho^{-1}}{x_1^{-1}\rho^{-1}} \right) + \\
&\quad + x_1^{-1}\rho e^{-\frac{2}{3}\pi i} \text{Ln} \left(\frac{1+x_1^{-1}\rho e^{-\frac{2}{3}\pi i}}{x_1^{-1}\rho e^{-\frac{2}{3}\pi i}} \right) + x_1^{-1}\rho e^{\frac{2}{3}\pi i} \text{Ln} \left(\frac{1+x_1^{-1}\rho e^{\frac{2}{3}\pi i}}{x_1^{-1}\rho e^{\frac{2}{3}\pi i}} \right) + \\
&\quad + x_1\rho^{-1}e^{\frac{2}{3}\pi i} \text{Ln} \left(\frac{1+x_1\rho^{-1}e^{\frac{2}{3}\pi i}}{x_1\rho^{-1}e^{\frac{2}{3}\pi i}} \right) + x_1\rho^{-1}e^{-\frac{2}{3}\pi i} \text{Ln} \left(\frac{1+x_1\rho^{-1}e^{-\frac{2}{3}\pi i}}{x_1\rho^{-1}e^{-\frac{2}{3}\pi i}} \right) + \\
&\quad \left. - 2x_1^{-1}\rho^{-1}e^{-\frac{2}{3}\pi i} \text{Ln} \left(\frac{1+x_1^{-1}\rho^{-1}e^{-\frac{2}{3}\pi i}}{x_1^{-1}\rho^{-1}e^{-\frac{2}{3}\pi i}} \right) - 2x_1^{-1}\rho^{-1}e^{\frac{2}{3}\pi i} \text{Ln} \left(\frac{1+x_1^{-1}\rho^{-1}e^{\frac{2}{3}\pi i}}{x_1^{-1}\rho^{-1}e^{\frac{2}{3}\pi i}} \right) \right) \\
&= \frac{\pi}{8x_1\rho} \left(\frac{1-x_1\rho}{1+x_1\rho} + \frac{1-x_1\rho e^{-\frac{2}{3}\pi i}}{1+x_1\rho e^{-\frac{2}{3}\pi i}} + \frac{1-x_1\rho e^{\frac{2}{3}\pi i}}{1+x_1\rho e^{\frac{2}{3}\pi i}} + \right. \\
&\quad + x_1^{-1}\rho \text{Ln}(1+x_1\rho^{-1}) + x_1\rho^{-1} \text{Ln}(1+x_1^{-1}\rho) - 2x_1^{-1}\rho^{-1} \text{Ln}(1+x_1\rho) + \\
&\quad + x_1^{-1}\rho e^{-\frac{2}{3}\pi i} \text{Ln} \left(1+x_1\rho^{-1}e^{\frac{2}{3}\pi i} \right) + x_1^{-1}\rho e^{\frac{2}{3}\pi i} \text{Ln} \left(1+x_1\rho^{-1}e^{-\frac{2}{3}\pi i} \right) + \\
&\quad + x_1\rho^{-1}e^{\frac{2}{3}\pi i} \text{Ln} \left(1+x_1^{-1}\rho e^{-\frac{2}{3}\pi i} \right) + x_1\rho^{-1}e^{-\frac{2}{3}\pi i} \text{Ln} \left(1+x_1^{-1}\rho e^{\frac{2}{3}\pi i} \right) + \\
&\quad \left. - 2x_1^{-1}\rho^{-1}e^{-\frac{2}{3}\pi i} \text{Ln} \left(1+x_1\rho e^{\frac{2}{3}\pi i} \right) - 2x_1^{-1}\rho^{-1}e^{\frac{2}{3}\pi i} \text{Ln} \left(1+x_1\rho e^{-\frac{2}{3}\pi i} \right) \right) \quad (94)
\end{aligned}$$

- From 0 to ρ :

The part that is integrated without distinction in formula from 0 to ρ we denote

$$\int_{r=0}^{\rho} I_7(x_1, \rho; r) r dr, \quad (95)$$

where

$$I_7(x_1, \rho; r) = \#(89).$$

Hence

$$\begin{aligned}
& \int_{r=0}^{\rho} I_7(x_1, \rho; r) r dr = \\
& = \int_{r=0}^{\rho} \left(\frac{\pi}{4\rho x_1} \left(\frac{x_1\rho}{1+x_1\rho} - \frac{\rho}{x_1+\rho} + \frac{x_1\rho(2x_1\rho-1)}{1-x_1\rho+x_1^2\rho^2} - \frac{\rho(2\rho-x_1)}{x_1^2-x_1\rho+\rho^2} \right) + \right. \\
& \quad + \frac{\pi}{4\rho x_1} \left(\frac{x_1^{-1}\rho}{r^2+x_1^{-1}\rho} - \frac{x_1\rho}{r^2+x_1\rho} \right) + \\
& \quad + \frac{\pi}{4\rho x_1} \left(\frac{x_1^{-1}\rho e^{-\frac{2}{3}\pi i}}{r^2+x_1^{-1}\rho e^{-\frac{2}{3}\pi i}} - \frac{x_1\rho e^{-\frac{2}{3}\pi i}}{r^2+x_1\rho e^{-\frac{2}{3}\pi i}} \right) + \\
& \quad \left. + \frac{\pi}{4\rho x_1} \left(\frac{x_1^{-1}\rho e^{\frac{2}{3}\pi i}}{r^2+x_1^{-1}\rho e^{\frac{2}{3}\pi i}} - \frac{x_1\rho e^{\frac{2}{3}\pi i}}{r^2+x_1\rho e^{\frac{2}{3}\pi i}} \right) \right) r dr \\
& = \frac{\pi}{8\rho x_1} \left[\left(\frac{x_1\rho}{1+x_1\rho} - \frac{\rho}{x_1+\rho} + \frac{x_1\rho(2x_1\rho-1)}{1-x_1\rho+x_1^2\rho^2} - \frac{\rho(2\rho-x_1)}{x_1^2-x_1\rho+\rho^2} \right) r^2 + \right. \\
& \quad + x_1^{-1}\rho \text{Ln}(r^2+x_1^{-1}\rho) - x_1\rho \text{Ln}(r^2+x_1\rho) + \\
& \quad + x_1^{-1}\rho e^{-\frac{2}{3}\pi i} \text{Ln}(r^2+x_1^{-1}\rho e^{-\frac{2}{3}\pi i}) - x_1\rho e^{-\frac{2}{3}\pi i} \text{Ln}(r^2+x_1\rho e^{-\frac{2}{3}\pi i}) + \\
& \quad \left. + x_1^{-1}\rho e^{\frac{2}{3}\pi i} \text{Ln}(r^2+x_1^{-1}\rho e^{\frac{2}{3}\pi i}) - x_1\rho e^{\frac{2}{3}\pi i} \text{Ln}(r^2+x_1\rho e^{\frac{2}{3}\pi i}) \right]_{r=0}^{\rho} \\
& = \frac{\pi}{8\rho x_1} \left(\left(\frac{x_1\rho}{1+x_1\rho} - \frac{\rho}{x_1+\rho} + \frac{x_1\rho(2x_1\rho-1)}{1-x_1\rho+x_1^2\rho^2} - \frac{\rho(2\rho-x_1)}{x_1^2-x_1\rho+\rho^2} \right) \rho^2 + \right. \\
& \quad + x_1^{-1}\rho \text{Ln} \left(\frac{\rho^2+x_1^{-1}\rho}{x_1^{-1}\rho} \right) - x_1\rho \text{Ln} \left(\frac{\rho^2+x_1\rho}{x_1\rho} \right) + \\
& \quad + x_1^{-1}\rho e^{-\frac{2}{3}\pi i} \text{Ln} \left(\frac{\rho^2+x_1^{-1}\rho e^{-\frac{2}{3}\pi i}}{x_1^{-1}\rho e^{-\frac{2}{3}\pi i}} \right) - x_1\rho e^{-\frac{2}{3}\pi i} \text{Ln} \left(\frac{\rho^2+x_1\rho e^{-\frac{2}{3}\pi i}}{x_1\rho e^{-\frac{2}{3}\pi i}} \right) + \\
& \quad \left. + x_1^{-1}\rho e^{\frac{2}{3}\pi i} \text{Ln} \left(\frac{\rho^2+x_1^{-1}\rho e^{\frac{2}{3}\pi i}}{x_1^{-1}\rho e^{\frac{2}{3}\pi i}} \right) - x_1\rho e^{\frac{2}{3}\pi i} \text{Ln} \left(\frac{\rho^2+x_1\rho e^{\frac{2}{3}\pi i}}{x_1\rho e^{\frac{2}{3}\pi i}} \right) \right)
\end{aligned}$$

$$\begin{aligned}
&= \frac{\pi}{8\rho x_1} \left(\left(\frac{x_1\rho}{1+x_1\rho} - \frac{\rho}{x_1+\rho} - \frac{\rho(2\rho-x_1)}{x_1^2-x_1\rho+\rho^2} + \frac{x_1\rho(2x_1\rho-1)}{1-x_1\rho+x_1^2\rho^2} \right) \rho^2 + \right. \\
&\quad - x_1\rho \text{Ln}(1+x_1^{-1}\rho) + x_1^{-1}\rho \text{Ln}(1+x_1\rho) + \\
&\quad - x_1\rho e^{-\frac{2}{3}\pi i} \text{Ln}\left(1+x_1^{-1}\rho e^{\frac{2}{3}\pi i}\right) + x_1^{-1}\rho e^{-\frac{2}{3}\pi i} \text{Ln}\left(1+x_1\rho e^{\frac{2}{3}\pi i}\right) + \\
&\quad \left. - x_1\rho e^{\frac{2}{3}\pi i} \text{Ln}\left(1+x_1^{-1}\rho e^{-\frac{2}{3}\pi i}\right) + x_1^{-1}\rho e^{\frac{2}{3}\pi i} \text{Ln}\left(1+x_1\rho e^{-\frac{2}{3}\pi i}\right) \right) \quad (96)
\end{aligned}$$

- From 0 to x_1 :

The part of the formula which only concerns the interval $(0, x_1)$ we denote as

$$\int_{r=0}^{x_1} I_8(x_1, \rho; r) r dr, \quad (97)$$

where

$$I_8(x_1, \rho; r) = \#(80) + \#(83) + \#(86).$$

$$\begin{aligned}
&\int_{r=0}^{x_1} I_8(x_1, \rho; r) r dr = \\
&= \frac{\pi}{4x_1\rho} \int_{r=0}^{x_1} \left(\frac{x_1\rho}{1+x_1\rho} - \frac{x_1}{\rho+x_1} + \frac{x_1\rho^{-1}}{r^2+x_1\rho^{-1}} - \frac{x_1\rho}{r^2+x_1\rho} \right) r dr + \\
&\quad + \frac{\pi}{4x_1\rho} \int_{r=0}^{x_1} \left(\frac{x_1\rho e^{\frac{2}{3}\pi i}}{1+x_1\rho e^{\frac{2}{3}\pi i}} - \frac{x_1}{\rho e^{-\frac{2}{3}\pi i}+x_1} + \frac{x_1\rho^{-1}e^{\frac{2}{3}\pi i}}{r^2+x_1\rho^{-1}e^{\frac{2}{3}\pi i}} - \frac{x_1\rho e^{\frac{2}{3}\pi i}}{r^2+x_1\rho e^{\frac{2}{3}\pi i}} \right) r dr + \\
&\quad + \frac{\pi}{4x_1\rho} \int_{r=0}^{x_1} \left(\frac{x_1\rho}{e^{\frac{2}{3}\pi i}+x_1\rho} - \frac{x_1}{\rho e^{\frac{2}{3}\pi i}+x_1} + \frac{x_1\rho^{-1}e^{-\frac{2}{3}\pi i}}{r^2+x_1\rho^{-1}e^{-\frac{2}{3}\pi i}} - \frac{x_1\rho e^{-\frac{2}{3}\pi i}}{r^2+x_1\rho e^{-\frac{2}{3}\pi i}} \right) r dr \\
&= \frac{\pi}{4x_1\rho} \int_{r=0}^{x_1} \left(\frac{x_1\rho}{1+x_1\rho} - \frac{x_1}{\rho+x_1} + \frac{x_1\rho e^{\frac{2}{3}\pi i}}{1+x_1\rho e^{\frac{2}{3}\pi i}} + \frac{x_1\rho e^{-\frac{2}{3}\pi i}}{1+x_1\rho e^{-\frac{2}{3}\pi i}} - \frac{x_1}{\rho e^{-\frac{2}{3}\pi i}+x_1} - \frac{x_1}{\rho e^{\frac{2}{3}\pi i}+x_1} \right) r dr + \\
&\quad + \frac{\pi}{4x_1\rho} \int_{r=0}^{x_1} \left(\frac{x_1\rho^{-1}}{r^2+x_1\rho^{-1}} - \frac{x_1\rho}{r^2+x_1\rho} \right) r dr + \\
&\quad + \frac{\pi}{4x_1\rho} \int_{r=0}^{x_1} \left(\frac{x_1\rho^{-1}e^{\frac{2}{3}\pi i}}{r^2+x_1\rho^{-1}e^{\frac{2}{3}\pi i}} - \frac{x_1\rho e^{\frac{2}{3}\pi i}}{r^2+x_1\rho e^{\frac{2}{3}\pi i}} \right) r dr + \\
&\quad + \frac{\pi}{4x_1\rho} \int_{r=0}^{x_1} \left(\frac{x_1\rho^{-1}e^{-\frac{2}{3}\pi i}}{r^2+x_1\rho^{-1}e^{-\frac{2}{3}\pi i}} - \frac{x_1\rho e^{-\frac{2}{3}\pi i}}{r^2+x_1\rho e^{-\frac{2}{3}\pi i}} \right) r dr
\end{aligned}$$

$$\begin{aligned}
&= \frac{\pi}{8x_1\rho} \left(\frac{x_1\rho}{1+x_1\rho} - \frac{x_1}{\rho+x_1} + \frac{(2x_1\rho-1)x_1\rho}{1-x_1\rho+x_1^2\rho^2} - \frac{(2x_1-\rho)x_1}{\rho^2-x_1\rho+x_1^2} \right) x_1^2 + \\
&\quad + \frac{\pi}{8x_1\rho} \left(x_1\rho^{-1} \text{Ln} \left(\frac{x_1^2+x_1\rho^{-1}}{x_1\rho^{-1}} \right) - x_1\rho \text{Ln} \left(\frac{x_1^2+x_1\rho}{x_1\rho} \right) \right) + \\
&\quad + \frac{\pi}{8x_1\rho} \left(x_1\rho^{-1} e^{\frac{2}{3}\pi i} \text{Ln} \left(\frac{x_1^2+x_1\rho^{-1} e^{\frac{2}{3}\pi i}}{x_1\rho^{-1} e^{\frac{2}{3}\pi i}} \right) - x_1\rho e^{\frac{2}{3}\pi i} \text{Ln} \left(\frac{x_1^2+x_1\rho e^{\frac{2}{3}\pi i}}{x_1\rho e^{\frac{2}{3}\pi i}} \right) \right) + \\
&\quad + \frac{\pi}{8x_1\rho} \left(x_1\rho^{-1} e^{-\frac{2}{3}\pi i} \text{Ln} \left(\frac{x_1^2+x_1\rho^{-1} e^{-\frac{2}{3}\pi i}}{x_1\rho^{-1} e^{-\frac{2}{3}\pi i}} \right) - x_1\rho e^{-\frac{2}{3}\pi i} \text{Ln} \left(\frac{x_1^2+x_1\rho e^{-\frac{2}{3}\pi i}}{x_1\rho e^{-\frac{2}{3}\pi i}} \right) \right) \\
&= \frac{\pi}{8x_1\rho} \left(\frac{x_1\rho}{1+x_1\rho} - \frac{x_1}{\rho+x_1} + \frac{(2x_1\rho-1)x_1\rho}{1-x_1\rho+x_1^2\rho^2} - \frac{(2x_1-\rho)x_1}{\rho^2-x_1\rho+x_1^2} \right) x_1^2 + \\
&\quad + \frac{\pi}{8x_1\rho} (x_1\rho^{-1} \text{Ln}(1+x_1\rho) - x_1\rho \text{Ln}(1+x_1\rho^{-1})) + \\
&\quad + \frac{\pi}{8x_1\rho} \left(x_1\rho^{-1} e^{\frac{2}{3}\pi i} \text{Ln} \left(1+x_1\rho e^{-\frac{2}{3}\pi i} \right) - x_1\rho e^{\frac{2}{3}\pi i} \text{Ln} \left(1+x_1\rho^{-1} e^{-\frac{2}{3}\pi i} \right) \right) + \\
&\quad + \frac{\pi}{8x_1\rho} \left(x_1\rho^{-1} e^{-\frac{2}{3}\pi i} \text{Ln} \left(1+x_1\rho e^{\frac{2}{3}\pi i} \right) - x_1\rho e^{-\frac{2}{3}\pi i} \text{Ln} \left(1+x_1\rho^{-1} e^{\frac{2}{3}\pi i} \right) \right) \\
&= \frac{\pi}{8x_1\rho} x_1^2 \left(\frac{x_1\rho}{1+x_1\rho} - \frac{x_1}{\rho+x_1} + \frac{(2x_1\rho-1)x_1\rho}{1-x_1\rho+x_1^2\rho^2} - \frac{(2x_1-\rho)x_1}{\rho^2-x_1\rho+x_1^2} \right) + \\
&\quad + \frac{\pi}{8x_1\rho} x_1\rho^{-1} \left(\text{Ln}(1+x_1\rho) + e^{\frac{2}{3}\pi i} \text{Ln} \left(1+x_1\rho e^{-\frac{2}{3}\pi i} \right) + e^{-\frac{2}{3}\pi i} \text{Ln} \left(1+x_1\rho e^{\frac{2}{3}\pi i} \right) \right) + \\
&\quad - \frac{\pi}{8x_1\rho} x_1\rho \left(\text{Ln}(1+x_1\rho^{-1}) + e^{\frac{2}{3}\pi i} \text{Ln} \left(1+x_1\rho^{-1} e^{-\frac{2}{3}\pi i} \right) + e^{-\frac{2}{3}\pi i} \text{Ln} \left(1+x_1\rho^{-1} e^{\frac{2}{3}\pi i} \right) \right). \quad (98)
\end{aligned}$$

- From x_1 to 1 :

The part that is integrated without distinction in formula from x_1 to 1 we denote as

$$\int_{r=x_1}^1 I_9(x_1, \rho; r) r dr, \quad (99)$$

where

$$I_9(x_1, \rho; r) = \#_{(81)} + \#_{(84)} + \#_{(87)}.$$

$$\begin{aligned}
&\int_{r=x_1}^1 I_9(x_1, \rho; r) r dr = \\
&= \frac{\pi}{4x_1\rho} \int_{r=x_1}^1 \left(\frac{1}{x_1\rho+1} - \frac{\rho}{x_1+\rho} + \frac{x_1\rho}{r^2+x_1\rho} - \frac{x_1\rho^{-1}}{r^2+x_1\rho^{-1}} \right) r dr + \\
&\quad + \frac{\pi}{4x_1\rho} \int_{r=x_1}^1 \left(\frac{1}{x_1\rho e^{-\frac{2}{3}\pi i}+1} - \frac{\rho}{x_1 e^{-\frac{2}{3}\pi i}+\rho} + \frac{x_1\rho e^{-\frac{2}{3}\pi i}}{r^2+x_1\rho e^{-\frac{2}{3}\pi i}} - \frac{x_1\rho^{-1} e^{-\frac{2}{3}\pi i}}{r^2+x_1\rho^{-1} e^{-\frac{2}{3}\pi i}} \right) r dr + \\
&\quad + \frac{\pi}{4x_1\rho} \int_{r=x_1}^1 \left(\frac{1}{x_1\rho e^{\frac{2}{3}\pi i}+1} - \frac{\rho}{x_1 e^{\frac{2}{3}\pi i}+\rho} + \frac{x_1\rho e^{\frac{2}{3}\pi i}}{r^2+x_1\rho e^{\frac{2}{3}\pi i}} - \frac{x_1\rho^{-1} e^{\frac{2}{3}\pi i}}{r^2+x_1\rho^{-1} e^{\frac{2}{3}\pi i}} \right) r dr
\end{aligned}$$

$$\begin{aligned}
&= \frac{\pi}{4x_1\rho} \int_{r=x_1}^1 \left(\frac{1}{x_1\rho+1} - \frac{\rho}{x_1+\rho} + \frac{1}{x_1\rho e^{-\frac{2}{3}\pi i}+1} + \frac{1}{1+x_1\rho e^{\frac{2}{3}\pi i}} - \frac{\rho}{x_1 e^{-\frac{2}{3}\pi i}+\rho} - \frac{\rho}{x_1 e^{\frac{2}{3}\pi i}+\rho} \right) r dr + \\
&+ \frac{\pi}{4x_1\rho} \int_{r=x_1}^1 \left(\frac{x_1\rho}{r^2+x_1\rho} - \frac{x_1\rho^{-1}}{r^2+x_1\rho^{-1}} \right) r dr + \\
&+ \frac{\pi}{4x_1\rho} \int_{r=x_1}^1 \left(\frac{x_1\rho e^{-\frac{2}{3}\pi i}}{r^2+x_1\rho e^{-\frac{2}{3}\pi i}} - \frac{x_1\rho^{-1} e^{-\frac{2}{3}\pi i}}{r^2+x_1\rho^{-1} e^{-\frac{2}{3}\pi i}} \right) r dr + \\
&+ \frac{\pi}{4x_1\rho} \int_{r=x_1}^1 \left(\frac{x_1\rho e^{\frac{2}{3}\pi i}}{r^2+x_1\rho e^{\frac{2}{3}\pi i}} - \frac{x_1\rho^{-1} e^{\frac{2}{3}\pi i}}{r^2+x_1\rho^{-1} e^{\frac{2}{3}\pi i}} \right) r dr \\
&= \frac{\pi}{8x_1\rho} \left(\frac{1}{x_1\rho+1} - \frac{\rho}{x_1+\rho} + \frac{2-x_1\rho}{1-x_1\rho+x_1^2\rho^2} - \frac{(2\rho-x_1)\rho}{x_1^2-x_1\rho+\rho^2} \right) (1-x_1^2) + \\
&+ \frac{\pi}{8x_1\rho} \left(x_1\rho \operatorname{Ln} \left(\frac{1+x_1\rho}{x_1^2+x_1\rho} \right) - x_1\rho^{-1} \operatorname{Ln} \left(\frac{1+x_1\rho^{-1}}{x_1^2+x_1\rho^{-1}} \right) \right) + \\
&+ \frac{\pi}{8x_1\rho} \left(x_1\rho e^{-\frac{2}{3}\pi i} \operatorname{Ln} \left(\frac{1+x_1\rho e^{-\frac{2}{3}\pi i}}{x_1^2+x_1\rho e^{-\frac{2}{3}\pi i}} \right) - x_1\rho^{-1} e^{-\frac{2}{3}\pi i} \operatorname{Ln} \left(\frac{1+x_1\rho^{-1} e^{-\frac{2}{3}\pi i}}{x_1^2+x_1\rho^{-1} e^{-\frac{2}{3}\pi i}} \right) \right) + \\
&+ \frac{\pi}{8x_1\rho} \left(x_1\rho e^{\frac{2}{3}\pi i} \operatorname{Ln} \left(\frac{1+x_1\rho e^{\frac{2}{3}\pi i}}{x_1^2+x_1\rho e^{\frac{2}{3}\pi i}} \right) - x_1\rho^{-1} e^{\frac{2}{3}\pi i} \operatorname{Ln} \left(\frac{1+x_1\rho^{-1} e^{\frac{2}{3}\pi i}}{x_1^2+x_1\rho^{-1} e^{\frac{2}{3}\pi i}} \right) \right). \tag{100}
\end{aligned}$$

• From ρ to 1 :

The last remaining part is

$$\int_{r=\rho}^1 I_{10}(x_1, \rho; r) r dr, \tag{101}$$

where

$$I_{10}(x_1, \rho; r) = \#(92) =$$

$$\begin{aligned}
&= \frac{-\pi}{4\rho x_1} \left(\frac{x_1}{\rho+x_1} - \frac{1}{x_1\rho+1} - \frac{1}{1+x_1\rho e^{\frac{2}{3}\pi i}} - \frac{1}{1+x_1\rho e^{-\frac{2}{3}\pi i}} + \frac{x_1 e^{-\frac{2}{3}\pi i}}{\rho+x_1 e^{-\frac{2}{3}\pi i}} + \frac{x_1 e^{\frac{2}{3}\pi i}}{\rho+x_1 e^{\frac{2}{3}\pi i}} \right) + \\
&+ \frac{-\pi}{4\rho x_1} \left(\frac{x_1^{-1}\rho}{r^2+x_1^{-1}\rho} - \frac{x_1\rho}{r^2+x_1\rho} \right) \\
&+ \frac{-\pi}{4\rho x_1} \left(\frac{x_1^{-1}\rho e^{\frac{2}{3}\pi i}}{r^2+x_1^{-1}\rho e^{\frac{2}{3}\pi i}} - \frac{x_1\rho e^{\frac{2}{3}\pi i}}{r^2+x_1\rho e^{\frac{2}{3}\pi i}} \right) + \\
&+ \frac{-\pi}{4\rho x_1} \left(\frac{x_1^{-1}\rho e^{-\frac{2}{3}\pi i}}{r^2+x_1^{-1}\rho e^{-\frac{2}{3}\pi i}} - \frac{x_1\rho e^{-\frac{2}{3}\pi i}}{r^2+x_1\rho e^{-\frac{2}{3}\pi i}} \right).
\end{aligned}$$

We obtain

$$\begin{aligned}
& \int_{r=\rho}^1 I_{10}(x_1, \rho; r) r dr = \\
&= \frac{\pi}{4\rho x_1} \int_{r=\rho}^1 \left(\frac{1}{x_1\rho + 1} - \frac{x_1}{\rho + x_1} + \frac{2 - x_1\rho}{1 - x_1\rho + x_1^2\rho^2} + \frac{(\rho - 2x_1)x_1}{x_1^2 - x_1\rho + \rho^2} + \right. \\
&\quad \left. + \frac{x_1\rho}{r^2 + x_1\rho} - \frac{x_1^{-1}\rho}{r^2 + x_1^{-1}\rho} + \frac{x_1\rho e^{\frac{2}{3}\pi i}}{r^2 + x_1\rho e^{\frac{2}{3}\pi i}} + \right. \\
&\quad \left. - \frac{x_1^{-1}\rho e^{\frac{2}{3}\pi i}}{r^2 + x_1^{-1}\rho e^{\frac{2}{3}\pi i}} + \frac{x_1\rho e^{-\frac{2}{3}\pi i}}{r^2 + x_1\rho e^{-\frac{2}{3}\pi i}} - \frac{x_1^{-1}\rho e^{-\frac{2}{3}\pi i}}{r^2 + x_1^{-1}\rho e^{-\frac{2}{3}\pi i}} \right) r dr \\
&= \frac{\pi}{8\rho x_1} \left[\left(\frac{1}{x_1\rho + 1} - \frac{x_1}{\rho + x_1} + \frac{2 - x_1\rho}{1 - x_1\rho + x_1^2\rho^2} + \frac{(\rho - 2x_1)x_1}{x_1^2 - x_1\rho + \rho^2} \right) r^2 + \right. \\
&\quad + x_1\rho \text{Ln}(r^2 + x_1\rho) - x_1^{-1}\rho \text{Ln}(r^2 + x_1^{-1}\rho) + \\
&\quad + x_1\rho e^{\frac{2}{3}\pi i} \text{Ln}(r^2 + x_1\rho e^{\frac{2}{3}\pi i}) - x_1^{-1}\rho e^{\frac{2}{3}\pi i} \text{Ln}(r^2 + x_1^{-1}\rho e^{\frac{2}{3}\pi i}) + \\
&\quad \left. + x_1\rho e^{-\frac{2}{3}\pi i} \text{Ln}(r^2 + x_1\rho e^{-\frac{2}{3}\pi i}) - x_1^{-1}\rho e^{-\frac{2}{3}\pi i} \text{Ln}(r^2 + x_1^{-1}\rho e^{-\frac{2}{3}\pi i}) \right]_{r=\rho}^1 \\
&= \frac{-\pi}{8\rho x_1} \left(\left(\frac{1}{x_1\rho + 1} - \frac{x_1}{\rho + x_1} + \frac{2 - x_1\rho}{1 - x_1\rho + x_1^2\rho^2} + \frac{(\rho - 2x_1)x_1}{x_1^2 - x_1\rho + \rho^2} \right) (1 - \rho^2) + \right. \\
&\quad + x_1\rho \text{Ln}\left(\frac{1 + x_1\rho}{\rho^2 + x_1\rho}\right) - x_1^{-1}\rho \text{Ln}\left(\frac{1 + x_1^{-1}\rho}{\rho^2 + x_1^{-1}\rho}\right) + \\
&\quad + x_1\rho e^{\frac{2}{3}\pi i} \text{Ln}\left(\frac{1 + x_1\rho e^{\frac{2}{3}\pi i}}{\rho^2 + x_1\rho e^{\frac{2}{3}\pi i}}\right) - x_1^{-1}\rho e^{\frac{2}{3}\pi i} \text{Ln}\left(\frac{1 + x_1^{-1}\rho e^{\frac{2}{3}\pi i}}{\rho^2 + x_1^{-1}\rho e^{\frac{2}{3}\pi i}}\right) + \\
&\quad \left. + x_1\rho e^{-\frac{2}{3}\pi i} \text{Ln}\left(\frac{1 + x_1\rho e^{-\frac{2}{3}\pi i}}{\rho^2 + x_1\rho e^{-\frac{2}{3}\pi i}}\right) - x_1^{-1}\rho e^{-\frac{2}{3}\pi i} \text{Ln}\left(\frac{1 + x_1^{-1}\rho e^{-\frac{2}{3}\pi i}}{\rho^2 + x_1^{-1}\rho e^{-\frac{2}{3}\pi i}}\right) \right) \\
&= \frac{\pi}{8\rho x_1} \left(\left(\frac{1}{x_1\rho + 1} - \frac{x_1}{\rho + x_1} + \frac{2 - x_1\rho}{1 - x_1\rho + x_1^2\rho^2} + \frac{(\rho - 2x_1)x_1}{x_1^2 - x_1\rho + \rho^2} \right) (1 - \rho^2) + \right. \\
&\quad + x_1\rho \text{Ln}\left(\frac{1 + x_1\rho}{(\rho + x_1)\rho}\right) - x_1^{-1}\rho \text{Ln}\left(\frac{\rho + x_1}{(1 + x_1\rho)\rho}\right) + \\
&\quad + x_1\rho e^{\frac{2}{3}\pi i} \text{Ln}\left(\frac{1 + x_1\rho e^{\frac{2}{3}\pi i}}{(\rho + x_1 e^{\frac{2}{3}\pi i})\rho}\right) - x_1^{-1}\rho e^{\frac{2}{3}\pi i} \text{Ln}\left(\frac{\rho + x_1 e^{-\frac{2}{3}\pi i}}{(1 + x_1\rho e^{-\frac{2}{3}\pi i})\rho}\right) + \\
&\quad \left. + x_1\rho e^{-\frac{2}{3}\pi i} \text{Ln}\left(\frac{1 + x_1\rho e^{-\frac{2}{3}\pi i}}{(\rho + x_1 e^{-\frac{2}{3}\pi i})\rho}\right) - x_1^{-1}\rho e^{-\frac{2}{3}\pi i} \text{Ln}\left(\frac{\rho + x_1 e^{\frac{2}{3}\pi i}}{(1 + x_1\rho e^{\frac{2}{3}\pi i})\rho}\right) \right) \tag{102}
\end{aligned}$$

7.5 Conclusion of Case 4)

Adding the formulae in (94), (96), (98), (100) and (102) we obtain

$$\begin{aligned}
& \frac{8x_1\rho}{\pi}\pi^2 \int_S \left(\lim_{x_2 \downarrow 0} \frac{G_S(x, z)}{x_2} \right) \left(\lim_{\substack{\psi \uparrow \frac{1}{3}\pi \\ y=(\rho \cos \psi, \rho \sin \psi)}} \frac{G_S(z, y)}{\rho \sin \left(\frac{1}{3}\pi - \psi\right)} \right) dz = \\
& = \frac{1-x_1\rho}{1+x_1\rho} + \frac{1-x_1\rho e^{-\frac{2}{3}\pi i}}{1+x_1\rho e^{-\frac{2}{3}\pi i}} + \frac{1-x_1\rho e^{\frac{2}{3}\pi i}}{1+x_1\rho e^{\frac{2}{3}\pi i}} + \\
& + x_1^{-1}\rho \text{Ln}(1+x_1\rho^{-1}) + x_1\rho^{-1}\text{Ln}(1+x_1^{-1}\rho) - 2x_1^{-1}\rho^{-1}\text{Ln}(1+x_1\rho) + \\
& + x_1^{-1}\rho e^{-\frac{2}{3}\pi i}\text{Ln}\left(1+x_1\rho^{-1}e^{\frac{2}{3}\pi i}\right) + x_1^{-1}\rho e^{\frac{2}{3}\pi i}\text{Ln}\left(1+x_1\rho^{-1}e^{-\frac{2}{3}\pi i}\right) + \\
& + x_1\rho^{-1}e^{\frac{2}{3}\pi i}\text{Ln}\left(1+x_1^{-1}\rho e^{-\frac{2}{3}\pi i}\right) + x_1\rho^{-1}e^{-\frac{2}{3}\pi i}\text{Ln}\left(1+x_1^{-1}\rho e^{\frac{2}{3}\pi i}\right) + \\
& - 2x_1^{-1}\rho^{-1}e^{-\frac{2}{3}\pi i}\text{Ln}\left(1+x_1\rho e^{\frac{2}{3}\pi i}\right) - 2x_1^{-1}\rho^{-1}e^{\frac{2}{3}\pi i}\text{Ln}\left(1+x_1\rho e^{-\frac{2}{3}\pi i}\right) + \\
& + \left(-\frac{\rho}{x_1+\rho} + \frac{x_1\rho}{1+x_1\rho} - \frac{\rho(2\rho-x_1)}{x_1^2-x_1\rho+\rho^2} + \frac{x_1\rho(2x_1\rho-1)}{1-x_1\rho+x_1^2\rho^2}\right)\rho^2 + \\
& - x_1\rho \text{Ln}(1+x_1^{-1}\rho) + x_1^{-1}\rho \text{Ln}(1+x_1\rho) + \\
& - x_1\rho e^{-\frac{2}{3}\pi i}\text{Ln}\left(1+x_1^{-1}\rho e^{\frac{2}{3}\pi i}\right) + x_1^{-1}\rho e^{-\frac{2}{3}\pi i}\text{Ln}\left(1+x_1\rho e^{\frac{2}{3}\pi i}\right) + \\
& - x_1\rho e^{\frac{2}{3}\pi i}\text{Ln}\left(1+x_1^{-1}\rho e^{-\frac{2}{3}\pi i}\right) + x_1^{-1}\rho e^{\frac{2}{3}\pi i}\text{Ln}\left(1+x_1\rho e^{-\frac{2}{3}\pi i}\right) + \\
& + \frac{x_1^3\rho}{1+x_1\rho} - \frac{x_1^3}{\rho+x_1} + \frac{(2x_1\rho-1)x_1^3\rho}{1-x_1\rho+x_1^2\rho^2} - \frac{(2x_1-\rho)x_1^3}{\rho^2-x_1\rho+x_1^2} + \\
& + x_1\rho^{-1}\text{Ln}(1+x_1\rho) + x_1\rho^{-1}e^{\frac{2}{3}\pi i}\text{Ln}\left(1+x_1\rho e^{-\frac{2}{3}\pi i}\right) + x_1\rho^{-1}e^{-\frac{2}{3}\pi i}\text{Ln}\left(1+x_1\rho e^{\frac{2}{3}\pi i}\right) + \\
& - x_1\rho \text{Ln}(1+x_1\rho^{-1}) - x_1\rho e^{\frac{2}{3}\pi i}\text{Ln}\left(1+x_1\rho^{-1}e^{-\frac{2}{3}\pi i}\right) - x_1\rho e^{-\frac{2}{3}\pi i}\text{Ln}\left(1+x_1\rho^{-1}e^{\frac{2}{3}\pi i}\right) + \\
& + \left(\frac{1}{x_1\rho+1} - \frac{\rho}{x_1+\rho} + \frac{2-x_1\rho}{1-x_1\rho+x_1^2\rho^2} - \frac{(2\rho-x_1)\rho}{x_1^2-x_1\rho+\rho^2}\right)(1-x_1^2) + \\
& + x_1\rho \text{Ln}\left(\frac{1+x_1\rho}{x_1^2+x_1\rho}\right) - x_1\rho^{-1}\text{Ln}\left(\frac{1+x_1\rho^{-1}}{x_1^2+x_1\rho^{-1}}\right) + \\
& + x_1\rho e^{-\frac{2}{3}\pi i}\text{Ln}\left(\frac{1+x_1\rho e^{-\frac{2}{3}\pi i}}{x_1^2+x_1\rho e^{-\frac{2}{3}\pi i}}\right) - x_1\rho^{-1}e^{-\frac{2}{3}\pi i}\text{Ln}\left(\frac{1+x_1\rho^{-1}e^{-\frac{2}{3}\pi i}}{x_1^2+x_1\rho^{-1}e^{-\frac{2}{3}\pi i}}\right) + \\
& + x_1\rho e^{\frac{2}{3}\pi i}\text{Ln}\left(\frac{1+x_1\rho e^{\frac{2}{3}\pi i}}{x_1^2+x_1\rho e^{\frac{2}{3}\pi i}}\right) - x_1\rho^{-1}e^{\frac{2}{3}\pi i}\text{Ln}\left(\frac{1+x_1\rho^{-1}e^{\frac{2}{3}\pi i}}{x_1^2+x_1\rho^{-1}e^{\frac{2}{3}\pi i}}\right) +
\end{aligned}$$

$$\begin{aligned}
& + \left(\frac{1}{x_1\rho + 1} - \frac{x_1}{\rho + x_1} + \frac{2 - x_1\rho}{1 - x_1\rho + x_1^2\rho^2} + \frac{(\rho - 2x_1)x_1}{x_1^2 - x_1\rho + \rho^2} \right) (1 - \rho^2) + \\
& + x_1\rho \text{Ln} \left(\frac{1 + x_1\rho}{(\rho + x_1)\rho} \right) - x_1^{-1}\rho \text{Ln} \left(\frac{\rho + x_1}{(1 + x_1\rho)\rho} \right) + \\
& + x_1\rho e^{\frac{2}{3}\pi i} \text{Ln} \left(\frac{1 + x_1\rho e^{\frac{2}{3}\pi i}}{(\rho + x_1 e^{\frac{2}{3}\pi i})\rho} \right) - x_1^{-1}\rho e^{\frac{2}{3}\pi i} \text{Ln} \left(\frac{\rho + x_1 e^{-\frac{2}{3}\pi i}}{(1 + x_1\rho e^{-\frac{2}{3}\pi i})\rho} \right) + \\
& + x_1\rho e^{-\frac{2}{3}\pi i} \text{Ln} \left(\frac{1 + x_1\rho e^{-\frac{2}{3}\pi i}}{(\rho + x_1 e^{-\frac{2}{3}\pi i})\rho} \right) - x_1^{-1}\rho e^{-\frac{2}{3}\pi i} \text{Ln} \left(\frac{\rho + x_1 e^{\frac{2}{3}\pi i}}{(1 + x_1\rho e^{\frac{2}{3}\pi i})\rho} \right)
\end{aligned} \tag{103}$$

$$\begin{aligned}
= & - \frac{\rho^3}{x_1 + \rho} - \frac{x_1^3}{\rho + x_1} - \frac{\rho(1 - x_1^2)}{x_1 + \rho} - \frac{x_1(1 - \rho^2)}{\rho + x_1} + \\
& + \frac{1 - x_1\rho}{1 + x_1\rho} + \frac{x_1\rho^3}{1 + x_1\rho} + \frac{x_1^3\rho}{1 + x_1\rho} + \frac{1 - x_1^2}{x_1\rho + 1} + \frac{1 - \rho^2}{x_1\rho + 1} + \\
& - \frac{(2x_1 - \rho)x_1^3}{\rho^2 - x_1\rho + x_1^2} - \frac{\rho^3(2\rho - x_1)}{x_1^2 - x_1\rho + \rho^2} - \frac{(2\rho - x_1)\rho(1 - x_1^2)}{x_1^2 - x_1\rho + \rho^2} + \frac{(\rho - 2x_1)x_1(1 - \rho^2)}{x_1^2 - x_1\rho + \rho^2} + \\
& + \frac{1 - x_1\rho e^{-\frac{2}{3}\pi i}}{1 + x_1\rho e^{-\frac{2}{3}\pi i}} + \frac{1 - x_1\rho e^{\frac{2}{3}\pi i}}{1 + x_1\rho e^{\frac{2}{3}\pi i}} + \frac{x_1\rho^3(2x_1\rho - 1)}{1 - x_1\rho + x_1^2\rho^2} + \\
& + \frac{(2x_1\rho - 1)x_1^3\rho}{1 - x_1\rho + x_1^2\rho^2} + \frac{(2 - x_1\rho)(1 - x_1^2)}{1 - x_1\rho + x_1^2\rho^2} + \frac{(2 - x_1\rho)(1 - \rho^2)}{1 - x_1\rho + x_1^2\rho^2} + \\
& - x_1\rho \text{Ln} (1 + x_1^{-1}\rho) + x_1\rho \text{Ln} \left(\frac{1 + x_1\rho}{x_1^2 + x_1\rho} \right) - x_1\rho \text{Ln} (1 + x_1\rho^{-1}) + x_1\rho \text{Ln} \left(\frac{1 + x_1\rho}{(\rho + x_1)\rho} \right) + \\
& - x_1\rho e^{\frac{2}{3}\pi i} \text{Ln} \left(1 + x_1\rho^{-1}e^{-\frac{2}{3}\pi i} \right) - x_1\rho e^{\frac{2}{3}\pi i} \text{Ln} \left(1 + x_1^{-1}\rho e^{-\frac{2}{3}\pi i} \right) + \\
& + x_1\rho e^{\frac{2}{3}\pi i} \text{Ln} \left(\frac{1 + x_1\rho e^{\frac{2}{3}\pi i}}{x_1^2 + x_1\rho e^{\frac{2}{3}\pi i}} \right) + x_1\rho e^{\frac{2}{3}\pi i} \text{Ln} \left(\frac{1 + x_1\rho e^{\frac{2}{3}\pi i}}{(\rho + x_1 e^{\frac{2}{3}\pi i})\rho} \right) + \\
& - x_1\rho e^{-\frac{2}{3}\pi i} \text{Ln} \left(1 + x_1\rho^{-1}e^{\frac{2}{3}\pi i} \right) - x_1\rho e^{-\frac{2}{3}\pi i} \text{Ln} \left(1 + x_1^{-1}\rho e^{\frac{2}{3}\pi i} \right) + \\
& + x_1\rho e^{-\frac{2}{3}\pi i} \text{Ln} \left(\frac{1 + x_1\rho e^{-\frac{2}{3}\pi i}}{x_1^2 + x_1\rho e^{-\frac{2}{3}\pi i}} \right) + x_1\rho e^{-\frac{2}{3}\pi i} \text{Ln} \left(\frac{1 + x_1\rho e^{-\frac{2}{3}\pi i}}{(\rho + x_1 e^{-\frac{2}{3}\pi i})\rho} \right) +
\end{aligned}$$

$$\begin{aligned}
& + x_1^{-1} \rho \operatorname{Ln} (1 + x_1 \rho^{-1}) + x_1^{-1} \rho \operatorname{Ln} (1 + x_1 \rho) - x_1^{-1} \rho \operatorname{Ln} \left(\frac{\rho + x_1}{(1 + x_1 \rho) \rho} \right) + \\
& + x_1^{-1} \rho e^{\frac{2}{3} \pi i} \operatorname{Ln} \left(1 + x_1 \rho e^{-\frac{2}{3} \pi i} \right) + x_1^{-1} \rho e^{\frac{2}{3} \pi i} \operatorname{Ln} \left(1 + x_1 \rho^{-1} e^{-\frac{2}{3} \pi i} \right) - x_1^{-1} \rho e^{\frac{2}{3} \pi i} \operatorname{Ln} \left(\frac{\rho + x_1 e^{-\frac{2}{3} \pi i}}{\left(1 + x_1 \rho e^{-\frac{2}{3} \pi i} \right) \rho} \right) + \\
& + x_1^{-1} \rho e^{-\frac{2}{3} \pi i} \operatorname{Ln} \left(1 + x_1 \rho e^{\frac{2}{3} \pi i} \right) + x_1^{-1} \rho e^{-\frac{2}{3} \pi i} \operatorname{Ln} \left(1 + x_1 \rho^{-1} e^{\frac{2}{3} \pi i} \right) - x_1^{-1} \rho e^{-\frac{2}{3} \pi i} \operatorname{Ln} \left(\frac{\rho + x_1 e^{\frac{2}{3} \pi i}}{\left(1 + x_1 \rho e^{\frac{2}{3} \pi i} \right) \rho} \right) + \\
& + x_1 \rho^{-1} \operatorname{Ln} (1 + x_1^{-1} \rho) + x_1 \rho^{-1} \operatorname{Ln} (1 + x_1 \rho) - x_1 \rho^{-1} \operatorname{Ln} \left(\frac{1 + x_1 \rho^{-1}}{x_1^2 + x_1 \rho^{-1}} \right) + \\
& + x_1 \rho^{-1} e^{\frac{2}{3} \pi i} \operatorname{Ln} \left(1 + x_1^{-1} \rho e^{-\frac{2}{3} \pi i} \right) + x_1 \rho^{-1} e^{\frac{2}{3} \pi i} \operatorname{Ln} \left(1 + x_1 \rho e^{-\frac{2}{3} \pi i} \right) - x_1 \rho^{-1} e^{\frac{2}{3} \pi i} \operatorname{Ln} \left(\frac{1 + x_1 \rho^{-1} e^{\frac{2}{3} \pi i}}{x_1^2 + x_1 \rho^{-1} e^{\frac{2}{3} \pi i}} \right) + \\
& + x_1 \rho^{-1} e^{-\frac{2}{3} \pi i} \operatorname{Ln} \left(1 + x_1^{-1} \rho e^{\frac{2}{3} \pi i} \right) + x_1 \rho^{-1} e^{-\frac{2}{3} \pi i} \operatorname{Ln} \left(1 + x_1 \rho e^{\frac{2}{3} \pi i} \right) - x_1 \rho^{-1} e^{-\frac{2}{3} \pi i} \operatorname{Ln} \left(\frac{1 + x_1 \rho^{-1} e^{-\frac{2}{3} \pi i}}{x_1^2 + x_1 \rho^{-1} e^{-\frac{2}{3} \pi i}} \right) + \\
& - 2x_1^{-1} \rho^{-1} \operatorname{Ln} (1 + x_1 \rho) - 2x_1^{-1} \rho^{-1} e^{-\frac{2}{3} \pi i} \operatorname{Ln} \left(1 + x_1 \rho e^{\frac{2}{3} \pi i} \right) - 2x_1^{-1} \rho^{-1} e^{\frac{2}{3} \pi i} \operatorname{Ln} \left(1 + x_1 \rho e^{-\frac{2}{3} \pi i} \right) \\
& = - (\rho - x_1)^2 + \frac{(1 - x_1 \rho) (2 - \rho^2 - x_1^2)}{1 + x_1 \rho} - 2 \frac{(x_1 - \rho)^2 (x_1 + \rho)^2}{\rho^2 - x_1 \rho + x_1^2} + 2 \frac{(1 + x_1 \rho) (1 - x_1 \rho) (2 - x_1^2 - \rho^2)}{1 - x_1 \rho + x_1^2 \rho^2} + \\
& + x_1 \rho \operatorname{Ln} \left(\frac{1 + x_1 \rho}{x_1^2 + x_1 \rho} \frac{1 + x_1 \rho}{(\rho + x_1) \rho} \frac{1}{1 + x_1^{-1} \rho} \frac{1}{1 + x_1 \rho^{-1}} \right) + \\
& + x_1 \rho e^{\frac{2}{3} \pi i} \operatorname{Ln} \left(\frac{1 + x_1 \rho e^{\frac{2}{3} \pi i}}{x_1^2 + x_1 \rho e^{\frac{2}{3} \pi i}} \frac{1 + x_1 \rho e^{\frac{2}{3} \pi i}}{\left(\rho + x_1 e^{\frac{2}{3} \pi i} \right) \rho} \frac{1}{1 + x_1 \rho^{-1} e^{-\frac{2}{3} \pi i}} \frac{1}{1 + x_1^{-1} \rho e^{-\frac{2}{3} \pi i}} \right) + \\
& + x_1 \rho e^{-\frac{2}{3} \pi i} \operatorname{Ln} \left(\frac{1 + x_1 \rho e^{-\frac{2}{3} \pi i}}{x_1^2 + x_1 \rho e^{-\frac{2}{3} \pi i}} \frac{1 + x_1 \rho e^{-\frac{2}{3} \pi i}}{\left(\rho + x_1 e^{-\frac{2}{3} \pi i} \right) \rho} \frac{1}{1 + x_1^{-1} \rho e^{\frac{2}{3} \pi i}} \frac{1}{1 + x_1 \rho^{-1} e^{\frac{2}{3} \pi i}} \right) + \\
& + x_1^{-1} \rho \operatorname{Ln} \left(\frac{(1 + x_1 \rho^{-1}) (1 + x_1 \rho) (1 + x_1 \rho) \rho}{\rho + x_1} \right) + \\
& + x_1^{-1} \rho e^{\frac{2}{3} \pi i} \operatorname{Ln} \left(\frac{\left(1 + x_1 \rho e^{-\frac{2}{3} \pi i} \right) \left(1 + x_1 \rho^{-1} e^{-\frac{2}{3} \pi i} \right) \left(1 + x_1 \rho e^{-\frac{2}{3} \pi i} \right) \rho}{\rho + x_1 e^{-\frac{2}{3} \pi i}} \right) +
\end{aligned}$$

$$\begin{aligned}
& + x_1^{-1} \rho e^{-\frac{2}{3}\pi i} \text{Ln} \left(\frac{\left(1 + x_1 \rho e^{\frac{2}{3}\pi i}\right) \left(1 + x_1 \rho^{-1} e^{\frac{2}{3}\pi i}\right) \left(1 + x_1 \rho e^{\frac{2}{3}\pi i}\right) \rho}{\rho + x_1 e^{\frac{2}{3}\pi i}} \right) + \\
& + x_1 \rho^{-1} \text{Ln} \left(\frac{(1 + x_1^{-1} \rho) (1 + x_1 \rho) (x_1^2 + x_1 \rho^{-1})}{1 + x_1 \rho^{-1}} \right) + \\
& + x_1 \rho^{-1} e^{\frac{2}{3}\pi i} \text{Ln} \left(\frac{\left(1 + x_1^{-1} \rho e^{-\frac{2}{3}\pi i}\right) \left(1 + x_1 \rho e^{-\frac{2}{3}\pi i}\right) \left(x_1^2 + x_1 \rho^{-1} e^{\frac{2}{3}\pi i}\right)}{1 + x_1 \rho^{-1} e^{\frac{2}{3}\pi i}} \right) + \\
& + x_1 \rho^{-1} e^{-\frac{2}{3}\pi i} \text{Ln} \left(\frac{\left(1 + x_1^{-1} \rho e^{\frac{2}{3}\pi i}\right) \left(1 + x_1 \rho e^{\frac{2}{3}\pi i}\right) \left(x_1^2 + x_1 \rho^{-1} e^{-\frac{2}{3}\pi i}\right)}{1 + x_1 \rho^{-1} e^{-\frac{2}{3}\pi i}} \right) + \\
& - 2x_1^{-1} \rho^{-1} \text{Ln} (1 + x_1 \rho) - 2x_1^{-1} \rho^{-1} e^{-\frac{2}{3}\pi i} \text{Ln} \left(1 + x_1 \rho e^{\frac{2}{3}\pi i}\right) - 2x_1^{-1} \rho^{-1} e^{\frac{2}{3}\pi i} \text{Ln} \left(1 + x_1 \rho e^{-\frac{2}{3}\pi i}\right) \\
= & -(\rho - x_1)^2 + \frac{(1 - x_1 \rho) (2 - \rho^2 - x_1^2)}{1 + x_1 \rho} - 2 \frac{(x_1 - \rho)^2 (x_1 + \rho)^2}{\rho^2 - x_1 \rho + x_1^2} + 2 \frac{(1 + x_1 \rho) (1 - x_1 \rho) (2 - x_1^2 - \rho^2)}{1 - x_1 \rho + x_1^2 \rho^2} + \\
& + x_1 \rho \text{Ln} \left(\frac{(x_1 \rho + 1)^2}{(x_1 + \rho)^4} \right) + \\
& + x_1 \rho e^{\frac{2}{3}\pi i} \text{Ln} \left(\frac{\left(1 + x_1 \rho e^{\frac{2}{3}\pi i}\right)^2}{\left(\rho + x_1 e^{\frac{2}{3}\pi i}\right)^2 \left(\rho + x_1 e^{-\frac{2}{3}\pi i}\right)^2} \right) + \\
& + x_1 \rho e^{-\frac{2}{3}\pi i} \text{Ln} \left(\frac{\left(1 + x_1 \rho e^{-\frac{2}{3}\pi i}\right)^2}{\left(\rho + x_1 e^{\frac{2}{3}\pi i}\right)^2 \left(\rho + x_1 e^{-\frac{2}{3}\pi i}\right)^2} \right) + \\
& + (x_1^{-1} \rho + x_1 \rho^{-1}) \text{Ln} \left((1 + x_1 \rho)^2 \right) + \\
& + (x_1^{-1} \rho + x_1 \rho^{-1}) e^{\frac{2}{3}\pi i} \text{Ln} \left(\left(1 + x_1 \rho e^{-\frac{2}{3}\pi i}\right)^2 \right) + \\
& + (x_1^{-1} \rho + x_1 \rho^{-1}) e^{-\frac{2}{3}\pi i} \text{Ln} \left(\left(1 + x_1 \rho e^{\frac{2}{3}\pi i}\right)^2 \right) + \\
& - 2x_1^{-1} \rho^{-1} \text{Ln} (1 + x_1 \rho) - 2x_1^{-1} \rho^{-1} e^{-\frac{2}{3}\pi i} \text{Ln} \left(1 + x_1 \rho e^{\frac{2}{3}\pi i}\right) - 2x_1^{-1} \rho^{-1} e^{\frac{2}{3}\pi i} \text{Ln} \left(1 + x_1 \rho e^{-\frac{2}{3}\pi i}\right)
\end{aligned}$$

$$\begin{aligned}
&= -(\rho - x_1)^2 + \frac{(1 - x_1\rho)(2 - \rho^2 - x_1^2)}{1 + x_1\rho} - 2\frac{(x_1 - \rho)^2(x_1 + \rho)^2}{\rho^2 - x_1\rho + x_1^2} + 2\frac{(1 + x_1\rho)(1 - x_1\rho)(2 - x_1^2 - \rho^2)}{1 - x_1\rho + x_1^2\rho^2} + \\
&\quad + 2x_1\rho\text{Ln}(x_1\rho + 1) - 4x_1\rho\text{Ln}(x_1 + \rho) + \\
&\quad + 2x_1\rho\left(e^{\frac{2}{3}\pi i}\text{Ln}\left(1 + x_1\rho e^{\frac{2}{3}\pi i}\right) + e^{-\frac{2}{3}\pi i}\text{Ln}\left(1 + x_1\rho e^{-\frac{2}{3}\pi i}\right)\right) + \\
&\quad - 2x_1\rho\left(e^{-\frac{2}{3}\pi i} + e^{\frac{2}{3}\pi i}\right)\text{Ln}\left(\left(\rho + x_1 e^{\frac{2}{3}\pi i}\right)\left(\rho + x_1 e^{-\frac{2}{3}\pi i}\right)\right) + \\
&\quad + 2(x_1^{-1}\rho + x_1\rho^{-1})\text{Ln}(1 + x_1\rho) + \\
&\quad + 2(x_1^{-1}\rho + x_1\rho^{-1})\left(e^{\frac{2}{3}\pi i}\text{Ln}\left(1 + x_1\rho e^{-\frac{2}{3}\pi i}\right) + e^{-\frac{2}{3}\pi i}\text{Ln}\left(1 + x_1\rho e^{\frac{2}{3}\pi i}\right)\right) + \\
&\quad - 2x_1^{-1}\rho^{-1}\text{Ln}(1 + x_1\rho) - 2x_1^{-1}\rho^{-1}\left(e^{-\frac{2}{3}\pi i}\text{Ln}\left(1 + x_1\rho e^{\frac{2}{3}\pi i}\right) + e^{\frac{2}{3}\pi i}\text{Ln}\left(1 + x_1\rho e^{-\frac{2}{3}\pi i}\right)\right) \\
&= -(\rho - x_1)^2 + \frac{(1 - x_1\rho)(2 - \rho^2 - x_1^2)}{1 + x_1\rho} - 2\frac{(x_1 - \rho)^2(x_1 + \rho)^2}{\rho^2 - x_1\rho + x_1^2} + 2\frac{(1 + x_1\rho)(1 - x_1\rho)(2 - x_1^2 - \rho^2)}{1 - x_1\rho + x_1^2\rho^2} + \\
&\quad + 2x_1\rho\ln(x_1\rho + 1) - 4x_1\rho\ln(x_1 + \rho) + \\
&\quad + 2x_1\rho\left(\left(e^{\frac{2}{3}\pi i} + e^{-\frac{2}{3}\pi i}\right)\ln\left|1 + x_1\rho e^{\frac{2}{3}\pi i}\right| + i\left(e^{\frac{2}{3}\pi i} - e^{-\frac{2}{3}\pi i}\right)\arctan\left(\frac{x_1\rho\sin\frac{2}{3}\pi}{1 + x_1\rho\cos\frac{2}{3}\pi}\right)\right) + \\
&\quad + 2x_1\rho\ln(\rho^2 - x_1\rho + x_1^2) + \\
&\quad + 2(x_1^{-1}\rho + x_1\rho^{-1})\ln(1 + x_1\rho) + \\
&\quad + 2(x_1^{-1}\rho + x_1\rho^{-1})\left(\left(e^{\frac{2}{3}\pi i} + e^{-\frac{2}{3}\pi i}\right)\ln\left|1 + x_1\rho e^{-\frac{2}{3}\pi i}\right| + i\left(e^{-\frac{2}{3}\pi i} - e^{\frac{2}{3}\pi i}\right)\arctan\left(\frac{x_1\rho\sin\frac{2}{3}\pi}{1 + x_1\rho\cos\frac{2}{3}\pi}\right)\right) + \\
&\quad - 2x_1^{-1}\rho^{-1}\ln(1 + x_1\rho) + \\
&\quad - 2x_1^{-1}\rho^{-1}\left(\left(e^{-\frac{2}{3}\pi i} + e^{\frac{2}{3}\pi i}\right)\ln\left|1 + x_1\rho e^{\frac{2}{3}\pi i}\right| + i\left(e^{-\frac{2}{3}\pi i} - e^{\frac{2}{3}\pi i}\right)\arctan\left(\frac{x_1\rho\sin\frac{2}{3}\pi}{1 + x_1\rho\cos\frac{2}{3}\pi}\right)\right) \\
&= -(\rho - x_1)^2 + \frac{(1 - x_1\rho)(2 - \rho^2 - x_1^2)}{1 + x_1\rho} - 2\frac{(x_1 - \rho)^2(x_1 + \rho)^2}{\rho^2 - x_1\rho + x_1^2} + 2\frac{(1 + x_1\rho)(1 - x_1\rho)(2 - x_1^2 - \rho^2)}{1 - x_1\rho + x_1^2\rho^2} + \\
&\quad + 2x_1\rho\ln(x_1\rho + 1) - 4x_1\rho\ln(x_1 + \rho) + \\
&\quad + 2x_1\rho\left(-\ln\sqrt{(1 - x_1\rho + x_1^2\rho^2)} - \sqrt{3}\arctan\left(\frac{x_1\rho\sin\frac{2}{3}\pi}{1 + x_1\rho\cos\frac{2}{3}\pi}\right)\right) + \\
&\quad + 2x_1\rho\ln(\rho^2 - x_1\rho + x_1^2) + \\
&\quad + 2(x_1^{-1}\rho + x_1\rho^{-1} - x_1^{-1}\rho^{-1})\ln(1 + x_1\rho) + \\
&\quad + 2(x_1^{-1}\rho + x_1\rho^{-1})\left(-\ln\sqrt{(1 - x_1\rho + x_1^2\rho^2)} + \sqrt{3}\arctan\left(\frac{x_1\rho\sin\frac{2}{3}\pi}{1 + x_1\rho\cos\frac{2}{3}\pi}\right)\right) + \\
&\quad - 2x_1^{-1}\rho^{-1}\left(-\ln\sqrt{(1 - x_1\rho + x_1^2\rho^2)} + \sqrt{3}\arctan\left(\frac{x_1\rho\sin\frac{2}{3}\pi}{1 + x_1\rho\cos\frac{2}{3}\pi}\right)\right) \\
&= -(\rho - x_1)^2 + \frac{(1 - x_1\rho)(2 - \rho^2 - x_1^2)}{1 + x_1\rho} - 2\frac{(x_1 - \rho)^2(x_1 + \rho)^2}{\rho^2 - x_1\rho + x_1^2} + 2\frac{(1 + x_1\rho)(1 - x_1\rho)(2 - x_1^2 - \rho^2)}{1 - x_1\rho + x_1^2\rho^2} + \\
&\quad + 2x_1\rho\ln\left(\frac{(x_1\rho + 1)^2}{(x_1 + \rho)^2}\frac{\rho^2 - x_1\rho + x_1^2}{1 - x_1\rho + x_1^2\rho^2}\right) + \\
&\quad + 2(x_1\rho - x_1^{-1}\rho - x_1\rho^{-1} + x_1^{-1}\rho^{-1})\left(\ln\sqrt{(1 - x_1\rho + x_1^2\rho^2)} - \ln(1 + x_1\rho) - \sqrt{3}\arctan\left(\frac{x_1\rho\sqrt{3}}{2 - x_1\rho}\right)\right).
\end{aligned}$$

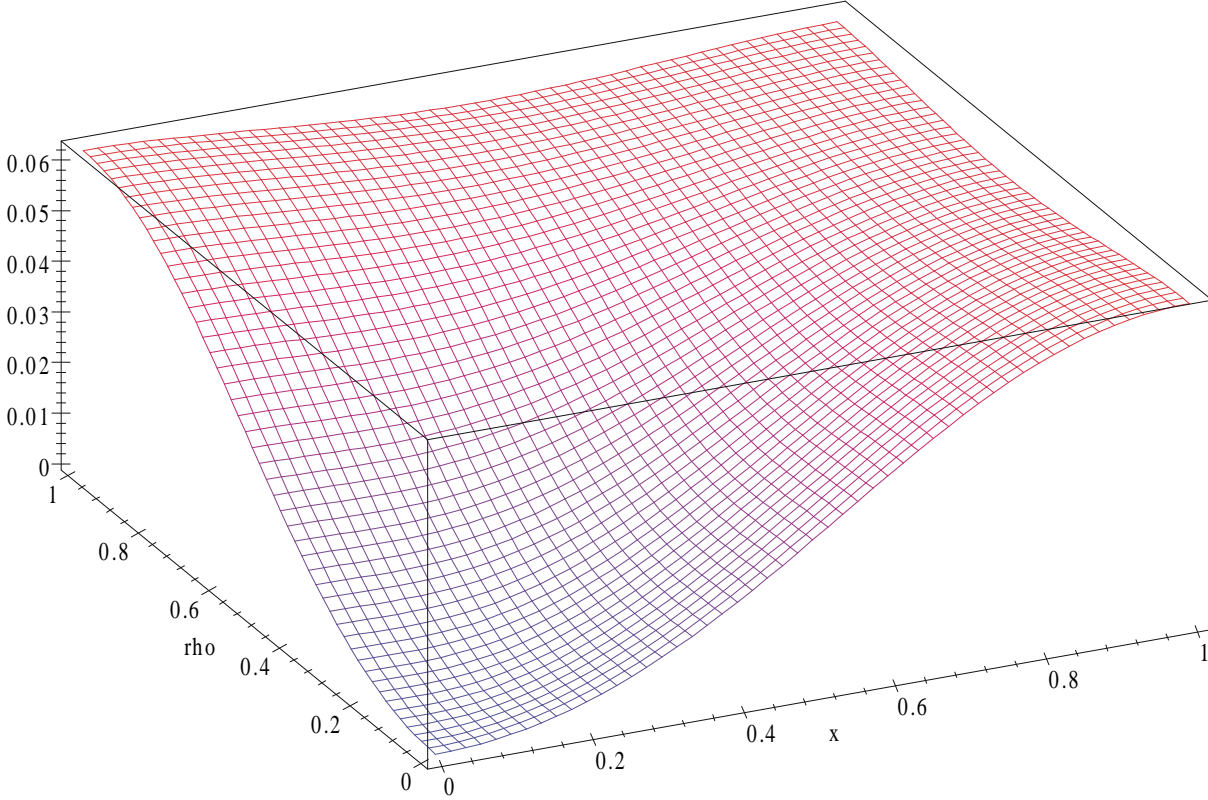


Figure 4: $x \in \Gamma_1$ and $y \in \Gamma_3$ with $|y| = \rho$: $T_{13}(x_1, \rho)$

After this very tedious calculation the enumerator of (77) can be listed as

$$\begin{aligned}
& \frac{1}{8x_1\rho\pi} \left(-(\rho - x_1)^2 + \frac{(1 - x_1\rho)(2 - \rho^2 - x_1^2)}{1 + x_1\rho} - 2\frac{(x_1 - \rho)^2(x_1 + \rho)^2}{\rho^2 - x_1\rho + x_1^2} + \right. \\
& + 2\frac{(1 - x_1^2\rho^2)(2 - x_1^2 - \rho^2)}{1 - x_1\rho + x_1^2\rho^2} + 2x_1\rho \ln \left(\frac{(x_1\rho + 1)^2}{(x_1 + \rho)^2} \frac{\rho^2 - x_1\rho + x_1^2}{1 - x_1\rho + x_1^2\rho^2} \right) + \\
& \left. + \left(x_1\rho - \frac{\rho}{x_1} - \frac{x_1}{\rho} + \frac{1}{x_1\rho} \right) \left(\ln \frac{1 - x_1\rho + x_1^2\rho^2}{(1 + x_1\rho)^2} - 2\sqrt{3} \arctan \left(\frac{x_1\rho\sqrt{3}}{2 - x_1\rho} \right) \right) \right). \tag{104}
\end{aligned}$$

Now an inspection of T_{13} reveals that it has three local maxima, namely at $(0, 1)$, $(1, 0)$ and $(1, 1)$. See Figure 4. These points have already been considered in the previous cases. This concludes the proof of Theorem 1.

Note added in proof: We would like to thank R. Bañuelos for pointing out that our result also provides a counterexample to a conjecture that recently appeared in [1].

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